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PREFACE

Dear Conference Participants,

Welcome to the *Eighth International Conference on Intuitionistic Fuzzy Sets and Contemporary Mathematics* (**IFSCOM-2022**). The aim of our conference is to bring together significant mathematician researchers with different mathematical interests from all over the world. This conference is one of the leading international conferences for presenting novel and fundamental advances in different fields of Mathematics. We want to offer a suitable environment where researchers can exchange ideas, discuss their recent research findings, and collaborate to produce new different ideas. We are pleased to have exceptional researchers in different areas including Algebra, Analysis, Applied Mathematics, Geometry, Graph Theory, Multi-Valued Mathematics, Topology, Statistics, and other fields related to engineering sciences and educational sciences, which are common fields of Mathematics.

It is also the aim of the conference that young researchers and graduate students engage in such exceptional event. Their inputs and participation in such event should encourage them to do more research activities in the future.

We would like to thank all participating scientists who made the most important contribution to this conference. Their contributions are the key ingredient to the success of the conference.
We are sincerely grateful to all participants who really value our work and efforts that we develop every year to improve this conference. We are so proud to reach this respected level of success. Indeed, this was not possible without the outstanding work, efforts and supports from the members of the conference team: Scientific Committee Members, Referee Committee Members and Local Organizing Committee Members.

We are very pleased to present the abstracts of the 8th International Conference on IFS and Contemporary Mathematics **IFSCOM-2022**. The conference was completed with **126** participants and **143** papers. The distribution of research papers delivered by the participants are classified by the following fields: Applied Mathematics (33), Algebra (29), Geometry (17), Topology (16), Analysis (14) Statistics (6) and other fields (23) such as Financial Mathematics, Fuzzy Sets, Game Theory, Geometric Computer Aided Design, Graph Theory, Intuitionistic Fuzzy, Machine Learning and Mathematical Modeling.

Ten invited speakers attended the conference to share information about current studies in different fields with our participants. We have 126 participants participated from 19 countries: Algeria, Australia, Azerbaijan, Bulgaria, France, Greece, India, Indonesia, Kazakhstan, Kuwait, Kyrgyzstan, Mexico, Morocco, North Macedonia, Oman, Russia, Serbia, South Africa and Turkey.

This proceeding booklet contains the titles and proceedings of presented talks during the conference. Many submitted articles to this conference are considered in the following listed journals and books:

Journals:
- Journal of Universal Mathematics (JUM)
- Notes on Intuitionistic Fuzzy Sets (Notes on IFS)
- Sakarya University Journal of Science

Books:
- IFSCOM2022 Abstract Book with an ISBN number
- IFSCOM2022 Proceeding Book with an ISBN number
- SPRINGER Book

We wish that all participants participate in all sessions, ask questions and be active in the conference. We also wish that this conference is a great place where you meet new friends, gain some knowledge, and get yourself involved in some research collaborations.

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Assoc. Prof. Abdullah ALAZEMI **Assoc. Prof. Gökhan ÇUVALCIOĞLU**

June 2022

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A GENERAL FACTOR THEOREM ON MATRIX SUMMABILITY

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Abstract. This article is devoted to the absolute summability of the series $\sum a_n \lambda_n X_n$. The known theorem of Sulaiman [1], which deals with $|A|_k$ summability, is generalized. The sufficient conditions for the $\varphi - \mid A, \beta; \delta \mid_k$ summability of the series $\sum a_n \lambda_n X_n$ are established.

1. Introduction

Let $\sum a_n$ be an infinite series with its partial sums (s_n). Let $A = (a_{nv})$ be a normal matrix, i.e., a lower triangular matrix of non-zero diagonal entries and

$$A_n(s) = \sum_{v=0}^{n} a_{nv} s_v, \quad n = 0, 1, ...$$

Let (φ_n) be any sequence of positive real numbers. The series $\sum a_n$ is said to be summable $\varphi - \mid A, \beta; \delta \mid_k$, $k \geq 1$, $\delta \geq 0$ and β is a real number, if (see [2])

$$\sum_{n=1}^{\infty} \varphi_n^{\beta(\delta k + k - 1)} |A_n(s) - A_{n-1}(s)|^k < \infty.$$

For $\beta = 1$, $\delta = 0$ and $\varphi_n = n$, $\varphi - \mid A, \beta; \delta \mid_k$ summability reduces to the $|A|_k$ summability (see [3]).

2. Known Results

In [1], Sulaiman has obtained the following results.

Lemma 1. *If $\sum n^{-1} \lambda_n$ is convergent, then (λ_n) is non-negative and decreasing, $\lambda_n \log n = O(1)$, and $n \Delta \lambda_n = O(1/(log n)^2)$.*

Lemma 2. *If $\sum n^{-1} \lambda_n X_n$ is convergent, and the conditions*

(1) $$n \Delta \lambda_n = O(\lambda_n) \quad as \quad n \to \infty,$$

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$$\sum_{v=1}^{n} \lambda_v = O(n\lambda_n) \quad as \quad n \to \infty \tag{2}$$

are satisfied, then

$$n\lambda_n \Delta X_n = O(1), \tag{3}$$

$$\sum_{n=1}^{m} \lambda_n \Delta X_n = O(1) \quad as \quad m \to \infty, \tag{4}$$

$$\sum_{n=1}^{m} n\lambda_n \Delta^2 X_n = O(1) \quad as \quad m \to \infty. \tag{5}$$

Let $A = (a_{nv})$ be a normal matrix, then two lower semimatrices $\bar{A} = (\bar{a}_{nv})$ and $\hat{A} = (\hat{a}_{nv})$ are defined as follows:

$$\bar{a}_{nv} = \sum_{i=v}^{n} a_{ni}, \quad n, v = 0, 1, ...$$

$$\hat{a}_{00} = \bar{a}_{00} = a_{00}, \quad \hat{a}_{nv} = \bar{a}_{nv} - \bar{a}_{n-1,v}, \quad n = 1, 2, ...$$

$$A_n(s) = \sum_{v=0}^{n} \bar{a}_{nv} a_v \quad \text{and} \quad \bar{\Delta} A_n(s) = \sum_{v=0}^{n} \hat{a}_{nv} a_v. \tag{6}$$

Theorem 1. *Let (λ_n), (X_n) be two sequences such that $\sum n^{-1}\lambda_n X_n$ is convergent, and the conditions (1) and (2) are satisfied. Let $A = (a_{nv})$ be a normal matrix with non-negative entries satisfying*

$$\bar{a}_{n0} = 1, \quad n = 0, 1, ..., \tag{7}$$

$$a_{n-1,v} \geq a_{nv}, \quad for \quad n \geq v + 1, \tag{8}$$

$$na_{nn} = O(1), \quad 1 = O(na_{nn}) \tag{9}$$

$$\sum_{v=1}^{n-1} a_{vv} \hat{a}_{nv} = O(a_{nn}). \tag{10}$$

If $t_v^k = O(1)(C, 1)$, where $t_v = \frac{1}{v+1} \sum_{r=1}^{v} r a_r$, then the series $\sum a_n \lambda_n X_n$ is summable $|A|_k, k \geq 1$.

3. Main Result

The concern of this paper is to get a more general theorem on $\varphi - |A, \beta; \delta|_k$ summability method. For more results on the topic, see [4–17].

Theorem 2. *Let (λ_n), (X_n) be two sequences such that $\sum n^{-1}\lambda_n X_n$ is convergent, and the conditions (1), (2), (7)-(10) are satisfied. Let (φ_n) be any sequence such that*

$$\varphi_n a_{nn} = O(1), \quad 1 = O(\varphi_n a_{nn})$$

$$\sum_{n=v+1}^{m+1} \varphi_n^{\beta(\delta k + k - 1) - k + 1} |\Delta_v (\hat{a}_{nv})| = O\left(\varphi_v^{\beta(\delta k + k - 1) - k}\right) \quad as \quad m \to \infty,$$

$$\sum_{n=v+1}^{m+1} \varphi_n^{\beta(\delta k+k-1)-k+1} \hat{a}_{n,v+1} = O\left(\varphi_v^{\beta(\delta k+k-1)-k+1}\right) \quad as \quad m \to \infty$$

$$\sum_{v=1}^{n-1} a_{vv} \hat{a}_{n,v+1} = O(a_{nn}).$$

If $\varphi_v^{\beta(\delta k+k-1)-k+1} t_v^k = O(1)(C,1)$, *where* (t_v) *as in Theorem 1, then the series* $\sum a_n \lambda_n X_n$ *is summable* $\varphi - \mid A, \beta; \delta \mid_k$, $k \geq 1$, $\delta \geq 0$ *and* $-\beta(\delta k+k-1)+k > 0$.

4. Proof of Theorem 2

Let $\theta_n = \lambda_n X_n$ and (M_n) be the A-transform of the series $\sum a_n \theta_n$. By (6), we get

$$\bar{\Delta} M_n = \sum_{v=1}^{n} \frac{\hat{a}_{nv} \theta_v}{v} v a_v.$$

Abel's transformation implies that

$$\begin{aligned}
\bar{\Delta} M_n &= \sum_{v=1}^{n-1} \Delta_v \left(\frac{\hat{a}_{nv} \theta_v}{v}\right) \sum_{r=1}^{v} r a_r + \frac{\hat{a}_{nn} \theta_n}{n} \sum_{v=1}^{n} v a_v \\
&= \frac{n+1}{n} a_{nn} \theta_n t_n + \sum_{v=1}^{n-1} \Delta_v \left(\hat{a}_{nv}\right) \theta_v t_v + \sum_{v=1}^{n-1} \hat{a}_{n,v+1} \Delta \theta_v t_v + \sum_{v=1}^{n-1} \frac{\hat{a}_{nv} \theta_v t_v}{v} \\
&= M_{n,1} + M_{n,2} + M_{n,3} + M_{n,4}.
\end{aligned}$$

By using the facts that $\varphi_n a_{nn} = O(1)$, $n a_{nn} = O(1)$, and $\theta_n^{k-1} = O(1)$, we have

$$\begin{aligned}
\sum_{n=1}^{m} \varphi_n^{\beta(\delta k+k-1)} \mid M_{n,1} \mid^k &= \sum_{n=1}^{m} \varphi_n^{\beta(\delta k+k-1)} \left| \frac{n+1}{n} a_{nn} \theta_n t_n \right|^k \\
&= O(1) \sum_{n=1}^{m} \varphi_n^{\beta(\delta k+k-1)} a_{nn}^k \theta_n^k t_n^k \\
&= O(1) \sum_{n=1}^{m} \varphi_n^{\beta(\delta k+k-1)-k+1} \left(\varphi_n a_{nn}\right)^{k-1} a_{nn} \theta_n^k t_n^k \\
&= O(1) \sum_{n=1}^{m} \varphi_n^{\beta(\delta k+k-1)-k+1} a_{nn} \theta_n^k t_n^k \\
&= O(1) \sum_{n=1}^{m} \frac{\varphi_n^{\beta(\delta k+k-1)-k+1} \theta_n \theta_n^{k-1} t_n^k}{n} \\
&= O(1) \sum_{n=1}^{m} \frac{\varphi_n^{\beta(\delta k+k-1)-k+1} \theta_n t_n^k}{n}.
\end{aligned}$$

Applying Abel's transformation to the above sum, we achieve

$$\begin{aligned}
\sum_{n=1}^{m} \varphi_n^{\beta(\delta k+k-1)} \mid M_{n,1} \mid^k &= O(1) \sum_{n=1}^{m-1} \left(\sum_{r=1}^{n} \varphi_r^{\beta(\delta k+k-1)-k+1} t_r^k \right) \Delta \left(\frac{\theta_n}{n} \right) \\
&+ O(1) \left(\sum_{n=1}^{m} \varphi_n^{\beta(\delta k+k-1)-k+1} t_n^k \right) \frac{\theta_m}{m} \\
&= O(1) \sum_{n=1}^{m-1} \Delta\theta_n + O(1) \sum_{n=1}^{m-1} \frac{\lambda_n X_n}{n} + O(1)\lambda_m X_m \\
&= O(1) \quad as \quad m \to \infty,
\end{aligned}$$

by using the hypotheses of Theorem 2 and Lemma 2. Then, by using Hölder's inequality, we obtain

$$\begin{aligned}
\sum_{n=2}^{m+1} \varphi_n^{\beta(\delta k+k-1)} \mid M_{n,2} \mid^k &\leq \sum_{n=2}^{m+1} \varphi_n^{\beta(\delta k+k-1)} \sum_{v=1}^{n-1} \left| \Delta_v \left(\hat{a}_{nv} \right) \right| \theta_v^k t_v^k \left(\sum_{v=1}^{n-1} \left| \Delta_v \left(\hat{a}_{nv} \right) \right| \right)^{k-1} \\
&\leq \sum_{n=2}^{m+1} \varphi_n^{\beta(\delta k+k-1)-k+1} \left(\varphi_n a_{nn} \right)^{k-1} \sum_{v=1}^{n-1} \left| \Delta_v \left(\hat{a}_{nv} \right) \right| \theta_v^k t_v^k \\
&= O(1) \sum_{v=1}^{m} \theta_v^k t_v^k \sum_{n=v+1}^{m+1} \varphi_n^{\beta(\delta k+k-1)-k+1} \left| \Delta_v \left(\hat{a}_{nv} \right) \right| \\
&= O(1) \sum_{v=1}^{m} \varphi_v^{\beta(\delta k+k-1)-k+1} a_{vv} \theta_v^k t_v^k \\
&= O(1) \quad as \quad m \to \infty,
\end{aligned}$$

as in $M_{n,1}$.
Again using Hölder's inequality, and the conditions of Theorem 2, we get

$$\begin{aligned}
\sum_{n=2}^{m+1} \varphi_n^{\beta(\delta k+k-1)} \mid M_{n,3} \mid^k &\leq \sum_{n=2}^{m+1} \varphi_n^{\beta(\delta k+k-1)} \sum_{v=1}^{n-1} \hat{a}_{n,v+1} (\Delta\theta_v)^k t_v^k a_{vv}^{1-k} \left(\sum_{v=1}^{n-1} a_{vv} \hat{a}_{n,v+1} \right)^{k-1} \\
&= O(1) \sum_{n=2}^{m+1} \varphi_n^{\beta(\delta k+k-1)-k+1} \left(\varphi_n a_{nn} \right)^{k-1} \sum_{v=1}^{n-1} \hat{a}_{n,v+1} (\Delta\theta_v)^k t_v^k a_{vv}^{1-k} \\
&= O(1) \sum_{v=1}^{m} (\Delta\theta_v)^k t_v^k a_{vv}^{1-k} \sum_{n=v+1}^{m+1} \varphi_n^{\beta(\delta k+k-1)-k+1} \hat{a}_{n,v+1} \\
&= O(1) \sum_{v=1}^{m} \varphi_v^{\beta(\delta k+k-1)-k+1} (\Delta\theta_v)^k t_v^k v^{k-1} \\
&= O(1) \sum_{v=1}^{m} \varphi_v^{\beta(\delta k+k-1)-k+1} (\Delta\theta_v) t_v^k \\
&= O(1) \sum_{v=1}^{m} \varphi_v^{\beta(\delta k+k-1)-k+1} t_v^k (\Delta\lambda_v X_v + \lambda_{v+1} \Delta X_v) \\
&= O(1) \sum_{v=1}^{m} \varphi_v^{\beta(\delta k+k-1)-k+1} t_v^k \Delta\lambda_v X_v + O(1) \sum_{v=1}^{m} \varphi_v^{\beta(\delta k+k-1)-k+1} t_v^k \lambda_{v+1} \Delta X_v.
\end{aligned}$$

For the first part, using the condition (1), and using Abel's transformation, we get

$$\begin{aligned}
\sum_{v=1}^{m} \varphi_v^{\beta(\delta k+k-1)-k+1} t_v^k \Delta\lambda_v X_v &= O(1) \sum_{v=1}^{m-1} \left(\sum_{r=1}^{v} \varphi_r^{\beta(\delta k+k-1)-k+1} t_r^k \right) \Delta \left(\frac{\lambda_v X_v}{v} \right) \\
&+ O(1) \left(\sum_{v=1}^{m} \varphi_v^{\beta(\delta k+k-1)-k+1} t_v^k \right) \frac{\lambda_m X_m}{m} \\
&= O(1) \sum_{v=1}^{m-1} \frac{\lambda_v X_v}{v} + O(1) \sum_{v=1}^{m-1} \Delta\lambda_v X_v + O(1) \sum_{v=1}^{m-1} \lambda_{v+1} \Delta X_v \\
&+ O(1) \lambda_m X_m \\
&= O(1) \quad as \quad m \to \infty.
\end{aligned}$$

For the second part, again using Abel's transformation, and the conditions (1), (4), (5), (3), we achieve

$$\begin{aligned}
\sum_{v=1}^{m} \varphi_v^{\beta(\delta k+k-1)-k+1} t_v^k \lambda_v \Delta X_v &= \sum_{v=1}^{m-1} \left(\sum_{r=1}^{v} \varphi_r^{\beta(\delta k+k-1)-k+1} t_r^k \right) \Delta \left(\lambda_v \Delta X_v \right) \\
&+ \left(\sum_{v=1}^{m} \varphi_v^{\beta(\delta k+k-1)-k+1} t_v^k \right) \lambda_m \Delta X_m \\
&= O(1) \sum_{v=1}^{m-1} \lambda_v \Delta X_v + \sum_{v=1}^{m-1} v \lambda_{v+1} \Delta^2 X_v + O(1) m \lambda_m \Delta X_m \\
&= O(1) \quad as \quad m \to \infty.
\end{aligned}$$

Eventually, we get $\sum_{n=2}^{m+1} \varphi_n^{\beta(\delta k+k-1)} \mid M_{n,3} \mid^k = O(1) \quad as \quad m \to \infty$.
Finally, as in $M_{n,1}$, we obtain

$$\begin{aligned}
\sum_{n=2}^{m+1} \varphi_n^{\beta(\delta k+k-1)} \mid M_{n,4} \mid^k &\leq \sum_{n=2}^{m+1} \varphi_n^{\beta(\delta k+k-1)} \sum_{v=1}^{n-1} \left(\frac{1}{v} \right)^k \hat{a}_{nv} \theta_v^k t_v^k a_{vv}^{1-k} \left(\sum_{v=1}^{n-1} a_{vv} \hat{a}_{nv} \right)^{k-1} \\
&= O(1) \sum_{n=2}^{m+1} \varphi_n^{\beta(\delta k+k-1)-k+1} \left(\varphi_n a_{nn} \right)^{k-1} \sum_{v=1}^{n-1} \hat{a}_{nv} \theta_v^k t_v^k a_{vv} \\
&= O(1) \sum_{v=1}^{m} \theta_v^k t_v^k a_{vv} \sum_{n=v+1}^{m+1} \varphi_n^{\beta(\delta k+k-1)-k+1} \hat{a}_{nv} \\
&= O(1) \sum_{v=1}^{m} \varphi_v^{\beta(\delta k+k-1)-k+1} a_{vv} \theta_v^k t_v^k = O(1) \quad as \quad m \to \infty.
\end{aligned}$$

Therefore, $\sum_{n=1}^{\infty} \varphi_n^{\beta(\delta k+k-1)} \mid M_{n,r} \mid^k < \infty$ for $r = 1, 2, 3$ and $r = 4$ are obtained, and the proof of Theorem 2 is finished.

5. Conclusions

If we take $\beta = 1$, $\delta = 0$ and $\varphi_n = n$, Theorem 2 reduces to Theorem 1. In addition to this, if we take $\beta = 1$, then we get a theorem which is already proved in [18].

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A SCHRÖDER POLYNOMIAL SOLUTION TO NONLINEAR MICRO-ELECTROMECHANICAL OSCILLATOR EQUATION

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0000-0002-3987-7171

Abstract. This study is dedicated to solving a quadratic nonlinear differential equation arising in the micro-electromechanical oscillator model by means of the matrix-colllocation method based on the Schröder polynomial. The method essentially generates a fundamental matrix equation made up of the matrix expansions of the linear and nonlinear terms of the model using the collocation points. With the elimination of this equation along with the initial conditions, the desired numerical solution is immediately obtained. For sake of overseeing the efficiency and precision of the method, some obtained solutions are differently established according to an excitation parameter. To do this, numerical and graphical instruments are included. Upon investigation of the outcomes, one can admit that the method is very proper to handle the model in question.

1. Introduction

In recent years, nonlinear differential equations have continued to catch much attention in the vast scientific models of mathematics, electricity, acoustics, physics, equation of motion, fluid dynamics, mechanics, etc. [1, 2, 3]. One of which appears in the micro-electromechanical oscillator model (MEOE) as second order, quadratic and rational nonlinear differential equation exposed to magneto-static excitation, which was previously studied in [4]. Essentially, it governs the physical motion of a current-conveying wire and its schematic background is mathematically described in [4].

In this study, by employing the matrix-collocation method endowed with the Schröder polynomial, we shall consider a numerical solution of MEOE in the form (see [4]):

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$$y''(t) + y(t) - \frac{K}{1 - y(t)} = 0,\ t \in [0, T],\ y(t) < 1, \tag{1}$$

subject to the homogeneous initial conditions

$$y(0) = y'(0) = 0, \tag{2}$$

where $K(> 0)$ is the excitation parameter, which determines the periodic solutions of the system according to its threshold.

In case the problem (1)–(2) is solved via the proposed method, it can be smoothed as

$$y''(t) + y(t) - y(t)y''(t) - y^2(t) = K,\ t \in [0, T],\ y(t) < 1, \tag{3}$$

which transforms to a second order quadratic nonlinear differential equation.

A desired Schröder polynomial solution to Eq. (3) is processed as

$$y_N(t) = \sum_{n=0}^{N} a_n S_n(t), \tag{4}$$

where a_n's are the unknown coefficients that have to be determined by the method and $S_n(t)$ is the Schröder polynomial that is explicitly defined to be (see [5])

$$S_n(t) = \sum_{k=0}^{n} \frac{(-1)^{(n-k)}}{k+1} \binom{2k}{k} \binom{n+k}{n-k} t^k,$$

and whose first three elements possess

$$\{S_0(t),\ S_1(t),\ S_2(t)\} = \left\{1,\ t-1,\ 2t^2 - 3t + 1\right\}.$$

One can refer to [5] for more details about Schröder polynomial.

Thereby, our object is here to solve MEOE (3), stating its Schröder polynomial solution by way of the matrix-collocation method.

2. Method of solution based on Schröder polynomial

In this section, the matrix-collocation method is proposed via the matrix expansions of the linear and nonlinear terms at the collocation points. In doing so, let us start with newly constructing the main matrix relation of the Schröder polynomial solution (4) to Eq. (3), as

$$y(t) = \boldsymbol{S}(t)\boldsymbol{A}, \tag{5}$$

where

$$\boldsymbol{S}(t) = \begin{bmatrix} S_0(t) & S_1(t) & \cdots & S_N(t) \end{bmatrix},$$

and

$$\boldsymbol{A} = \begin{bmatrix} a_0 & a_1 & \cdots & a_N \end{bmatrix}^T,$$

The main matrix relation (5) of the differentiated form can be stated as

$$y'(t) = \boldsymbol{S}'(t)\,\boldsymbol{A} \text{ and } y''(t) = \boldsymbol{S}''(t)\,\boldsymbol{A}, \tag{6}$$

where

$$\boldsymbol{S}'(t) = \begin{bmatrix} S_0'(t) & S_1'(t) & \cdots & S_N'(t) \end{bmatrix},$$

and

$$\boldsymbol{S}''(t) = \begin{bmatrix} S_0''(t) & S_1''(t) & \cdots & S_N''(t) \end{bmatrix}.$$

On the other hand, let us now construct the matrix relations of nonlinear terms in Eq. (3). Using the matrix relations (5) and (6), it follows that

$$[y(t)\,y''(t)] = \boldsymbol{S}(t)\,\overline{\boldsymbol{S}''(t)}\;\overline{\boldsymbol{A}}, \tag{7}$$

where

$$\overline{\boldsymbol{S}''(t)} = \mathrm{diag}\big[\boldsymbol{S}''(t)\big]_{(N+1)\times(N+1)^2},$$

and

$$\overline{\boldsymbol{A}} = \begin{bmatrix} a_0\boldsymbol{A} & a_1\boldsymbol{A} & \cdots & a_N\boldsymbol{A} \end{bmatrix}^T_{1\times(N+1)^2}.$$

Analogously, the matrix relation for quadratic nonlinear term can be extracted as

$$\big[y^2(t)\big] = \boldsymbol{S}(t)\,\overline{\boldsymbol{S}(t)}\;\overline{\boldsymbol{A}}, \tag{8}$$

where

$$\overline{\boldsymbol{S}(t)} = \mathrm{diag}[\boldsymbol{S}(t)]_{(N+1)\times(N+1)^2}.$$

Once the standard collocation points

$$t_i = \frac{Ti}{N},\ i = 0, 1, \ldots, N,\ \ t_0 = 0 < t_1 < \ldots < t_N = T,$$

are inserted separately into the matrix relations (5)-(8), this leads to a fundamental matrix equation

$$\big\{\boldsymbol{S}''(t_i) + \boldsymbol{S}(t_i)\big\}\,\boldsymbol{A} - \Big\{\boldsymbol{S}(t_i)\,\overline{\boldsymbol{S}''(t_i)} + \boldsymbol{S}(t_i)\,\overline{\boldsymbol{S}(t_i)}\Big\}\,\overline{\boldsymbol{A}} = \boldsymbol{K},$$

or, briefly,

$$\boldsymbol{W}\boldsymbol{A} + \boldsymbol{Z}\overline{\boldsymbol{A}} = \boldsymbol{K} \Rightarrow [\boldsymbol{W};\ \boldsymbol{Z} : \boldsymbol{K}], \tag{9}$$

where

$$\boldsymbol{W} = \boldsymbol{S}'' + \boldsymbol{S},\ \boldsymbol{Z} = -\boldsymbol{S}\overline{\boldsymbol{S}''} - \boldsymbol{S}\overline{\boldsymbol{S}},$$

and

$$\boldsymbol{K} = \begin{bmatrix} K & K & \cdots & K \end{bmatrix}^T_{1\times(N+1)},$$

such that

$$\boldsymbol{S} = \begin{bmatrix} \boldsymbol{S}(t_0) \\ \boldsymbol{S}(t_1) \\ \vdots \\ \boldsymbol{S}(t_N) \end{bmatrix} = \begin{bmatrix} S_0(t_0) & S_1(t_0) & \cdots & S_N(t_0) \\ S_0(t_1) & S_1(t_1) & \cdots & S_N(t_1) \\ \vdots & \vdots & \ddots & \vdots \\ S_0(t_N) & S_1(t_N) & \cdots & S_N(t_N) \end{bmatrix},$$

$$\boldsymbol{S}'' = \begin{bmatrix} \boldsymbol{S}''(t_0) \\ \boldsymbol{S}''(t_1) \\ \vdots \\ \boldsymbol{S}''(t_N) \end{bmatrix} = \begin{bmatrix} S''_0(t_0) & S''_1(t_0) & \cdots & S''_N(t_0) \\ S''_0(t_1) & S''_1(t_1) & \cdots & S''_N(t_1) \\ \vdots & \vdots & \ddots & \vdots \\ S''_0(t_N) & S''_1(t_N) & \cdots & S''_N(t_N) \end{bmatrix},$$

$$\overline{\boldsymbol{S}} = \mathrm{diag}[\boldsymbol{S}(t_i)]_{(N+1)\times(N+1)^2},\ \overline{\boldsymbol{S}''} = \mathrm{diag}\big[\boldsymbol{S}''(t_i)\big]_{(N+1)\times(N+1)^2},\ i = 0, 1, \ldots, N.$$

On the other hand, the matrix relations of the homogeneous initial conditions (2) are established using the matrix relations (5) and (6), as

$$\begin{aligned} & y\,(0) = \boldsymbol{S}(0)\boldsymbol{A} \equiv 0 \Rightarrow [\boldsymbol{U}_1;\ 0] \Rightarrow \begin{bmatrix} S_0(0) & S_1(0) & \cdots & S_N(0) & : & 0 \end{bmatrix}, \\ & y'\,(0) = \boldsymbol{S}'(0)\boldsymbol{A} \equiv 0 \Rightarrow [\boldsymbol{U}_2;\ 0] \Rightarrow \begin{bmatrix} S'_0(0) & S'_1(0) & \cdots & S'_N(0) & : & 0 \end{bmatrix}. \end{aligned} \tag{10}$$

As of this point of view, the augmented matrix system is now enabled to be stated by the fundamental matrix equation (9) and the conditional matrix forms (10). Indeed, this system is constructed by adding the conditional matrix forms (10) into the places, from which the last two rows of $\boldsymbol{W}$ and $\boldsymbol{Z}$ are removed. Hence, the augmented matrix system holds

$$[\boldsymbol{W}^*\ ;\ \boldsymbol{Z}^*\ :\ \boldsymbol{K}^*],$$

which can be readily eliminated by *Solve* command on Mathematica.

As soon as this system is solved, the unknown coefficients (4) are determined and later, the Schröder polynomial solution is acquired.

3. Micro-electromechanical oscillator model

In this section, a micro-electromechanical oscillator (MEOE) model (3) is approximately treated by the Schröder matrix-collocation method with respect to N and K. To do this, a computer program dependent upon the infrastructure of the method is formed on Mathematica 13. Numerical and graphical illustrations are, thus, precisely obtained.

An Example
The approximate periodic solution (APS) of MEOE (3) was previously given by (see [4])

$$y(t) = \left(1 - \sqrt{1-4K}\right) \sin^2 \left(\frac{1}{2} \sqrt{\frac{\sqrt{3-12K} - 4\sqrt{1-4K} - \sqrt{3}}{\sqrt{3-12K} - 2\sqrt{1-4K} - 2 - \sqrt{3}}}\; t \right).$$

Notice here that the Schröder polynomial solution and APS are tested through the Mathematica solution, which is run by *NDSolve* command on Mathematica, since it has no exact solution. This procedure yields the absolute errors corresponding to the mentioned solutions. Therefore, the robust comparisons are made properly.

Consider MEOE (3) for different values of $K (\in (0, 0.15])$. Implementing the proposed method along with *NDSolve* module for various N and T, the Schröder polynomial solutions are indicated in Figs. 1 and 2. Frankly, in Fig. 1, the Schröder polynomial solution coincides well with the Mathematica solution for relatively high K. Also, in Fig. 2, the Schröder polynomial solution follows the same profile as the Mathematica solution, in spite of the fact that they are on long time interval $T = 5$. For sake of comparison, Table 3 reveals that the the Schröder polynomial solution is in very good agreement with APS. On the other hand, as N is increased, the error function in Fig. 3 decays and this situation can also be noticed in Table 3. In addition, the CPU timing values in Table 3 show very remarkable amount of time in seconds.

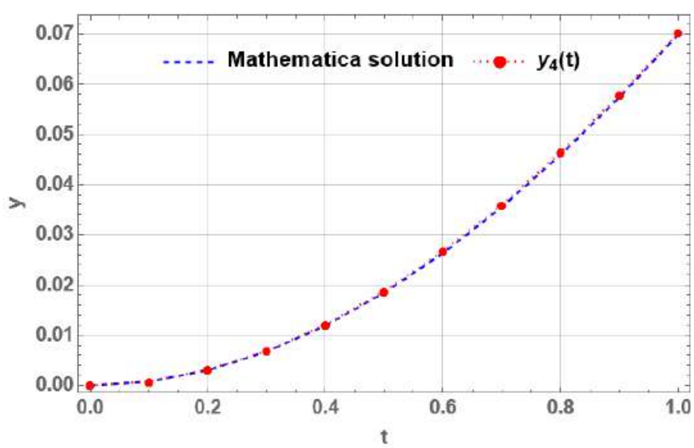

FIGURE 1. Consistent profile between Schröder polynomial and Mathematica solutions versus $K = 0.15$ and $T = 1$.

4. Concluding Remarks

A Schröder matrix-collocation method has been introduced to solve MEOE (3), establishing a sustainable solution form via the Schröder polynomial. The fundamental matrix equation has consistently worked out the method of solution for different N and K. Thereby, the obtained solutions have achieved very good approximation to the Mathematica solution for both normal and long time intervals

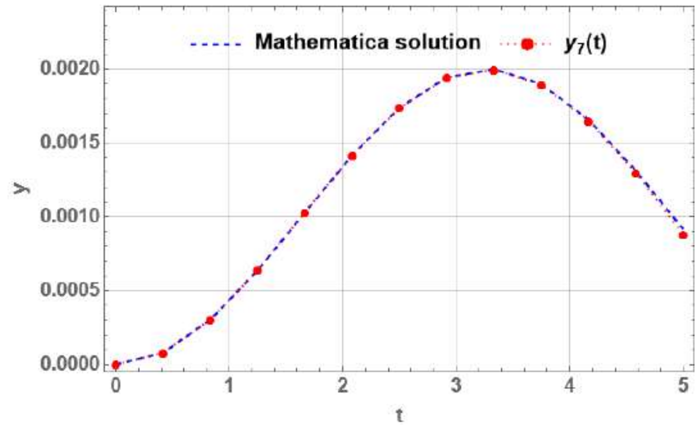

FIGURE 2. Consistent profile between Schröder polynomial and Mathematica solutions versus $K = 0.001$ and $T = 5$.

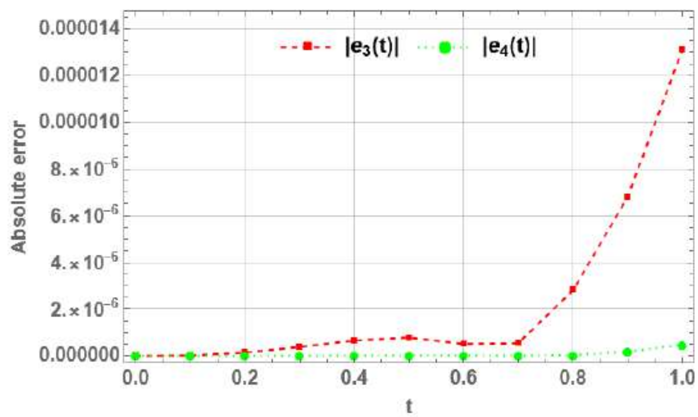

FIGURE 3. Error profiles of the absolute error functions in terms of N, $K = 0.001$ and $T = 1$.

t_i	$N = 7$	APS [4]
0.2	1.8239e–05	1.6623e–05
0.4	5.4594e–05	6.1493e–05
0.6	5.0969e–05	1.2055e–04
0.8	8.9203e–05	1.7344e–04
1.0	4.9651e–04	1.9784e–04

TABLE 1. Comparison of the absolute error values of the Schröder polynomial solution and APS versus $K = 0.15$ and $T = 1$.

as in Figs. 1 and 2. It is also evident from Figs. 1 and 2 that the motion of a solution has been affected due to the interference of the excitation parameter K

t_i	$N = 3$	$N = 4$
0.2	$1.4159e{-}07$	$5.0164e{-}09$
0.4	$6.5479e{-}07$	$1.3662e{-}08$
0.6	$5.0880e{-}07$	$2.2612e{-}08$
0.8	$2.8013e{-}06$	$4.0230e{-}08$
1.0	$1.3092e{-}05$	$4.7820e{-}07$
Timing	0.2969	0.3906

Table 2. Evolution of the absolute error and timing values versus the initial N's, $K = 0.001$ and $T = 1$.

in the method. Numerical values in Tables 3 and 3 validate the precision of the method.

Having investigated the deductions above, one can admit that the proposed method is very proper to handle MEOE (3) and other nonlinear differential equations of cubic or quartic type.

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ON q,ω-CONVOLUTION AND APPLICATIONS

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Abstract. In the previous paper [11], the author presented the q,ω-Laplace transform and its properties and also defined the q,ω-convolution of two functions and proved the q,ω-convolution theorem. In this paper, we deal with the same q,ω-convolution and investigate its other properties and give some examples as applications.

1. Introduction and Preliminaries

Recently, with the definition of different difference operators, such as q-difference oparator (or Jackson difference operator), q,ω-difference operator (or Hahn difference operator), β-general difference operator (see, respectively, $[1-3]$), studies on the classical derivative operator have started to be done for these operators as well.

In $[4]$, the authors studied the properties of the q-Laplace transform and connections to the other functions and integral transforms, such as Mittag-Leffler function, hypergeometric and H-function, Mellin transform, Hankel transform, etc. The q-convolution theorem of the q-Sumudu transform was established in $[5]$. The β-convolution associated with the general quantum difference operator was defined and investigated some properties and applications in $[6]$. More results for q-Laplace transform and related topics can be seen $[7-10]$.

Let $q\in(0,1)$, $\omega>0$, $\omega_0:=\frac{\omega}{1-q}$ and I be an interval of $\mathbb{R}$ containing ω_0. Also let $h(x)=qx+\omega,\ x\in I$ and kth order iteration of $h(x)$ is given by $h^k(x)=q^kx+\omega[k]_q$ where $[k]_q=\frac{1-q^k}{1-q}$. The Hahn difference operator (see $[2,12]$) of f which is defined on I is given by

$$(1)\qquad D_{q,\omega}f(x)=\begin{cases}\frac{f(h(x))-f(x)}{h(x)-x},\ x\neq\omega_0,\\ f'(\omega_0),\ x=\omega_0.\end{cases}$$

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In [11], the q,ω-Laplace transform of a function is defined the following integral representation:

$$L_{q,\omega}\left(f\left(x\right)\right)=F_{q,\omega}\left(s\right):=\frac{1}{s-\omega_0}\int\limits_{\omega_0}^{\omega_0+[\infty]_q}E_{q,-\omega}\left(-h\left(x\right)\right)f\left(\frac{x}{s-\omega_0}\right)d_{q,\omega}x, \tag{2}$$

and alternatively if $f\left(x\right)$ has the q,ω-Taylor expansion at ω_0 of the form $f\left(x\right)=\sum\limits_{n=0}^{\infty}\frac{a_n(x-\omega_0)^n}{[n]_q!}$, then its q,ω-Laplace transform is defined by

$$F_{q,\omega}\left(s\right):=\sum_{n=0}^{\infty}\frac{a_n}{\left(s-\omega_0\right)^{n+1}}, \tag{3}$$

and also q,ω-analogs of its basic properties similar to classical and q-analogs are obtained. Similarly, the definition of q,ω-convolution of two functions and the q,ω-convolution theorem are given in [11], by the followings:

Definition 1.1. The q,ω-analogue of convolution of f and g functions is defined as

$$\left(f*g\right)\left(x\right)=\int\limits_{\omega_0}^{x}f\left(t\right)g\left(x-\left(qt+\omega_0\right)\right)d_{q,\omega}t, \tag{4}$$

where $g\left(x-\left(qt+\omega_0\right)\right)=\left(x-\omega_0-q\left(t-\omega_0\right)\right)_q$ and $\left(1-a\right)_q^n=\prod\limits_{k=0}^{n-1}\left(1-aq^k\right)$, $\left(1-a\right)_q^0=1$.

Theorem 1.2. *Let f and g be two functions and $F_{q,\omega}\left(s\right)$ and $G_{q,\omega}\left(s\right)$ be their q,ω-Laplace transform. Then*

$$L_{q,\omega}\left(\left(f*g\right)\left(x\right)\right)=F_{q,\omega}\left(s\right)G_{q,\omega}\left(s\right). \tag{5}$$

In this paper, we will make a more detailed analysis of the q,ω-convolution and give some properties and applications. Throughout this paper, we will use some results and properties of q,ω-Laplace transform given in [11].

2. Some Properties and Applications of q,ω-Convolution

Theorem 2.1. *Let f, g and h be the q,ω-integrable functions and $c\in\mathbb{C}$, then the following properties hold:*

- **i:** $c\left(f*g\right)=cf*g=f*cg$,
- **ii:** $\left(f+g\right)*h=f*g+g*h$.

Proof. According to (4) and the q,ω-integral definition (see [12]) the proof will be omitted. □

Definition 2.2. The partial q,ω-derivative for a multivariable continuous function $f\left(x,t,...\right)$ is given by

$$\partial_{q,\omega}^{x}f\left(x,t,...\right)=\frac{f\left(h\left(x\right),t,...\right)-f\left(x,t,...\right)}{h\left(x\right)-x},\ x\neq\omega_0. \tag{6}$$

where $\partial_{q,\omega}^{x}:=\frac{\partial_{q,\omega}}{\partial_{q,\omega}x}$.

Lemma 2.3. *If* $F(x) := \int\limits_{\omega_0}^{x} f(x,t)\, d_{q,\omega}t$, *then* $D_{q,\omega}F(x)$ *at* $x \neq \omega_0$ *exists and is given by*

$$D_{q,\omega}F(x) = f(h(x),x) + \int\limits_{\omega_0}^{x} \partial_{q,\omega}^{x} f(x,t)\, d_{q,\omega}t. \tag{7}$$

Proof. From the q,ω-integral definition (see [12])

$$\begin{aligned}
\int\limits_{\omega_0}^{x} f(x,t)\, d_{q,\omega}t &= (x(1-q)-\omega) \sum_{k=0}^{\infty} q^k f\left(x, q^k x + \omega [k]_q\right) \\
&= (1-q)(x-\omega_0) \sum_{k=0}^{\infty} q^k f\left(x, h^k(x)\right),
\end{aligned}$$

and from (1), we have

$$\begin{aligned}
&D_{q,\omega}\left(\int\limits_{\omega_0}^{x} f(x,t)\, d_{q,\omega}t\right) \\
&= \frac{(1-q)(h(x)-\omega_0) \sum\limits_{k=0}^{\infty} q^k f\left(h(x), h^{k+1}(x)\right) - (1-q)(x-\omega_0) \sum\limits_{k=0}^{\infty} q^k f\left(x, h^k(x)\right)}{h(x)-x} \\
&= \frac{(1-q)(x-\omega_0) \sum\limits_{k=0}^{\infty} q^k \left\{ f\left(h(x), h^k(x)\right) - f\left(x, h^k(x)\right) \right\}}{h(x)-x} + f(h(x),x) \\
&= f(h(x),x) + (1-q)(x-\omega_0) \sum_{k=0}^{\infty} q^k \partial_{q,\omega}^{x} f\left(x, h^k(x)\right) \\
&= f(h(x),x) + \int\limits_{\omega_0}^{x} \partial_{q,\omega}^{x} f(x,t)\, d_{q,\omega}t.
\end{aligned}$$

□

Lemma 2.4. *Let* f *and* g *be the* q,ω*-integrable functions, then we have*

$$D_{q,\omega}(f * g)(x) = f(x)\, g(\omega_0) + (f * \partial_{q,\omega} g)(x). \tag{8}$$

Proof. By Definition 1.1 and Lemma 2.3 and using the equality $h(x) - q(x-\omega_0) = \omega_0$, we get

$$\begin{aligned}
D_{q,\omega}(f * g)(x) &= f(x)\, g(h(x) - q(x-\omega_0)) + \int\limits_{\omega_0}^{x} f(t)\, \partial_{q,\omega}^{x} g(x - q(t-\omega_0))\, d_{q,\omega}t \\
&= f(x)\, g(\omega_0) + \left(f * \partial_{q,\omega}^{x} g\right)(x).
\end{aligned}$$

□

In view of the equality ([11])

$$L_{q,\omega}(D_{q,\omega} f(x)) = (s-\omega_0)\, F_{q,\omega}(s) - f(\omega_0), \tag{9}$$

and using Theorem 1.2, we get

$$\begin{aligned} L_{q,\omega}\left(D_{q,\omega}\left(f*g\right)(x)\right) &= \left(s-\omega_0\right)L_{q,\omega}\left(\left(f*g\right)(x)\right)-\left(f*g\right)\left(\omega_0\right) \\ &= \left(s-\omega_0\right)F_{q,\omega}\left(s\right)G_{q,\omega}\left(s\right). \end{aligned} \tag{10}$$

If we take the inverse q,ω-Laplace transform and using Lemma 2.4, then we have

$$\begin{aligned} &L^{-1}_{q,\omega}\left(\left(s-\omega_0\right)F_{q,\omega}\left(s\right)G_{q,\omega}\left(s\right)\right) \\ &= f\left(x\right)g\left(\omega_0\right)+\int_{\omega_0}^{x} f\left(t\right)\partial^{x}_{q,\omega}g\left(x-q(t-\omega_0)\right)d_{q,\omega}t. \end{aligned} \tag{11}$$

Now we give some applications of q,ω-convolution theorem.

Example 2.5. Using the q,ω-convolution theorem, find the followings for $\alpha,\beta\in\mathbb{R}$

i: $\left(x-\omega_0\right)^{\alpha}*\left(x-\omega_0\right)^{\beta}$
ii: $e_{q,\alpha\omega}\left(\alpha x\right)*e_{q,\beta\omega}\left(\beta x\right)$

For i: Using Theorem 1.2, we have

$$\begin{aligned} L_{q,\omega}\left(\left(x-\omega_0\right)^{\alpha}*\left(x-\omega_0\right)^{\beta}\right) &= L_{q,\omega}\left(x-\omega_0\right)^{\alpha}L_{q,\omega}\left(x-\omega_0\right)^{\beta} \\ &= \frac{\Gamma_{q,\omega}\left(\alpha+1\right)}{\left(s-\omega_0\right)^{\alpha+1}}\frac{\Gamma_{q,\omega}\left(\beta+1\right)}{\left(s-\omega_0\right)^{\beta+1}} \\ &= \frac{[\alpha]_q!\,[\beta]_q!}{\left(s-\omega_0\right)^{\alpha+\beta+2}}. \end{aligned}$$

Then by taking the inverse q,ω-Laplace transform we get

$$\begin{aligned} \left(x-\omega_0\right)^{\alpha}*\left(x-\omega_0\right)^{\beta} &= [\alpha]_q!\,[\beta]_q!L^{-1}_{q,\omega}\left(\frac{1}{\left(s-\omega_0\right)^{\alpha+\beta+2}}\right) \\ &= [\alpha]_q!\,[\beta]_q!\frac{\left(x-\omega_0\right)^{\alpha+\beta+1}}{[\alpha+\beta+1]_q!} \\ &= \left(x-\omega_0\right)^{\alpha+\beta+1}B_{q,\omega}\left(\alpha+1,\beta+1\right), \end{aligned}$$

where $B_{q,\omega}\left(t,s\right)=\int_{\omega_0}^{\omega_0+1}\left(x-\omega_0\right)^{t-1}\left(1-q\left(t-\omega_0\right)\right)^{s-1}_q d_{q,\omega}t$ (see [11]).

For ii: Using Theorem 1, we have

$$\begin{aligned} L_{q,\omega}\left(e_{q,\alpha\omega}\left(\alpha x\right)*e_{q,\beta\omega}\left(\beta x\right)\right) &= \frac{1}{s-\omega_0-\alpha}\frac{1}{s-\omega_0-\beta} \\ &= \frac{1}{\alpha-\beta}\left\{\frac{1}{s-\omega_0-\alpha}-\frac{1}{s-\omega_0-\beta}\right\}. \end{aligned}$$

Then appliying the inverse q,ω-Laplace transform we obtain

$$\begin{aligned} &e_{q,\alpha\omega}\left(\alpha x\right)*e_{q,\beta\omega}\left(\beta x\right) \\ &= \frac{1}{\alpha-\beta}\left\{L^{-1}_{q,\omega}\left(\frac{1}{s-\omega_0-\alpha}\right)-L^{-1}_{q,\omega}\left(\frac{1}{s-\omega_0-\beta}\right)\right\} \\ &= \frac{1}{\alpha-\beta}\left\{e_{q,\alpha\omega}\left(\alpha x\right)-e_{q,\beta\omega}\left(\beta x\right)\right\}. \end{aligned}$$

Example 2.6. Using the q,ω-Laplace transform and q,ω-convolution theorem, find the solution of the q,ω-initial value problem

$$D_{q,\omega}^{2}y\left(x\right)+c^{2}y\left(x\right)=c^{2}f\left(x\right),$$
$$y\left(\omega_{0}\right)=0,\quad D_{q,\omega}y\left(\omega_{0}\right)=0.$$

By taking the q,ω-Laplace transform and using its properties (see [11]), we have

$$\left(s-\omega_{0}\right)^{2}Y_{q,\omega}\left(s\right)+c^{2}Y_{q,\omega}\left(s\right)=c^{2}F_{q,\omega}\left(s\right),$$

where $L_{q,\omega}y\left(x\right)=Y_{q,\omega}\left(s\right)$ and $L_{q,\omega}f\left(x\right)=F_{q,\omega}\left(s\right)$,so that

$$\begin{aligned} Y_{q,\omega}\left(s\right) &= \frac{c^{2}F_{q,\omega}\left(s\right)}{\left(s-\omega_{0}\right)^{2}+c^{2}} \\ &= L_{q,\omega}\left(c\sin_{q,c\omega}\left(cx\right)\right)F_{q,\omega}\left(s\right) \\ &= L_{q,\omega}\left(c\sin_{q,c\omega}\left(cx\right)*f\left(x\right)\right), \end{aligned}$$

hence

$$y\left(x\right)=c\sin_{q,c\omega}\left(cx\right)*f\left(x\right).$$

3. Conclusion

The present paper continues the paper [11] which based on q,ω-Laplace transform and its properties. Also the q,ω-convolution definition and theorem was given in [11] . The aim of the present paper is to investigate the other properties and applications of q,ω-convolution.

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(ψ, φ)-CONTRACTION ON Δ-SYMMETRIC QUASI-METRIC SPACE

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ABSTRACT. In this paper, in the setting of Δ-symmetric quasi-metric spaces, we introduce the notion of (ψ, φ)-contraction and we prove a fixed point theorem for (ψ, φ)-contractive mappings.

1. INTRODUCTION

Banach fixed-point theorem [4] has plenty of extension. One of them is the following theorem, given by Boyd and Wong.

Theorem 1.1. [5] *Let (X, d) be a complete metric space and $T : X \to X$ a mapping which satisfies the contractive type condition:*

$$d(Tx, Ty) \leq \varphi(d(x, y)) for all x, y \in X,$$

where $\varphi : \mathbb{R}^+ \to \mathbb{R}$ is a functions such that

(i) $\varphi(t) < t$ *for all* $t > 0$,
(ii) $\limsup_{s \to t+} \varphi(s) < t$ *for all* $t > 0$.

Then, T has a unique fixed point $z \in X$ and $T^n(x_0) \to z$ for each $x_0 \in X$, as $n \to \infty$.

The significant result of Proinov is the following:

Theorem 1.2. [1] *Let (X, d) be a complete metric space and $T : X \to X$ a mapping which satisfies the contractive type condition:*

$$\psi(d(Tx, Ty)) \leq \varphi(d(x, y)) for all x, y \in X with d(Tx, Ty) > 0,$$

where $\psi, \varphi : \mathbb{R}^+ \to \mathbb{R}$ are two functions such that

(i) $\varphi(t) < \psi(t)$ *for all* $t > 0$,
(ii) ψ *is non-decreasing,*
(iii) $\limsup_{t \to \epsilon+} \varphi(t) < \psi(\epsilon+)$ *for all* $t > 0$.

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Then, T has a unique fixed point $z \in X$ and $T^n(x_0) \to z$ for each $x_0 \in X$, as $n \to \infty$.

First of all we shall fix the basic notations: Throughout the paper, $\mathbb{N}$ and $\mathbb{N}_0$ denote the set of positive integers and the set of nonnegative integers. Similarly, let $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{R}_0^+$ represent the set of reals, positive reals and the set of nonnegative reals, respectively. Throughout the paper, all consider set $\mathcal{Q}$ is non-empty.

Definition 1.3. [2] A function $\mathcal{Q} \times \mathcal{Q} \to \mathbb{R}_0^+$ is called quasi-metric if the following hold:

(q1) $q(u,v) = q(v,u) = 0 \Leftrightarrow u = v$;
(q2) $q(u,w) \leq q(u,v) + q(v,w)$,for all $u, v, w \in \mathcal{Q}$

The pair $(\mathcal{Q}, q)$ is called a quasi-metric space.

Definition 1.4. [2] Let $(\mathcal{Q}, q)$ be a quasi metric space. We say that it is Δ-symmetric if there exists a positive real number $\Delta > 0$ such that:

$$q(u,v) \leq \Delta q(v,u) \ for \ all \ s,t \in \mathcal{Q}$$

The pair $(\mathcal{Q}, q_\delta)$ is called a Δ-symmetric quasi-metric space.

In the case of $\Delta = 1$, $(\mathcal{Q}, q_\delta)$ become a metric space.

Example 1.5. Let $(\mathcal{Q}, q)$ be a quasi metric space and a function $q : \mathcal{Q} \times \mathcal{Q} \to \mathbb{R}^+$ defined as follows:

$$q(u,v) = \begin{cases} 5d(v,u), & \text{if } v \geq u \\ d(u,v), & \text{otherwise} \end{cases}$$

$(\mathcal{Q}, q)$ is a five-symmetric quasi-metric space, but it is not a metric space

For recalling the main properties of $(\mathcal{Q}, q_\delta)$, we give the following:

Lemma 1.6. [3] *Suppose that $(\mathcal{Q}, q_\delta)$ be a Δ-symmetric quasi-metric space and $\{a_n\}$ be a sequence in $\mathcal{Q}$ and $a \in \mathcal{Q}$. Then,*

i) *$\{a_n\}$ right-converges to $a \Leftrightarrow \{a_n\}$ left-converges to $a \Leftrightarrow \{a_n\}$ converges to a.*
ii) *$\{a_n\}$ is right-Cauchy $\Leftrightarrow \{a_n\}$ is left-Cauchy$\Leftrightarrow \{a_n\}$ is Cauchy.*
iii) *If $\{b_n\}$ is a sequence in $\mathcal{Q}$ and $q_\delta(a_n, b_n) \to 0$ then $q_\delta(b_n, a_n) \to 0$.*

Lemma 1.7. *Let $(\mathcal{Q}, q_\delta)$ be a Δ-symmetric quasi-metric space and $\{x_n\}$ be a sequence in $\mathcal{Q}$ such that $q_\delta(x_n, x_{n+1}) \to 0$ as $n \to \infty$. If $\{x_n\}$ is not Cauchy sequence then there exists an $\epsilon > 0$ and sequence of positive integers $\{m_k\}$ and $\{n_k\}$ with $m_k > n_k$ such that*

i)

$$\lim_{k\to\infty} q_\delta(x_{n_{k+1}}, x_{m_{k+1}}) = \epsilon+ \tag{1.1}$$

ii)

$$\lim_{k\to\infty} q_\delta(x_{n_k}, x_{m_k}) = \lim_{k\to\infty} q_\delta(x_{n_{k+1}}, x_{m_k}) = \lim_{k\to\infty} q_\delta(x_{n_k}, x_{m_{k+1}}) = \epsilon \tag{1.2}$$

Proof. If If $\{x_n\}$ is not Cauchy sequence and $\lim_{k\to\infty} q_\delta(x_n, x_{n+1}) \to 0$ then there exists an $\epsilon > 0$ and N_0 such that for every $N > N_0$ there exists sequences of positive integers $\{m_k\}$ and $\{n_k\}$ with $m_k > n_k > N$ satisfying $q_\delta(x_{n_k+1}, x_{m_k+1}) > \epsilon$. By choosing m_k, the least positive integer satisfying $q_\delta(x_{n_k+1}, x_{m_k+1}) > \epsilon$ we have

$q_\delta(x_{n_k+1}, x_{m_k+1}) > \epsilon$ and $q_\delta(x_{n_k+1}, x_{m_k}) < \epsilon$ with $m_k > n_k > N$. By these inequalities and the triangular inequality, we have

$$\begin{aligned}\epsilon < q_\delta(x_{n_k+1}, x_{m_k+1}) &\le q_\delta(x_{n_k+1}, x_{m_k}) + q_\delta(x_{m_k}, x_{m_k+1})\\ &\le \epsilon + \Delta q_\delta(x_{m_k+1}, x_{m_k}), where \Delta > 0.\end{aligned}$$

We obtain the limit in 1.1.

We have $\lim_{k\to\infty} q_\delta(x_{n_{k+1}}, x_{m_k}) < \epsilon$. Hence,

$$\limsup_{k\to\infty} q_\delta(x_{n_{k+1}}, x_{m_k}) < \epsilon$$

By using triangular inequality, we have

$$\epsilon < q_\delta(x_{n_k+1}, x_{m_k+1}) \le q_\delta(x_{n_k+1}, x_{m_k}) + q_\delta(x_{m_k}, x_{m_k+1})$$

Hence,

$$\epsilon \le \liminf_{k\to\infty} q_\delta(x_{n_k+1}, x_{m_k}) \le \limsup_{k\to\infty} q_\delta(x_{n_k+1}, x_{m_k}) \le \epsilon$$

Thus we get $\lim_{k\to\infty} q_\delta(x_{n_{k+1}}, x_{m_k}) = \epsilon$ in 1.2.

By using triangular inequality, we have

$$\begin{aligned}\epsilon < q_\delta(x_{n_k+1}, x_{m_k+1}) &\le q_\delta(x_{n_k+1}, x_{n_k}) + q_\delta(x_{n_k}, x_{m_k+1})\\ &\le \Delta q_\delta(x_{n_k}, x_{n_k+1}) + q_\delta(x_{n_k}, x_{m_k+1})\end{aligned}$$

So we have, $\epsilon \le \liminf_{k\to\infty} q_\delta(x_{n_k}, x_{m_k+1})$ and from

$$q_\delta(x_{n_k}, x_{m_k+1}) \le q_\delta(x_{n_k}, x_{n_k+1}) + q_\delta(x_{n_k+1}, x_{m_k+1})$$

we get

$$\limsup_{k\to\infty} q_\delta(x_{n_k}, x_{m_k+1}) \le \epsilon.$$

Thus we get $\lim_{k\to\infty} q_\delta(x_{n_k}, x_{m_k+1}) = \epsilon$ in 1.2.

Now consider

$$\begin{aligned}\epsilon < q_\delta(x_{n_k+1}, x_{m_k+1}) &\le q_\delta(x_{n_k+1}, x_{n_k}) + q_\delta(x_{n_k}, x_{m_k}) + q_\delta(x_{m_k}, x_{m_k+1})\\ &\le \Delta q_\delta(x_{n_k}, x_{n_k+1}) + q_\delta(x_{n_k}, x_{m_k}) + q_\delta(x_{n_k}, x_{m_k+1})\end{aligned}$$

So we have, $\epsilon \le \liminf_{k\to\infty} q_\delta(x_{n_k}, x_{m_k})$ and from

$$q_\delta(x_{n_k}, x_{m_k}) \le q_\delta(x_{n_k}, x_{n_k+1}) + q_\delta(x_{n_k+1}, x_{m_k}),$$

we get

$$\limsup_{k\to\infty} q_\delta(x_{n_k}, x_{m_k}) \le \epsilon.$$

Thus we get $\lim_{k\to\infty} q_\delta(x_{n_k}, x_{m_k}) = \epsilon$ in 1.2.

□

2. Main result

Now we are ready to state our main theorem that is the extension of Theorem 1.2.

Theorem 2.1. *Let $(\mathcal{Q}, q_\delta)$ be a complete Δ-symmetric quasi-metric space and $T : \mathcal{Q} \to \mathcal{Q}$ a mapping which satisfies the contractive type condition:*

$$\psi(q_\delta(Tu, Tv)) \le \varphi(q_\delta(u, v)) \ for\ all\ u.v \in \mathcal{Q}\ with\ q_\delta(Tu, Tv) > 0, \tag{2.1}$$

where $\psi, \varphi : \mathbb{R}^+ \to \mathbb{R}$ are two functions such that

(i) *$\varphi(t) < \psi(t)$ for all $t > 0$,*
(ii) *ψ is non-decreasing,*
(iii) *$\limsup_{t\to\epsilon+} \varphi(t) < \psi(\epsilon+)$ for all $t > 0$.*

Then, T has a unique fixed point $z \in \mathcal{Q}$ and $T^n(x_0) \to z$ for each $x_0 \in \mathcal{Q}$, as $n \to \infty$.

Proof. Starting with the point $x_0 \in \mathcal{Q}$ we define sequence $\{x_n\}$

$$x_1 = Tx_0, x_2 = Tx_1, ..., Tx_{n+1} = Tx_n, ...$$

with $x_n \neq x_{n+1} \neq x_{n+2}$ $\forall n \in \mathbb{N}$. (Otherwise we can find $n_0 \in \mathbb{N}$ such that $x_{n_0} = x_{n_0+1}$ and we have x_{n_0} is the fixed point). Let $u = x_n$ and $v = x_{n+1}$ in 2.1 and $d(Tx_{n-1}, Tx_n) = q_\delta(x_n, x_{n+1}) > 0$ we get

$$\psi(q_\delta(x_n, x_{n+1})) \leq \psi(q_\delta(Tx_{n-1}, Tx_n)) \leq \varphi(q_\delta(x_{n-1}, x_n))$$

and by the condition [(i)] of Theorem 2.1 we have

$$\psi(q_\delta(x_n, x_{n+1})) \leq \psi(q_\delta(x_{n-1}, x_n))$$

and taking [(ii)] of Theorem 2.1 in to account we find that $q_\delta(x_n, x_{n+1}) < q_\delta(x_{n-1}, x_n)$. This shows us that the sequence x_n , where $x_n = q_\delta(x_n, x_{n+1})$ $\forall n \in \mathbb{N}$, is a non-increasing sequence of positive real numbers, so that there exists $\zeta \geq 0$ such that $\lim_{n\to\infty} x_n = \zeta$. We try to show that $\zeta = 0$. Let us assume that $\zeta > 0$, taking the limit superior in the equality

$$\psi(x_n) \leq \varphi(x_{n-1}) \leq \psi(x_{n-1})$$

and taking in to account [(iii)] we get

$$\psi(\zeta+) = \lim_{n\to\infty} \psi(x_n) \leq \limsup_{n\to\infty} \varphi(x_{n-1}) < \limsup_{n\to\infty} \psi(x_{n-1}) < \psi(\zeta+)$$

which is a contradiction so that

$$\lim_{n\to\infty} q_\delta(x_{n-1}, x_n) = 0.$$

Next, we prove that the sequence x_n is Cauchy. Assume that it is not. By Lemma 1.7, we can find $\epsilon > 0$ and two sequences of positive real numbers α_k and β_k, $\beta_k > \alpha_k > k$ such that $q_\delta(x_{\alpha_k+1}, x_{\beta_k+1}) > \epsilon$ for all $k \geq 1$. Consider 2.1 with $u = x_{\alpha_k}$ and $v = x_{\beta_k}$ we get

$$\psi(q_\delta(x_{\alpha_k+1}, x_{\beta_k+1})) \leq \varphi(q_\delta(x_{\alpha_k}, x_{\beta_k})). \tag{2.2}$$

Hence, taking in to account that $\varphi < \psi$, we obtain

$$\psi(q_\delta(x_{\alpha_k+1}, x_{\beta_k+1})) \leq \varphi(q_\delta(x_{\alpha_k}, x_{\beta_k})) < \psi(q_\delta(x_{\alpha_k}, x_{\beta_k})).$$

From the monotonicity of ψ, we have $q_\delta(x_{\alpha_k+1}, x_{\beta_k+1}) < q_\delta(x_{\alpha_k}, x_{\beta_k})$. It follows from Lemma 1.7 that $q_\delta(x_{\alpha_k+1}, x_{\beta_k+1}) \to \epsilon+$ and $q_\delta(x_{\alpha_k}, x_{\beta_k}) \to \epsilon+$. By taking limit superior in 2.2, we get

$$\psi(\epsilon+) = \lim_{k\to\infty} \psi(q_\delta(x_{\alpha_k+1}, x_{\beta_k+1})) \leq \limsup_{k\to\infty} \varphi(q_\delta(x_{\alpha_k}, x_{\beta_k})) < \psi(\epsilon+)$$

Which is a contradiction to [(ii)] of Theorem 2.1.

Since $(\mathcal{Q}, q_\delta)$ be a complete Δ-symmetric quasi-metric space, we can find $z \in \mathcal{Q}$ such that

$$\lim_{n\to\infty} (q_\delta(x_n, z)) = 0$$

Of course, if we suppose that $q_\delta(Tx_k, Tz) = 0$ for (infinetly) many values of k, then

$$q_\delta(z, Tz) \leq q_\delta(z, Tx_k) + q_\delta(Tx_k, Tz) = q_\delta(z, x_{k+1}) + \Delta q_\delta(z, x_{k+1}).$$

Letting $k \to \infty$ the above inequalities lead us to $q_\delta(z, Tz) \leq 0$, this means that is $Tz = z$, that is z is a fixed point of T.

Supposing that $z^* \in \mathcal{Q}$ is an other fixed point of T, but different than z we have that $q_\delta(Tz, Tz^*) > 0$ and then by 2.1 and [(i)] of Theorem 2.1,

$$\psi(q_\delta(z, z^*)) \leq \psi(q_\delta(Tz, Tz^*)) \leq \varphi(q_\delta(z, z^*)) < \psi(q_\delta(z, z^*))$$

and taking [(ii)] of Theorem 2.1 in to account and monotonicity of ψ, we obtain $q_\delta(z, z^*) < q_\delta(z, z^*)$, which is a contradiction. Thus we have $z = z^*$. □

Theorem 2.1 extended in the setting (α, ψ, φ)-contraction by [6].

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APPROXIMATION IN A VARIABLE BOUNDED INTERVAL

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Abstract. In this paper, the important approach features of a Kantorovich-type modified operator on a mobile range will be examined. Approximation results will be given practically with numerical calculations and graphics.

1. Introduction

After Korovkin's theorem proof that he gave the necessary conditions for the uniform convergence of linear positive operator sequences to a function, many operators and many spaces in which convergence were investigated were defined in this field. Approximation to functions by convenient operators for work in many fields includes a technique that makes things easier. With this intention, many operators have been given and made available to researchers. A few examples of these operators can be given as follows: [1], [2], [3], [4], [5], [6]. After the occur of quantum theory, q modifications of many known operators have been created. For example: [7], [8], [9]. With the emergence of post-quantum theory in many areas of mathematics, (p, q)versions of operators began to be defined. [10], [11], [12], [13] are a few examples of these operators. One of the important operators, whose modifications were defined and developed by different researchers at different times, is the Gadjiev Ibragimov operator. This operator was first defined in [14]. Here are some examples of later work: [15], [16], [17], [18], [19], [20], [21], [22], [23], [24], [25]. In the study, first of all, the necessary equations for the main theorem are proven, then important numerical calculations are made by showing that the operator provides a Korovkin type theorem. With the applications made later, approximation and rate of convergence of the operator were examined.

2. Basic Definitions and Theorems

Definition 2.1. Let $\mu, \psi \in \mathbb{R}^{+}$, ρ_r and ξ_r be sequences of real numbers provides the following properties

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$\lim_{r\to\infty}\xi_r=\infty$, $\lim_{r\to\infty}\frac{\rho_r}{\xi_r+r+\psi}=0$, $\lim_{r\to\infty}\frac{r+\mu}{\xi_r+r+\psi}=0$ and $\lim_{r\to\infty}\frac{\rho_r}{\xi_r+r+\psi}r$ $=1$. Also,let $R_{r,\theta}(x)$ be a function that satisfies the following conditions depending on θ and r parameters:

i) For all $r,\theta\in\mathbb{N}_0$ and all $x\in\left[0,\frac{r+\mu}{r+\psi}\right]$,

$$(-1)^{\theta}R_{r,\theta}(x)\geq 0.$$

ii) For all $x\in\left[0,\frac{r+\mu}{r+\psi}\right]$,

$$\sum_{\theta=0}^{\infty}R_{r,\theta}(x)\frac{(-\rho_r)^{\theta}}{\theta!}=1,$$

iii) For every $x\in\left[0,\frac{r+\mu}{r+\psi}\right]$,let m be an integer where $r+m$ is a natural number.

$$R_{r,\theta}(x)=-rxR_{r+m,\theta-1}(x)$$

With the help of this information, a generalization of the Gadjiev Ibragimov operator is defined by [27] as:

$$\hat{N}_r(f,x)=\sum_{\theta=0}^{\infty}f\left(\frac{\theta+r+\mu}{\xi_r+r+\psi}\right)R_{r,\theta}(x)\frac{(-\rho_r)^{\theta}}{\theta!}.$$

Lemma 2.2. *The following equations are valid for the $N_r(f,x)$ operator* [27].

$$\begin{aligned}
i)\hat{N}_r(1,x) &= 1,\\
ii)\hat{N}_r(t,x) &= \frac{r\rho_r x}{\xi_r+r+\psi}+\frac{r+\mu}{\xi_r+r+\psi},\\
iii)\hat{N}_r\left(t^2,x\right) &= \frac{\rho_r^2 r(r+m)}{(\xi_r+r+\psi)^2}x^2+\frac{(2(r+\mu)+1)r\rho_r}{(\xi_r+r+2)^2}x+\frac{(r+\mu)^2}{(\xi_r+r+\psi)^2}.
\end{aligned}$$

Theorem 2.3. *For all $f\in\left[0,\frac{r+\mu}{r+\psi}\right]$ and*

$$\hat{N}_r(f,x)=\sum_{\theta=0}^{\infty}f\left(\frac{\theta+r+\mu}{\xi_r+r+\psi}\right)R_{r,\theta}(x)\frac{(-\rho_r)^{\theta}}{\theta!},$$

$$\lim_{r\to\infty}||\hat{N}_r(f,x)-f(x)||_{C\left[0,\frac{r+\mu}{r+\psi}\right]}=0.$$

Theorem 2.4. *For all $f\in\left[0,\frac{r+\mu}{r+\psi}\right]$, for a sufficiently large number of r and a constant P independent of the sequences $(\rho_r),(\xi_r)$,*

$$||\hat{N}_r(f,x)-f(x)||_{C\left[0,\frac{r+\mu}{r+\psi}\right]}\leq P\omega\{f;\delta_r\}$$

inequality is valid. Here,

$$\delta_r:=\sqrt{\left(\frac{\rho_r r}{\xi_r+r+\psi}-1\right)^2+\frac{2\rho_r}{\xi_r+r+\psi}m+5}.$$

3. Materials and Methods

Definition 3.1. Let $f : L_1\left[0, \frac{r+\mu}{r+\psi}\right] \to C\left[0, \frac{r+\mu}{r+\psi}\right]$ .The operator defined as

$$\widehat{\hat{N}}_r(f,x) = (\xi_r + r + \psi) \sum_{\theta=0}^{\infty} R_{r,\theta}(x) \frac{(-\rho_r)^\theta}{\theta!} \int\limits_{\frac{\theta+r+\mu}{\xi_r+r+\psi}}^{\frac{\theta+r+\mu+1}{\xi_r+r+\psi}} f(p)\, dp$$

is called the Kantorovich generalization of the Gadjiev-Ibragimov type an operator. Here $R_{r,\theta}(x)$; satisfies the conditions in Definition 2.1

Theorem 3.2. *Let* $f : L_1\left[0, \frac{r+\mu}{r+\psi}\right] \to C\left[0, \frac{r+\mu}{r+\psi}\right]$ *and for operators*

$$\widehat{\hat{N}}_r(f,x) = (\xi_r + r + \psi) \sum_{\theta=0}^{\infty} R_{r,\theta}(x) \frac{(-\rho_r)^\theta}{\theta!} \int\limits_{\frac{\theta+r+\mu}{\xi_r+r+\psi}}^{\frac{\theta+r+\mu+1}{\xi_r+r+\psi}} f(p)\, dp$$

the following equations are provided.

$$\begin{aligned} i)\widehat{\hat{N}}_r(1,x) &= 1, \\ ii)\widehat{\hat{N}}_r(t,x) &= \frac{r\rho_r x}{\xi_r + r + \psi} + \frac{2r + 2\mu + 1}{2(\xi_r + r + \psi)}, \\ iii)\widehat{\hat{N}}_r\left(t^2,x\right) &= \frac{\rho_r^2 r(r+m)}{(\xi_r + r + \psi)^2} x^2 + \frac{2r^2\rho_r + 2r\rho_r + 2\mu r\rho_r}{(\xi_r + r + \psi)^2} x \\ &\quad + \frac{3r^2 + 6r\mu + 3r + 3\mu^2 + 3\mu + 1}{3(\xi_r + r + \psi)^2}. \end{aligned}$$

Proof. From the definition of the operator

$$\begin{aligned} i)\ \widehat{\hat{N}}_r(1,x) &= (\xi_r + r + \psi) \sum_{\theta=0}^{\infty} R_{r,\theta}(x) \frac{(-\rho_r)^\theta}{\theta!} \int\limits_{\frac{\theta+r+\mu}{\xi_r+r+\psi}}^{\frac{\theta+r+\mu+1}{\xi_r+r+\psi}} 1dp \\ &= (\xi_r + r + \psi) \sum_{\theta=0}^{\infty} R_{r,\theta}(x) \frac{(-\rho_r)^\theta}{\theta!} \left(\frac{1}{\xi_r + r + \psi}\right) = 1 \end{aligned}$$

equation is obtained.

ii) Description of the operator and from Lemma 2.2 *i)* and *ii)* using the $(\theta \to \theta + 1)$ transform

$$\widehat{\hat{N}}_r(t,x) = (\xi_r + r + \psi) \sum_{\theta=0}^{\infty} R_{r,\theta}(x) \frac{(-\rho_r)^\theta}{\theta!} \int\limits_{\frac{\theta+r+\mu}{\xi_r+r+\psi}}^{\frac{\theta+r+\mu+1}{\xi_r+r+\psi}} pdp$$

$$
\begin{aligned}
&= \frac{(\xi_r+r+\psi)}{2}\sum_{\theta=0}^{\infty} R_{r,\theta}(x)\frac{(-\rho_r)^\theta}{\theta!}\left[\left(\frac{\theta+r+\mu+1}{\xi_r+r+\psi}\right)^2-\left(\frac{\theta+r+\mu}{\xi_r+r+\psi}\right)^2\right] \\
&= \sum_{\theta=1}^{\infty}\left(\frac{-rx}{\xi_r+r+\psi}\right)R_{r+m,\theta-1}(x)\frac{(-\rho_r)^\theta}{(\theta-1)!}+\left(\frac{2r+2\mu+1}{2(\xi_r+r+\psi)}\right) \quad (\theta\to\theta+1,(r+m)\in\mathbb{N}) \\
&= \frac{r\rho_r x}{\xi_r+r+\psi}+\frac{2r+2\mu+1}{2(\xi_r+r+\psi)}
\end{aligned}
$$

is obtained.

$$
\begin{aligned}
iii)\ \hat{\hat{N}}_r(t^2,x) &= \sum_{\theta=0}^{\infty}\frac{\theta(\theta-1)}{(\xi_r+r+2)^2}R_{r,\theta}(x)\frac{(-\rho_r)^\theta}{\theta!}+\sum_{\theta=0}^{\infty}\frac{2\theta r}{(\xi_r+r+2)^2}R_{r,\theta}(x)\frac{(-\rho_r)^\theta}{\theta!} \\
&\quad+\sum_{\theta=0}^{\infty}\frac{2\theta\mu}{(\xi_r+r+2)^2}R_{r,\theta}(x)\frac{(-\rho_r)^\theta}{\theta!}+\sum_{\theta=0}^{\infty}\frac{2\theta}{(\xi_r+r+2)^2}R_{r,\theta}(x)\frac{(-\rho_r)^\theta}{\theta!} \\
&\quad+\sum_{\theta=0}^{\infty}\frac{3r^2+6r\mu+3r+3\mu^2+3\mu+1}{3(\xi_r+r+2)^2}R_{r,\theta}(x)\frac{(-\rho_r)^\theta}{\theta!} \\
&= \frac{\rho_r^2 r(r+m)}{(\xi_r+r+\psi)^2}x^2+\frac{2r^2\rho_r+2r\rho_r+2\mu r\rho_r}{(\xi_r+r+\psi)^2}x \\
&\quad+\frac{3r^2+6r\mu+3r+3\mu^2+3\mu+1}{3(\xi_r+r+\psi)^2}
\end{aligned}
$$

is achived.Thus the theorem is proven. □

Theorem 3.3. *For* $\hat{\hat{N}}_r(f,x)$ *and all* $f\in\left[0,\frac{r+\mu}{r+\psi}\right]$,

$$\lim_{r\to\infty}||\hat{N}_r(f,x)-f(x)||_{C\left[0,\frac{r+\mu}{r+\psi}\right]}=0$$

is valid.

Proof. The equations in Theorem 3.2 will be used for proof, we have

$$\lim_{r\to\infty}\left\|\hat{\hat{N}}_r(1,x)-1\right\|_{C\left[0,\frac{r+\mu}{r+\psi}\right]}=0$$

From Theorem 3.2 ii)

$$\left|\hat{\hat{N}}_r(t,x)-x\right|=\left|x\left(\frac{r\rho_r}{\xi_r+r+\psi}-1\right)\right|+\left|\frac{2r+2\mu+1}{2(\xi_r+r+\psi)}\right|$$

can be written. Here since $x\in\left[0,\frac{r+\mu}{r+\psi}\right]$ and $\lim_{r\to\infty}\frac{\rho_r}{\xi_r+r+\psi}r=1$, then

$$\max_{x\in\left[0,\frac{r+\mu}{r+\psi}\right]}\left|\hat{\hat{N}}_r(t,x)-x\right|\le\left|\frac{r+\mu}{r+\psi}\right|\left|\frac{r\rho_r}{\xi_r+r+\psi}-1\right|+\left|\frac{2r+2\mu+3}{2(\xi_r+r+\psi)}\right|$$

inequality is valid. If the limit of both sides is taken since

$$
\begin{aligned}
\lim_{r\to\infty}\left\|\hat{N}_r(t,x)-x\right\|_{C\left[0,\frac{r+\mu}{r+\psi}\right]} &\le \lim_{r\to\infty}\left|\frac{r+\mu}{r+\psi}\right|\lim_{r\to\infty}\left|\frac{r\rho_r}{\xi_r+r+\psi}-1\right| \\
&\quad+\lim_{r\to\infty}\left|\frac{2r+2\mu+1}{2(\xi_r+r+\psi)}\right|
\end{aligned}
$$

then

$$\lim_{r\to\infty}\left\|\hat{N}_r\left(t,x\right)-x\right\|_{C\left[0,\frac{r+\mu}{r+\psi}\right]}=0$$

is written. Here, if we use

$$\begin{aligned}\left|\widehat{\hat{N}}_r\left(t^2,x\right)-x^2\right| &= \left|\frac{\rho_r^2 r\left(r+m\right)}{\left(\xi_r+r+\psi\right)^2}x^2+\frac{2r^2\rho_r+2r\rho_r+2\mu r\rho_r}{\left(\xi_r+r+\psi\right)^2}x\right.\\ &\quad\left.+\frac{3r^2+6r\mu+3r+3\mu^2+3\mu+1}{3\left(\xi_r+r+\psi\right)^2}-x^2\right|\end{aligned}$$

so

$$\max_{x\in\left[0,\frac{r+\mu}{r+\psi}\right]}\left|\widehat{\hat{N}}_r\left(t^2,x\right)-x^2\right|\leq\left(\frac{r+\mu}{r+\psi}\right)^2\left|\frac{\rho_r^2 r\left(r+m\right)}{\left(\xi_r+r+\psi\right)^2}-1\right|$$
$$+\left|\frac{2r^2\rho_r+2r\rho_r+2\mu r\rho_r}{\left(\xi_r+r+\psi\right)^2}\right|\left|\frac{r+\mu}{r+\psi}\right|+\left|\frac{3r^2+6r\mu+3r+3\mu^2+3\mu+1}{3\left(\xi_r+r+\psi\right)^2}\right|$$

is obtained. Using

$$\lim_{r\to\infty}\frac{\rho_r^2 r(r+m)}{\left(\xi_r+r+\psi\right)^2}=\lim_{r\to\infty}\frac{\rho_r^2 r^2}{\left(\xi_r+r+\psi\right)^2}+\lim_{r\to\infty}\frac{\rho_r^2 rm}{\left(\xi_r+r+\psi\right)^2}=1$$
$$\lim_{r\to\infty}\frac{2r\rho_r\left(r+1\right)}{\left(\xi_r+r+\psi\right)^2}\left(\frac{r+\mu}{r+\psi}\right)=0,\ \lim_{r\to\infty}\frac{2\mu\rho_r r}{\left(\xi_r+r+\psi\right)^2}\left(\frac{r+\mu}{r+\psi}\right)=0$$

and

$$\lim_{r\to\infty}\frac{3r^2+6r\mu+3r+3\mu^2+3\mu+1}{3\left(\xi_r+r+\psi\right)^2}=0,$$

then we get

$$\lim_{r\to\infty}\left\|\widehat{\hat{N}}_r\left(t^2,x\right)-x^2\right\|_{C\left[0,\frac{r+\mu}{r+\psi}\right]}=0.$$

Hence using the Korovkin Theorem, we get

$$\lim_{r\to\infty}\left\|\widehat{\hat{N}}_r\left(f,x\right)-f\left(x\right)\right\|_{C\left[0,\frac{r+\mu}{r+\psi}\right]}=0.$$

□

Theorem 3.4. *Let $f\in C_\rho^0[0,\infty)$. Then*

$$\lim_{r\to\infty}\left\|\widehat{\hat{N}}_r\left(f,x\right)-f\left(x\right)\right\|_{\rho,\left[0,\frac{r+\mu}{r+\psi}\right]}=0$$

is valid.

Proof. Let $\frac{x}{1+x^2}<1$, the demonstration of convergence for the test functions is sufficient for proof. Firstly, it can be easily shown that

$$\left\|\widehat{\hat{N}}_r\left(1,x\right)-1\right\|_{\rho,\left[0,\frac{r+\mu}{r+\psi}\right]}=\lim_{r\to\infty}\sup_{x\in\left[0,\frac{r+\mu}{r+\psi}\right]}\frac{\left|\widehat{\hat{N}}_r\left(1,x\right)-1\right|}{1+x^2}=0$$

Similarly, from the definition of the operator

$$\lim_{r\to\infty}\left\|\widehat{\hat{N}}_r\left(t,x\right)-x\right\|_{\rho,\left[0,\frac{r+\mu}{r+\psi}\right]}\leq\lim_{r\to\infty}\left(\frac{r\rho_r}{\xi_r+r+\psi}-1+\frac{2r+2\mu+1}{2(\xi_r+r+\psi)}\right)=0,$$

then

$$\lim_{r\to\infty}\left\|\widehat{\hat{N}}_r\left(t,x\right)-x\right\|_{\rho,\left[0,\frac{r+\mu}{r+\psi}\right]}=0$$

is valid. Finally, if we take the limit of both sides of the

$$\begin{aligned}\left\|\widehat{\hat{N}}_r\left(t^2,x\right)-x^2\right\|_{\rho,\left[0,\frac{r+\mu}{r+\psi}\right]} \leq & \left(\frac{\rho_r^2 r\left(r+m\right)}{\left(\xi_r+r+\psi\right)^2}-1\right)+\frac{2r^2\rho_r+2r\rho_r+2\mu r\rho_r}{\left(\xi_r+r+\psi\right)^2}\\ &+\frac{3r^2+6r\mu+3r+3\mu^2+3\mu+1}{3\left(\xi_r+r+\psi\right)^2}.\end{aligned}$$

So, it will be that

$$\lim_{r\to\infty}\left\|\widehat{\hat{N}}_r\left(t^2,x\right)-x^2\right\|_{\rho,\left[0,\frac{r+\mu}{r+\psi}\right]}=\lim_{r\to\infty}\left(\frac{\rho_r^2 r(r+m)}{\left(\xi_r+r+\psi\right)^2}-1\right)$$

$$+\lim_{r\to\infty}\left(\frac{2r^2\rho_r+2r\rho_r+2\mu r\rho_r}{\left(\xi_r+r+\psi\right)^2}\right)+\lim_{r\to\infty}\left(\frac{3r^2+6r\mu+3r+3\mu^2+3\mu+1}{3\left(\xi_r+r+\psi\right)^2}\right)$$

Therefore,

$$\lim_{r\to\infty}\left\|\widehat{\hat{N}}_r\left(f,x\right)-f\left(x\right)\right\|_{\rho,\left[0,\frac{r+\mu}{r+\psi}\right]}=0$$

is obtained. Thus, the proof is complete. □

Lemma 3.5. *The following equations are valid for* $\widehat{\hat{N}}_r\left(f,x\right)$.

$$\begin{aligned} i)\widehat{\hat{N}}_r\left(t^3,x\right) =\;& \tfrac{\rho_r^3 r(r+m)(r+2m)}{(\xi_r+r+\psi)^3}x^3+\Big(\tfrac{18\rho_r^2 r(r+m)}{4(\xi_r+r+\psi)^3}+\tfrac{3\rho_r^2 r^2(r+m)}{(\xi_r+r+\psi)^3}\\ &+\tfrac{3\rho_r^2\mu r(r+m)}{(\xi_r+r+\psi)^3}\Big)x^2+\Big(\tfrac{\left(6r\mu\rho_r+6r^2\rho_r+3r^3\rho_r+6r^2\mu\rho_r+3r\mu^2\rho_r\right)}{(\xi_r+r+\psi)^3}\\ &+\tfrac{14r\rho_r}{4(\xi_r+r+\psi)^3}\Big)x+\tfrac{4r^3+6r^2+4r+4\mu^3+6\mu^2+4\mu+12r^2\mu+12r\mu^2+12r\mu+1}{4(\xi_r+r+\psi)^3},\end{aligned}$$

$$\begin{aligned} ii)\widehat{\hat{N}}_r\left(t^4,x\right) =\;& \tfrac{\rho_r^4 r(r+m)(r+2m)(r+3m)}{(\xi_r+r+\psi)^4}x^4\\ &+\tfrac{(4r+4\mu+8)\rho_r^3 r(r+m)(r+2m)}{(\xi_r+r+\psi)^4}x^3\\ &+\tfrac{\left(6r^2+18r+15+12r\mu+6\mu^2+18\mu\right)\rho_r^2 r(r+m)}{(\xi_r+r+\psi)^4}x^2\\ &+\tfrac{\left(4r^3+4\mu^3+12r^2+12\mu^2+12r^2\mu+12\mu^2 r+24r\mu+14r+14\mu+6\right)\rho_r r}{(\xi_r+r+\psi)^4}x\\ &+\tfrac{\left(5r^4+5\mu^4+10r^3+10\mu^3+20r^3\mu+20r\mu^3+10r^2+10\mu^2\right)}{5(\xi_r+r+\psi)^4}\\ &+\tfrac{\left(5r+5\mu+30r^2\mu+30r\mu^2+30r^2\mu^2+20r\mu+1\right)}{5(\xi_r+r+\psi)^4}.\end{aligned}$$

Proof. i) Using the equations

$\rho^4-b^4=(\rho-b)(\rho+b)\left(\rho^2+b^2\right), \theta^3=\theta(\theta-1)(\theta-2)+3\theta^2-2\theta$ and $\theta^2=\theta(\theta-1)+\theta$

then we write

$$\widehat{\widehat{N}}_r\left(t^3,x\right)=(\xi_r+r+\psi)\sum_{\theta=0}^{\infty}R_{r,\theta}(x)\frac{(-\rho_r)^\theta}{\theta!}\int\limits_{\frac{\theta+r+\mu}{\xi_r+r+\psi}}^{\frac{\theta+r+\mu+1}{\xi_r+r+\psi}}p^3dp$$

$$\begin{aligned}
&= \frac{(\xi_r+r+\psi)}{4}\sum_{\theta=0}^{\infty}R_{r,\theta}(x)\frac{(-\rho_r)^\theta}{\theta!}\left[\frac{(2\theta+2r+2\mu+1)}{(\xi_r+r+\psi)^2}\frac{\left(2\theta^2+4\theta r+4\theta\mu+2\theta\right)}{(\xi_r+r+\psi)^2}\right] \\
&\quad+\frac{(\xi_r+r+\psi)}{4}\sum_{\theta=0}^{\infty}R_{r,\theta}(x)\frac{(-\rho_r)^\theta}{\theta!}\left[\frac{(2\theta+2r+2\mu+1)}{(\xi_r+r+\psi)^2}\right. \\
&\quad\left.\times\frac{\left(2r^2+4r\mu+2r+2\mu^2+2\mu+1\right)}{(\xi_r+r+\psi)^2}\right]
\end{aligned}$$

$$\begin{aligned}
&= \frac{(\xi_r+r+\psi)}{4}\sum_{\theta=0}^{\infty}R_{r,\theta}(x)\frac{(-\rho_r)^\theta}{\theta!}\left[\frac{4\theta^3+12\theta^2r+12\theta^2\mu+6\theta^2+12\theta r^2}{(\xi_r+r+\psi)^4}\right] \\
&\quad+\frac{(\xi_r+r+\psi)}{4}\sum_{\theta=0}^{\infty}R_{r,\theta}(x)\frac{(-\rho_r)^\theta}{\theta!}\left[\frac{12\theta r+4\theta+24\theta r\mu+12\theta\mu^2+12\theta\mu}{(\xi_r+r+\psi)^4}\right. \\
&\quad\left.+\frac{4r^3+6r^2+4r+4\mu^3+6\mu^2+4\mu+12r^2\mu+12r\mu^2+12r\mu+1}{(\xi_r+r+\psi)^4}\right]
\end{aligned}$$

Using next equality

$$\begin{aligned}
&4\theta^3+12\theta^2r+12\theta^2\mu+6\theta^2+12\theta r^2+12\theta r+4\theta+24\theta r\mu+12\theta\mu^2 \\
&+12\theta\mu+4r^3+6r^2+4r+4\mu^3+6\mu^2+4\mu+12r^2\mu+12r\mu^2+12r\mu+1
\end{aligned}$$

$$\begin{aligned}
&= 4(\theta(\theta-1)(\theta-2)+3(\theta(\theta-1)+\theta)-2\theta)+12r(\theta(\theta-1)+\theta) \\
&\quad+12\mu(\theta(\theta-1)+\theta)+6(\theta(\theta-1)+\theta)+12\theta r^2+12\theta r+4\theta+24\theta r\mu \\
&\quad+12\theta\mu^2+12\theta\mu+4r^3+6r^2+4r+4\mu^3+6\mu^2+4\mu+12r^2\mu+12r\mu^2+12r\mu+1
\end{aligned}$$

we get

$$\begin{aligned}
\widehat{\widehat{N}}_r\left(t^3,x\right) &= \sum_{\theta=0}^{\infty}R_{r,\theta}(x)\frac{(-\rho_r)^\theta}{\theta!}\left[\tfrac{(4\theta(\theta-1)(\theta-2)+18\theta(\theta-1)+12r\theta(\theta-1)+12\mu\theta(\theta-1))}{4(\xi_r+r+\psi)^3}\right] \\
&\quad+\sum_{\theta=0}^{\infty}R_{r,\theta}(x)\frac{(-\rho_r)^\theta}{\theta!}\left[\tfrac{24\theta\mu+24\theta r+14\theta+12\theta r^2+24\theta r\mu+12\theta\mu^2}{4(\xi_r+r+\psi)^3}\right] \\
&\quad+\sum_{\theta=0}^{\infty}R_{r,\theta}(x)\frac{(-\rho_r)^\theta}{\theta!}\left[\tfrac{4r^3+6r^2+4r+4\mu^3+6\mu^2+4\mu+12r^2\mu+12r\mu^2+12r\mu+1}{4(\xi_r+r+\psi)^3}\right]
\end{aligned}$$

$$
\begin{aligned}
= & \sum_{\theta=0}^{\infty} R_{r,\theta}(x) \frac{(-\rho_r)^\theta}{\theta!} \left[\frac{\theta(\theta-1)(\theta-2)}{(\xi_r+r+\psi)^3}\right] + \sum_{\theta=0}^{\infty} R_{r,\theta}(x) \frac{(-\rho_r)^\theta}{\theta!} \left[\frac{18\theta(\theta-1)}{4(\xi_r+r+\psi)^3}\right] \\
& + \sum_{\theta=0}^{\infty} R_{r,\theta}(x) \frac{(-\rho_r)^\theta}{\theta!} \left[\frac{3r\theta(\theta-1)}{(\xi_r+r+\psi)^3}\right] + \sum_{\theta=0}^{\infty} R_{r,\theta}(x) \frac{(-\rho_r)^\theta}{\theta!} \left[\frac{3\mu\theta(\theta-1)}{(\xi_r+r+\psi)^3}\right] \\
& + \sum_{\theta=0}^{\infty} R_{r,\theta}(x) \frac{(-\rho_r)^\theta}{\theta!} \left[\frac{6\theta\mu}{(\xi_r+r+\psi)^3}\right] + \sum_{\theta=0}^{\infty} R_{r,\theta}(x) \frac{(-\rho_r)^\theta}{\theta!} \left[\frac{6\theta r}{(\xi_r+r+\psi)^3}\right] \\
& + \sum_{\theta=0}^{\infty} R_{r,\theta}(x) \frac{(-\rho_r)^\theta}{\theta!} \left[\frac{14\theta}{4(\xi_r+r+\psi)^3}\right] + \sum_{\theta=0}^{\infty} R_{r,\theta}(x) \frac{(-\rho_r)^\theta}{\theta!} \left[\frac{3\theta r^2}{(\xi_r+r+\psi)^3}\right] \\
& + \sum_{\theta=0}^{\infty} R_{r,\theta}(x) \frac{(-\rho_r)^\theta}{\theta!} \left[\frac{6\theta r\mu}{(\xi_r+r+\psi)^3}\right] + \sum_{\theta=0}^{\infty} R_{r,\theta}(x) \frac{(-\rho_r)^\theta}{\theta!} \left[\frac{3\theta\mu^2}{(\xi_r+r+\psi)^3}\right] \\
& + \sum_{\theta=0}^{\infty} R_{r,\theta}(x) \frac{(-\rho_r)^\theta}{\theta!} \left[\frac{4r^3+6r^2+4r+4\mu^3+6\mu^2+4\mu+12r^2\mu+12r\mu^2+12r\mu+1}{4(\xi_r+r+\psi)^3}\right]
\end{aligned}
$$

Then

$$
\begin{aligned}
\widehat{\widehat{N}}_r\left(t^3,x\right) = & \frac{\rho_r^3 r(r+m)(r+2m)}{(\xi_r+r+\psi)^3}x^3 + \left(\frac{18\rho_r^2 r(r+m)}{4(\xi_r+r+\psi)^3} + \frac{3\rho_r^2 r^2(r+m)}{(\xi_r+r+\psi)^3}\right. \\
& \left. + \frac{3\rho_r^2\mu r(r+m)}{(\xi_r+r+\psi)^3}\right)x^2 + \left(\frac{\left(6r\mu\rho_r+6r^2\rho_r+3r^3\rho_r+6r^2\mu\rho_r+3r\mu^2\rho_r\right)}{(\xi_r+r+\psi)^3}\right. \\
& \left. + \frac{14r\rho_r}{4(\xi_r+r+\psi)^3}\right)x + \frac{4r^3+6r^2+4r+4\mu^3+6\mu^2+4\mu+12r^2\mu+12r\mu^2+12r\mu+1}{4(\xi_r+r+\psi)^3},
\end{aligned}
$$

is written.

ii) Since the following equations can be written easily

$$
\begin{aligned}
c^5-d^5 &= (c+d)\left(c^4-d^4\right)-cd\left(c^3-d^3\right), \theta^4=\theta(\theta-1)(\theta-2)(\theta-3)+6\theta^3-11\theta^2+6\theta, \\
\theta^3 &= \theta(\theta-1)(\theta-2)+3\theta^2-2\theta, \theta^2=\theta(\theta-1)+\theta.
\end{aligned}
$$

Then,we get

$$
\widehat{\widehat{N}}_r\left(t^4,x\right) = (\xi_r+r+\psi)\sum_{\theta=0}^{\infty} R_{r,\theta}(x)\frac{(-\rho_r)^\theta}{\theta!}\int_{\frac{\theta+r+\mu}{\xi_r+r+\psi}}^{\frac{\theta+r+\mu+1}{\xi_r+r+\psi}} p^4\,dp
$$

$$
\begin{aligned}
= & \sum_{\theta=0}^{\infty} \frac{5\left(\theta\left(\theta-3\right)\left(\theta-2\right)\left(\theta-1\right)\right)}{5\left(\xi_r+r+\psi\right)^4} R_{r,\theta}\left(x\right) \frac{\left(-\rho_r\right)^\theta}{\theta!} \\
& +\sum_{\theta=0}^{\infty} \frac{20r\theta\left(\theta-1\right)\left(\theta-2\right)}{5\left(\xi_r+r+\psi\right)^4} R_{r,\theta}\left(x\right) \frac{\left(-\rho_r\right)^\theta}{\theta!} \\
& +\sum_{\theta=0}^{\infty} \frac{20\mu\theta\left(\theta-1\right)\left(\theta-2\right)}{5\left(\xi_r+r+\psi\right)^4} R_{r,\theta}\left(x\right) \frac{\left(-\rho_r\right)^\theta}{\theta!} \\
& +\sum_{\theta=0}^{\infty} \frac{40\theta\left(\theta-1\right)\left(\theta-2\right)}{5\left(\xi_r+r+\psi\right)^4} R_{r,\theta}\left(x\right) \frac{\left(-\rho_r\right)^\theta}{\theta!} \\
& +\sum_{\theta=0}^{\infty} \frac{75\theta\left(\theta-1\right)}{5\left(\xi_r+r+\psi\right)^4} R_{r,\theta}\left(x\right) \frac{\left(-\rho_r\right)^\theta}{\theta!}+\sum_{\theta=0}^{\infty} \frac{90r\theta\left(\theta-1\right)}{5\left(\xi_r+r+\psi\right)^4} R_{r,\theta}\left(x\right) \frac{\left(-\rho_r\right)^\theta}{\theta!} \\
& +\sum_{\theta=0}^{\infty} \frac{30r^2\theta\left(\theta-1\right)}{5\left(\xi_r+r+\psi\right)^4} R_{r,\theta}\left(x\right) \frac{\left(-\rho_r\right)^\theta}{\theta!}+\sum_{\theta=0}^{\infty} \frac{60r\mu\theta\left(\theta-1\right)}{5\left(\xi_r+r+\psi\right)^4} R_{r,\theta}\left(x\right) \frac{\left(-\rho_r\right)^\theta}{\theta!} \\
& +\sum_{\theta=0}^{\infty} \frac{30\mu^2\theta\left(\theta-1\right)}{5\left(\xi_r+r+\psi\right)^4} R_{r,\theta}\left(x\right) \frac{\left(-\rho_r\right)^\theta}{\theta!}+\sum_{\theta=0}^{\infty} \frac{90\mu\theta\left(\theta-1\right)}{5\left(\xi_r+r+\psi\right)^4} R_{r,\theta}\left(x\right) \frac{\left(-\rho_r\right)^\theta}{\theta!} \\
& +\sum_{\theta=0}^{\infty} \frac{30\theta}{5\left(\xi_r+r+\psi\right)^4} R_{r,\theta}\left(x\right) \frac{\left(-\rho_r\right)^\theta}{\theta!}+\sum_{\theta=0}^{\infty} \frac{70r\theta}{5\left(\xi_r+r+\psi\right)^4} R_{r,\theta}\left(x\right) \frac{\left(-\rho_r\right)^\theta}{\theta!} \\
& +\sum_{\theta=0}^{\infty} \frac{70\mu\theta}{5\left(\xi_r+r+\psi\right)^4} R_{r,\theta}\left(x\right) \frac{\left(-\rho_r\right)^\theta}{\theta!}+\sum_{\theta=0}^{\infty} \frac{20r^3\theta}{5\left(\xi_r+r+\psi\right)^4} R_{r,\theta}\left(x\right) \frac{\left(-\rho_r\right)^\theta}{\theta!} \\
& +\sum_{\theta=0}^{\infty} \frac{20\mu^3\theta}{5\left(\xi_r+r+\psi\right)^4} R_{r,\theta}\left(x\right) \frac{\left(-\rho_r\right)^\theta}{\theta!}+\sum_{\theta=0}^{\infty} \frac{60\mu^2\theta}{5\left(\xi_r+r+\psi\right)^4} R_{r,\theta}\left(x\right) \frac{\left(-\rho_r\right)^\theta}{\theta!} \\
& +\sum_{\theta=0}^{\infty} \frac{60r^2\theta}{5\left(\xi_r+r+\psi\right)^4} R_{r,\theta}\left(x\right) \frac{\left(-\rho_r\right)^\theta}{\theta!}+\sum_{\theta=0}^{\infty} \frac{60r^2\mu\theta}{5\left(\xi_r+r+\psi\right)^4} R_{r,\theta}\left(x\right) \frac{\left(-\rho_r\right)^\theta}{\theta!} \\
& +\sum_{\theta=0}^{\infty} \frac{60\mu^2r\theta}{5\left(\xi_r+r+\psi\right)^4} R_{r,\theta}\left(x\right) \frac{\left(-\rho_r\right)^\theta}{\theta!}+\sum_{\theta=0}^{\infty} \frac{120r\mu\theta}{5\left(\xi_r+r+\psi\right)^4} R_{r,\theta}\left(x\right) \frac{\left(-\rho_r\right)^\theta}{\theta!} \\
& +\sum_{\theta=0}^{\infty} \frac{5r^4+5\mu^4+10r^3+10\mu^3+20r^3\mu+20r\mu^3+10r^2+10\mu^2}{5\left(\xi_r+r+\psi\right)^4} R_{r,\theta}\left(x\right) \frac{\left(-\rho_r\right)^\theta}{\theta!} \\
& +\sum_{\theta=0}^{\infty} \frac{30r^2\mu+30r\mu^2+30r^2\mu^2+20r\mu+5r+5\mu+1}{5\left(\xi_r+r+\psi\right)^4} R_{r,\theta}\left(x\right) \frac{\left(-\rho_r\right)^\theta}{\theta!}
\end{aligned}
$$

The desired equality is obtained from the equations. □

In the following Lemma, some of the central moments of the operator $\widehat{\widehat{N}}_r\left(f,x\right)$ are calculated.

Lemma 3.6. *The following equations are valid for the first five central moments of the operator* $\widehat{\widehat{N}}_r\left(f,x\right)$.

$$i)\ \widehat{\widehat{N}}_r\left(\left(t-x\right)^0,x\right)=1,$$

$$ii)\widehat{\hat{N}}_r\left((t-x)^1, x\right) = \frac{r\rho_r - (\xi_r + r + \psi)}{\xi_r + r + \psi}x + \frac{2r + 2\mu + 1}{2(\xi_r + r + \psi)},$$

$$\begin{aligned} iii)\widehat{\hat{N}}_r\left((t-x)^2, x\right) &= \left(\frac{\rho_r^2 r\,(r+m)}{(\xi_r + r + \psi)^2} - \frac{2r\rho_r}{\xi_r + r + \psi} + 1\right)x^2 \\ &+ \left(\frac{2r^2\rho_r + 2r\rho_r + 2\mu r\rho_r}{(\xi_r + r + \psi)^2} - \frac{2r + 2\mu + 1}{\xi_r + r + \psi}\right)x \\ &+ \frac{3r^2 + 6r\mu + 3r + 3\mu^2 + 3\mu + 1}{3\,(\xi_r + r + \psi)^2}, \end{aligned}$$

$$\begin{aligned} iv)\widehat{\hat{N}}_r\left((t-x)^3, x\right) &= \left(\frac{\rho_r^3 r(r+m)(r+2m)}{(\xi_r+r+\psi)^3} - \frac{3r\rho_r}{\xi_r+r+\psi} + \frac{3\rho_r^2 r(r+m)}{(\xi_r+r+\psi)^2} - 1\right)x^3 \\ &+ \left(\frac{18\rho_r^2 r(r+m)}{4(\xi_r+r+\psi)^3} + \frac{3\rho_r^2 r^2(r+m)}{(\xi_r+r+\psi)^3} + \frac{3\rho_r^2\mu r(r+m)}{(\xi_r+r+\psi)^3}\right. \\ &\left. - \frac{6r+6\mu+3}{2(\xi_r+r+\psi)} + \frac{6r^2\rho_r+6r\rho_r+6\mu r\rho_r}{(\xi_r+r+\psi)^2}\right)x^2 \\ &+ \left(\frac{\left(6r\mu\rho_r+6r^2\rho_r+3r^3\rho_r+6r^2\mu\rho_r+3r\mu^2\rho_r\right)}{(\xi_r+r+\psi)^3}\right. \\ &\left. + \frac{14r\rho_r}{4(\xi_r+r+\psi)^3} + \frac{3r^2+6r\mu+3r+3\mu^2+3\mu+1}{(\xi_r+r+\psi)^2}\right)x \\ &+ \frac{4r^3+6r^2+4r+4\mu^3+6\mu^2+4\mu+12r^2\mu+12r\mu^2+12r\mu+1}{4(\xi_r+r+\psi)^3} \end{aligned}$$

$$\begin{aligned} v)\widehat{\hat{N}}_r\left((t-x)^4, x\right) &= \left(\frac{\rho_r^4 r(r+m)(r+2m)(r+3m)}{(\xi_r+r+\psi)^4} - \frac{4\rho_r^3 r(r+m)(r+2m)}{(\xi_r+r+\psi)^3} + \frac{6\rho_r^2 r(r+m)}{(\xi_r+r+\psi)^2}\right. \\ &\left. - \frac{4r\rho_r}{\xi_r+r+\psi} + 1\right)x^4 + \left(\frac{(4r+4\mu+8)\rho_r^3 r(r+m)(r+2m)}{(\xi_r+r+\psi)^4} - \frac{18\rho_r^2 r(r+m)}{(\xi_r+r+\psi)^3}\right. \\ &- \frac{12\rho_r^2 r^2(r+m)}{(\xi_r+r+\psi)^3} - \frac{12\rho_r^2\mu r(r+m)}{(\xi_r+r+\psi)^3} + \frac{12r^2\rho_r+12r\rho_r+12\mu r\rho_r}{(\xi_r+r+\psi)^2} \\ &\left. - \frac{4r+4\mu+2}{(\xi_r+r+\psi)}\right)x^3 + \left(\frac{\left(6r^2+18r+15+12r\mu+6\mu^2+18\mu\right)\rho_r^2 r(r+m)}{(\xi_r+r+\psi)^4}\right. \\ &- \frac{24r\mu\rho_r+24r^2\rho_r+12r^3\rho_r+24r^2\mu\rho_r+12r\mu^2\rho_r}{(\xi_r+r+\psi)^3} - \frac{14r\rho_r}{(\xi_r+r+\psi)^3} \\ &\left. + \frac{6r^2+12r\mu+6r+6\mu^2+6\mu+2}{(\xi_r+r+\psi)^2}\right)x^2 \\ &+ \left(\frac{\left(4r^3+4\mu^3+12r^2+12\mu^2+12r^2\mu+12\mu^2 r+24r\mu+14r+14\mu+6\right)\rho_r r}{(\xi_r+r+\psi)^4}\right. \\ &\left. - \frac{4r^3+6r^2+4r+4\mu^3+6\mu^2+4\mu+12r^2\mu+12r\mu^2+12r\mu+1}{(\xi_r+r+\psi)^3}\right)x \\ &+ \frac{\left(5r^4+5\mu^4+10r^3+10\mu^3+20r^3\mu+20r\mu^3+10r^2+10\mu^2\right)}{5(\xi_r+r+\psi)^4} \\ &+ \frac{\left(5r+5\mu+30r^2\mu+30r\mu^2+30r^2\mu^2+20r\mu+1\right)}{5(\xi_r+r+\psi)^4}. \end{aligned}$$

Proof. $i)$Clearly

$$\widehat{\hat{N}}_r\left((t-x)^0, x\right) = 1.$$

ii) Using definition of operators

$$\begin{aligned}
\widehat{\hat{N}}_r\left((t-x)^1,x\right) &= \widehat{\hat{N}}_r(t,x) - x\widehat{\hat{N}}_r(1,x) \\
&= \left(\frac{r\rho_r}{\xi_r+r+\psi}-1\right)x + \frac{2r+2\mu+1}{2(\xi_r+r+\psi)} \\
&= \frac{r\rho_r-(\xi_r+r+\psi)}{\xi_r+r+\psi}x + \frac{2r+2\mu+1}{2(\xi_r+r+\psi)}.
\end{aligned}$$

iii) We write

$$\begin{aligned}
\widehat{\hat{N}}_r\left((t-x)^2,x\right) &= \widehat{\hat{N}}_r\left(t^2,x\right) - 2x\widehat{\hat{N}}_r(t,x) + x^2\widehat{\hat{N}}_r(1,x) \\
&= \frac{\rho_r^2 r(r+m)}{(\xi_r+r+\psi)^2}x^2 + \frac{2r^2\rho_r+2r\rho_r+2\mu r\rho_r}{(\xi_r+r+\psi)^2}x \\
&\quad + \frac{3r^2+6r\mu+3r+3\mu^2+3\mu+1}{3(\xi_r+r+\psi)^2} \\
&\quad -2x\left(\frac{r\rho_r}{\xi_r+r+\psi}x + \frac{2r+2\mu+1}{2(\xi_r+r+\psi)}\right) + x^2 \\
&= \left(\frac{\rho_r^2 r(r+m)}{(\xi_r+r+\psi)^2} - \frac{2r\rho_r}{\xi_r+r+\psi} + 1\right)x^2 \\
&\quad + \left(\frac{2r^2\rho_r+2r\rho_r+2\mu r\rho_r}{(\xi_r+r+\psi)^2} - \frac{2r+2\mu+1}{\xi_r+r+\psi}\right)x \\
&\quad + \frac{3r^2+6r\mu+3r+3\mu^2+3\mu+1}{3(\xi_r+r+\psi)^2}.
\end{aligned}$$

iv)We get

$$\begin{aligned}
\widehat{\hat{N}}_r\left((t-x)^3,x\right) &= \widehat{\hat{N}}_r\left(t^3,x\right) - 3x^2\widehat{\hat{N}}_r(t,x) + 3x\widehat{\hat{N}}_r\left(t^2,x\right) - x^3\widehat{\hat{N}}_r(1,x) \\
&= \frac{\rho_r^3 r(r+m)(r+2m)}{(\xi_r+r+\psi)^3}x^3 + \left(\frac{18\rho_r^2 r(r+m)}{4(\xi_r+r+\psi)^3} + \frac{3\rho_r^2 r^2(r+m)}{(\xi_r+r+\psi)^3} + \frac{3\rho_r^2\mu r(r+m)}{(\xi_r+r+\psi)^3}\right)x^2 \\
&\quad + \left(\frac{\left(6r\mu\rho_r+6r^2\rho_r+3r^3\rho_r+6r^2\mu\rho_r+3r\mu^2\rho_r\right)}{(\xi_r+r+\psi)^3} + \frac{14r\rho_r}{4(\xi_r+r+\psi)^3}\right)x \\
&\quad + \frac{4r^3+6r^2+4r+4\mu^3+6\mu^2+4\mu+12r^2\mu+12r\mu^2+12r\mu+1}{4(\xi_r+r+\psi)^3} \\
&\quad -3x^2\left(\frac{r\rho_r}{\xi_r+r+\psi}x + \frac{2r+2\mu+1}{2(\xi_r+r+\psi)}\right) \\
&\quad +3x\left(\frac{\rho_r^2 r(r+m)}{(\xi_r+r+\psi)^2}x^2 + \frac{2r^2\rho_r+2r\rho_r+2\mu r\rho_r}{(\xi_r+r+\psi)^2}x\right. \\
&\quad \left. + \frac{3r^2+6r\mu+3r+3\mu^2+3\mu+1}{3(\xi_r+r+\psi)^2}\right) - x^3 \\
&= \left(\frac{\rho_r^3 r(r+m)(r+2m)}{(\xi_r+r+\psi)^3} - \frac{3r\rho_r}{\xi_r+r+\psi} + \frac{3\rho_r^2 r(r+m)}{(\xi_r+r+\psi)^2} - 1\right)x^3 \\
&\quad + \left(\frac{18\rho_r^2 r(r+m)}{4(\xi_r+r+\psi)^3} + \frac{3\rho_r^2 r^2(r+m)}{(\xi_r+r+\psi)^3} + \frac{3\rho_r^2\mu r(r+m)}{(\xi_r+r+\psi)^3}\right. \\
&\quad \left. - \frac{6r+6\mu+3}{2(\xi_r+r+\psi)} + \frac{6r^2\rho_r+6r\rho_r+6\mu r\rho_r}{(\xi_r+r+\psi)^2}\right)x^2
\end{aligned}$$

$$+\left(\frac{\left(6r\mu\rho_r+6r^2\rho_r+3r^3\rho_r+6r^2\mu\rho_r+3r\mu^2\rho_r\right)}{(\xi_r+r+\psi)^3}+\frac{14r\rho_r}{4(\xi_r+r+\psi)^3}\right.$$
$$\left.+\frac{3r^2+6r\mu+3r+3\mu^2+3\mu+1}{(\xi_r+r+\psi)^2}\right)x$$
$$+\frac{4r^3+6r^2+4r+4\mu^3+6\mu^2+4\mu+12r^2\mu+12r\mu^2+12r\mu+1}{4(\xi_r+r+\psi)^3}.$$

v)Finally,

$$\begin{aligned}
\widehat{\widehat{N}}_r\left((t-x)^4,x\right) &= \widehat{\widehat{N}}_r\left(t^4,x\right)-4x\widehat{\widehat{N}}_r\left(t^3,x\right)+6x^2\widehat{\widehat{N}}_r\left(t^2,x\right) \\
&\quad -4x^3\widehat{\widehat{N}}_r\left(t,x\right)+x^4\widehat{\widehat{N}}_r\left(1,x\right) \\
&= \frac{\rho_r^4 r(r+m)(r+2m)(r+3m)}{(\xi_r+r+\psi)^4}x^4+x^4 \\
&\quad +\frac{(4r+4\mu+8)\rho_r^3 r(r+m)(r+2m)}{(\xi_r+r+\psi)^4}x^3 \\
&\quad +\frac{\left(6r^2+18r+15+12r\mu+6\mu^2+18\mu\right)\rho_r^2 r(r+m)}{(\xi_r+r+\psi)^4}x^2 \\
&\quad +\frac{\left(4r^3+4\mu^3+12r^2+12\mu^2+12r^2\mu+12\mu^2r+24r\mu+14r+14\mu+6\right)\rho_r r}{(\xi_r+r+\psi)^4}x \\
&\quad +\frac{\left(5r^4+5\mu^4+10r^3+10\mu^3+20r^3\mu+20r\mu^3+10r^2+10\mu^2\right)}{5(\xi_r+r+\psi)^4} \\
&\quad +\frac{\left(5r+5\mu+30r^2\mu+30r\mu^2+30r^2\mu^2+20r\mu+1\right)}{5(\xi_r+r+\psi)^4} \\
&\quad -4x\left(\frac{\rho_r^3 r(r+m)(r+2m)}{(\xi_r+r+\psi)^3}x^3+\left(\frac{18\rho_r^2 r(r+m)}{4(\xi_r+r+\psi)^3}+\frac{3\rho_r^2 r^2(r+m)}{(\xi_r+r+\psi)^3}\right.\right. \\
&\quad \left.+\frac{3\rho_r^2\mu r(r+m)}{(\xi_r+r+\psi)^3}\right)x^2+\left(\frac{\left(6r\mu\rho_r+6r^2\rho_r+3r^3\rho_r+6r^2\mu\rho_r+3r\mu^2\rho_r\right)}{(\xi_r+r+\psi)^3}\right. \\
&\quad \left.+\frac{14r\rho_r}{4(\xi_r+r+\psi)^3}\right)x \\
&\quad \left.+\frac{4r^3+6r^2+4r+4\mu^3+6\mu^2+4\mu+12r^2\mu+12r\mu^2+12r\mu+1}{4(\xi_r+r+\psi)^3}\right) \\
&\quad +6x^2\left(\frac{\rho_r^2 r(r+m)}{(\xi_r+r+\psi)^2}x^2+\frac{2r^2\rho_r+2r\rho_r+2\mu r\rho_r}{(\xi_r+r+\psi)^2}x\right. \\
&\quad \left.+\frac{3r^2+6r\mu+3r+3\mu^2+3\mu+1}{3(\xi_r+r+\psi)^2}\right) \\
&\quad +4x^3\left(\frac{r\rho_r}{\xi_r+r+\psi}x+\frac{2r+2\mu+1}{2(\xi_r+r+\psi)}\right)
\end{aligned}$$

□

Theorem 3.7. *For all* $f\in\left[0,\frac{r+\mu}{r+\psi}\right]$,$\mu\leq\psi$ *and a sufficiently large* r *also for* K *is a constant independent of* (ρ_r) *and* (ξ_r)

$$\left\|\widehat{\widehat{N}}_r\left(f,x\right)-f(x)\right\|_{C\left[0,\frac{r+\mu}{r+\psi}\right]}\leq K\omega\left(f;\delta_r\right).$$

Here δ_r *is defined as* $\delta_r=\left\{\left(\frac{\rho_r r}{\xi_r+r+\psi}-1\right)^2+\frac{(r+\mu+1)}{(\xi_r+r+\psi)}+\frac{(r+\mu+1)^2}{(\xi_r+r+\psi)^2}\right\}^{1/2}$

Proof. Using

$$\left|\widehat{\hat{N}}_r(f,x)-f(x)\right| \leq \widehat{\hat{N}}_r(|f(t)-f(x)|;x)$$

we get

$$\left|\widehat{\hat{N}}_r(f,x)-f(x)\right| \leq (\xi_r+r+\psi)\sum_{\theta=0}^{\infty} R_{r,\theta}(x)\frac{(-\rho_r)^\theta}{\theta!}\int_{\frac{\theta+r+\mu}{\xi_r+r+\psi}}^{\frac{\theta+r+\mu+1}{\xi_r+r+\psi}} |f(p)-f(x)|\,dp$$

Then, from the modulus of continuity for all $\delta_r > 0$, we can write

$$|f(p)-f(x)| \leq \omega(f;\delta_r)\left(1+\frac{(p-x)^2}{\delta_r^2}\right).$$

So,

$$\begin{aligned}\left|\widehat{\hat{N}}_r(f,x)-f(x)\right| &\leq (\xi_r+r+\psi)\sum_{\theta=0}^{\infty} R_{r,\theta}(x)\tfrac{(-\rho_r)^\theta}{\theta!}\int_{\frac{\theta+r+\mu}{\xi_r+r+\psi}}^{\frac{\theta+r+\mu+1}{\xi_r+r+\psi}} |f(p)-f(x)|\,dp \\ &\leq \omega(f;\delta_r)(\xi_r+r+\psi)\sum_{\theta=0}^{\infty} R_{r,\theta}(x)\tfrac{(-\rho_r)^\theta}{\theta!}\int_{\frac{\theta+r+\mu}{\xi_r+r+\psi}}^{\frac{\theta+r+\mu+1}{\xi_r+r+\psi}}\left(1+\tfrac{(p-x)^2}{\delta_r^2}\right)dp.\end{aligned}$$

Thus, we have

$$\int_{\frac{\theta+r+\mu}{\xi_r+r+\psi}}^{\frac{\theta+r+\mu+1}{\xi_r+r+\psi}}\left(1+\frac{(p-x)^2}{\delta_r^2}\right)dp = \frac{1}{\xi_r+r+\psi}+\frac{1}{\delta_r^2}\left(\frac{p^3}{3}-xp^2+x^2p\right)\Bigg|_{\frac{\theta+r+\mu}{\xi_r+r+\psi}}^{\frac{\theta+r+\mu+1}{\xi_r+r+\psi}}.$$

We define $\frac{\theta+r+\mu}{\xi_r+r+\psi} := a$,

$$\begin{aligned}\left(\frac{p^3}{3}-xp^2+x^2p\right)\Bigg|_a^{a+\frac{1}{\xi_r+r+\psi}} &= \frac{a^3}{3}+\frac{a^2}{\xi_r+r+\psi}+\frac{a}{(\xi_r+r+\psi)^2}+\frac{1}{3(\xi_r+r+\psi)^3} \\ &\quad -xa^2-\frac{2ax}{\xi_r+r+\psi}-\frac{x}{(\xi_r+r+\psi)^2}+x^2a+\frac{x^2}{\xi_r+r+\psi}-\frac{a^3}{3}+xa^2-x^2a.\end{aligned}$$

So, we have

$$\begin{aligned}\int_{\frac{\theta+r+\mu}{\xi_r+r+\psi}}^{\frac{\theta+r+\mu+1}{\xi_r+r+\psi}}\left(1+\frac{(p-x)^2}{\delta_r^2}\right)dp &= \frac{1}{\xi_r+r+\psi}+\frac{1}{\delta_r^2}\left[\frac{\theta^2+2\theta r+r^2+2\theta\mu+2r\mu+\mu^2}{(\xi_r+r+\psi)^3}\right. \\ &\quad +\frac{\theta+r+\mu}{(\xi_r+r+\psi)^3}+\frac{1}{3(\xi_r+r+\psi)^3}-\frac{2\theta x+2rx+2\mu x}{(\xi_r+r+\psi)^2} \\ &\quad \left.-\frac{x}{(\xi_r+r+\psi)^2}+\frac{x^2}{\xi_r+r+\psi}\right].\end{aligned}$$

Since,

$$\begin{aligned}\left|\widehat{\hat{N}}_r(f,x)-f(x)\right| &\leq (\xi_r+r+\psi)\,\omega(f;\delta_r)\sum_{\theta=0}^{\infty}R_{r,\theta}(x)\frac{(-\rho_r)^\theta}{\theta!}\left[\frac{1}{\xi_r+r+\psi}\right.\\
&+\frac{1}{\delta^2}\left[\frac{\theta^2+2\theta r+r^2+2\theta\mu+2r\mu+\mu^2}{(\xi_r+r+\psi)^3}+\frac{\theta+r+\mu}{(\xi_r+r+\psi)^3}\right.\\
&+\frac{1}{3(\xi_r+r+\psi)^3}-\frac{2\theta x+2rx+2\mu x}{(\xi_r+r+\psi)^2}\\
&\left.\left.-\frac{x}{(\xi_r+r+\psi)^2}+\frac{x^2}{\xi_r+r+\psi}\right]\right],\end{aligned}$$

then we have

$$\begin{aligned}\left|\widehat{\hat{N}}_r(f,x)-f(x)\right| &\leq \omega(f;\delta_r)+\omega(f;\delta_r)\frac{1}{\delta^2}\left\{\left(\tfrac{\rho_r^2 r(r+m)}{(\xi_r+r+\psi)^2}-\tfrac{2\rho_r r}{(\xi_r+r+\psi)}+1\right)x^2\right.\\
&+\tfrac{1}{3(\xi_r+r+\psi)^2}+\left(\tfrac{2\rho_r r^2+2\rho_r\mu r+\rho_r r}{(\xi_r+r+\psi)^2}-\tfrac{2r+2\mu+1}{(\xi_r+r+\psi)}\right)x\\
&\left.+\tfrac{2\mu r+\mu^2+r+\mu+r^2}{(\xi_r+r+\psi)^2}\right\}\\
&\leq K\left(\omega(f;\delta_r)\frac{1}{\delta^2}\left\{\left(\frac{\rho_r r}{\xi_r+r+\psi}-1\right)^2\right.\right.\\
&\left.\left.+\frac{(r+\mu+1)}{(\xi_r+r+\psi)}+\frac{(r+\mu+1)^2}{(\xi_r+r+\psi)^2}\right\}\right).\end{aligned}$$

So

$$\delta_r=\left\{\left(\frac{\rho_r r}{\xi_r+r+\psi}-1\right)^2+\tfrac{(r+\mu+1)}{(\xi_r+r+\psi)}+\tfrac{(r+\mu+1)^2}{(\xi_r+r+\psi)^2}\right\}^{1/2}. \qquad \square$$

Example 3.8. Let $\widehat{\hat{N}}_r(f,x)=(\xi_r+r+\psi)\sum_{\theta=0}^{\infty}R_{r,\theta}(x)\frac{(-\rho_r)^\theta}{\theta!}\int_{\frac{\theta+r+\mu}{\xi_r+r+\psi}}^{\frac{\theta+r+\mu+1}{\xi_r+r+\psi}}f(p)\,dp, r=25, m=30$ and for $x\in\left[0,\frac{r+\mu}{r+\psi}\right]$, let $R_{r,\theta}(x)=(-1)^\theta(rx)^\theta e^{-rx\rho_r}$.In this case, the graph of the operator's approximation to the $f(x)=\frac{x^3+1}{e^{4x+1}}$ is given in grafics. The drawing is made for $\widehat{\hat{N}}_{10}(f,x)$ in green, $\widehat{\hat{N}}_{11}(f,x)$ in red, $\widehat{\hat{N}}_{12}(f,x)$ in black, $\widehat{\hat{N}}_{13}(f,x)$ in cyan, $\widehat{\hat{N}}_{14}(f,x)$ in magenta, $f(x)$ in blue. $\mu=20,\psi=10$ and $(\rho_r)=r$ and $(\xi_r)=r^2$ selected in Figure 1 a), $\mu=10,\psi=20$ and $(\rho_r)=r$ and $(\xi_r)=r^2$selected in Figure 1 b),$\mu=10,\psi=20$ and $(\rho_r)=r$ and $(\xi_r)=r^2+1$selected in Figure 1 c).

Example 3.9. *The error bound of* $f(x)=\frac{\sin 30}{(x^2+2)^2}+1$ *for* $x\in\left[0,\frac{r+3}{r+4}\right]$, $(\rho_r)=1$, $(\xi_r)=r$ *and* $(\rho_r)=r$, $(\xi_r)=r^2$. It can be seen from the table below that the error in approximation to this function is smaller when $(\rho_r)=r$, $(\xi_r)=r^2$.

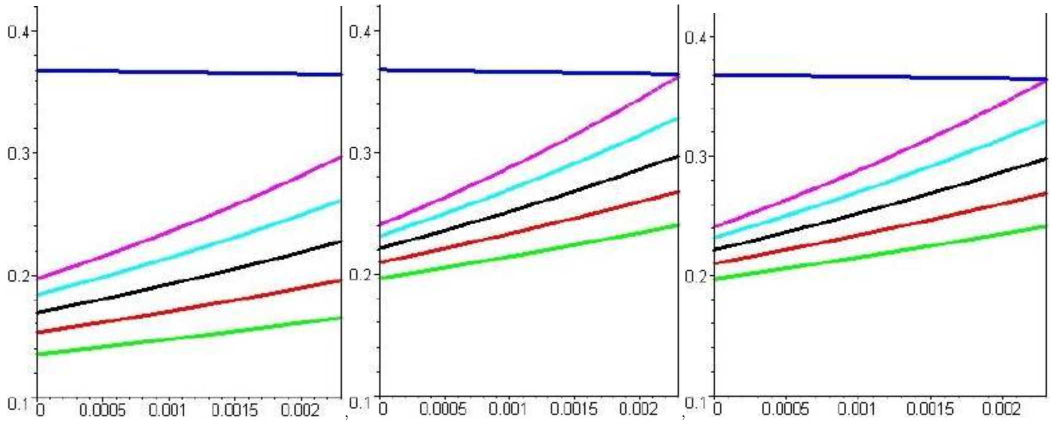

FIGURE 1. Approximation to function f

n	**Error Bounds for** $(\rho_r) = 1$, $(\xi_r) = r$	n	**Error Bound for**$(\rho_r) = r$, $(\xi_r) = r^2$
10	2.893844918	10	0.8267063506
10^2	2.964092648	10^2	0.2219790137
10^3	2.964094871	10^3	0.06877279786
10^4	2.964094872	10^4	0.02170248013
10^5	2.964094872	10^5	0.006861490344
10^6	2.964094872	10^6	0.001755626608
10^7	2.964094872	10^7	0.0006861332312
10^8	2.964094872	10^8	0.0002169743338
10^9	2.964094872	10^9	0.00006861330746

TABLE 1. The error bounds of f by different sequnces.

4. Conclusions

In this study, which is defined on a mobile range and examined the important approach features of the operator; It is thought that it will shape their studies by approaching the functions and will guide the researchers who cannot work on a fixed interval.

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6. Conflict of Interests

The authors stated that there are no conflict of interest in this article.

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ON DISCONTINUITY PROBLEM VIA SIMULATION FUNCTIONS

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Abstract. Metric fixed-point theory has been extensively studied with various perspectives. One of them is to generalize the used metric space such as S-metric and b-metric spaces. Another approach is to solve the raised open problems such as the Rhoades' discontinuity problem. We combine these two approaches and consider the set of simulation functions to present new solutions to the Rhoades' discontinuity problem on the existence of a self-mapping which has a fixed point but is not continuous at the fixed point on an S-metric and a b-metric space.

1. Introduction and Motivation

Recently, the set of simulation functions defined in [15] has been used for metric fixed-point theory and generalizations to solve some open problems (for example, see [6, 14, 15, 16, 17, 21, 25, 29, 31, 32]).

Recall that the function $\zeta : [0,\infty)\times[0,\infty) \to \mathbb{R}$ is said to be a simulation function in the Khojasteh et al.'s sense, if the following hold:

(ζ_1) $\zeta(0,0) = 0$,

(ζ_2) $\zeta(t,s) < s - t$ for all $s,t > 0$,

(ζ_3) If $\{t_n\}$, $\{s_n\}$ are sequences in $(0,\infty)$ such that

$$\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0,$$

then

$$\limsup_{n\to\infty} \zeta(t_n, s_n) < 0.$$

The set of all the simulation functions is denoted by $\mathcal{Z}$ (see [15] and [6] for more details).

In [31], Roldán-López-de-Hierro et al. modified this definition of simulation functions. For this purpose, only the condition (ζ_3) was replaced by the following condition $(\zeta_3)^*$ as follows:

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$(\zeta_3)^*$ If $\{t_n\}$, $\{s_n\}$ are sequences in $(0,\infty)$ such that

$$\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$$

and

$$t_n < s_n \text{ for all } n \in \mathbb{N},$$

then

$$\limsup_{n\to\infty} \zeta(t_n, s_n) < 0.$$

Every simulation function in the Khojasteh et al.'s sense is also a simulation function in the Roldán-López-de-Hierro et al.'s sense, the converse is not true (for example see Example 3.3 in [31]).

Some examples of simulation functions $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ are

- $\zeta(t,s) = \lambda s - t$, where $\lambda \in [0,1)$,
- $\zeta(t,s) = s - \varphi(s) - t$, where $\varphi : [0,\infty) \to [0,\infty)$ is a continuous function such that $\varphi(t) = 0$ if and only if $t = 0$,
- $\zeta(t,s) = s\phi(s) - t$, where $\phi : [0,\infty) \to [0,1)$ is a mapping such that $\limsup\limits_{t\to r^+} \phi(t) < 1$ for all $r > 0$,
- $\zeta(t,s) = \eta(s) - t$, where $\varphi : [0,\infty) \to [0,\infty)$ be an upper semi-continuous mapping such that $\eta(t) < t$ for all $t > 0$ and $\eta(0) = 0$,
- $\zeta(t,s) = s - \int\limits_0^t \psi(t)dt$, where $\psi : [0,\infty) \to [0,1)$ is a function such that $\int\limits_0^t \psi(t)dt$ exists and $\int\limits_0^\varepsilon \psi(u)du > \varepsilon$ for each $\varepsilon > 0$.

For study metric fixed-point theory, one of the used approaches is to generalize a metric space. For this aim, the notion of an S-metric space defined in [33] as follows:

An S-metric on a nonempty set X is a function $\mathcal{S} : X \times X \times X \to [0,\infty)$ satisfying the subsequent conditions:

$(S1)$ $\mathcal{S}(x,y,z) = 0$ if and only if $x = y = z$,

$(S2)$ $\mathcal{S}(x,y,z) \leq \mathcal{S}(x,x,a) + \mathcal{S}(y,y,a) + \mathcal{S}(z,z,a)$, for all $x,y,z,a \in X$.

The pair $(X,\mathcal{S})$ is known as an S-metric space.

The following theorem establishes the association of an S-metric with a b-metric [1].

Theorem 1.1. [34] *Let $(X,\mathcal{S})$ be an S-metric space and*

$$d^S(x,y) = \mathcal{S}(x,x,y),$$

$x,y \in X$. Then

1. *d^S is a b-metric on X,*
2. *$x_n \to x$ in $(X,\mathcal{S})$ if and only if $x_n \to x$ in (X,d^S),*
3. *$\{x_n\}$ is a Cauchy sequence in $(X,\mathcal{S})$ if and only if $\{x_n\}$ is a Cauchy sequence in (X,d^S).*

The b-metric d^S arises from S-metric $\mathcal{S}$.

Relationships between a metric and an S-metric have been studied in many works (see [10], [11], and [23] for more details). Let (X,d) be a metric space. Hieu et al. gave the following relation between a metric and an S-metric space [11]: The function $\mathcal{S}_d : X \times X \times X \to [0, \infty)$ defined by

$$\mathcal{S}_d(x, y, z) = d(x, z) + d(y, z),\ x, y, z \in X, \tag{1.1}$$

is an S-metric on X. Here $\mathcal{S}_d$ is called the S-metric that generated from the metric d [23]. But, there may exist an S-metric which is not obtained from a metric as seen in the following example:

Example 1.2. [23] Suppose the function $\mathcal{S} : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to [0, \infty)$ is defined on $\mathbb{R}$ as follows:

$$\mathcal{S}(x, y, z) = |x - z| + |x + z - 2y|\,,\ x, y, z \in X,$$

then $\mathcal{S}$ is an S-metric on X which is not generated from any standard metric d.

Now, consider an S-metric space $(X, \mathcal{S})$. Gupta [10] showed that each S-metric describes a metric such as:

$$d_S(x, y) = \mathcal{S}(x, x, y) + \mathcal{S}(y, x, x),\ x, y \in X. \tag{1.2}$$

However, the function d_S does not always describe a metric since all the elements of X do not verify the triangle inequality everywhere as seen in the following example (see, [23] for more details):

Example 1.3. [23] Suppose the function $\mathcal{S} : X \times X \times X \to [0, \infty)$ is defined on $X = \{1, 2, 3\}$ as:

$$\mathcal{S}(x, y, z) = \begin{cases} 1, & \text{if } x \neq y \neq z \\ 0, & \text{if } x = y = z \end{cases},$$

$$\mathcal{S}(1, 1, 2) = \mathcal{S}(2, 2, 1) = 5,\ \mathcal{S}(2, 2, 3) = \mathcal{S}(3, 3, 2) = \mathcal{S}(1, 1, 3) = \mathcal{S}(3, 3, 1) = 2,$$

$x, y, z \in X$, then $\mathcal{S}$ is an S-metric on X which does not satisfy the equality (1.2) for any standard metric d_S.

In this paper, we focus on the Rhoades' discontinuity problem (see, [30]) at fixed point using the properties of simulation functions on both S-metric and b-metric spaces. We obtain new solutions to this problem. Some recent solutions of this problem have been given in the literature using different techniques (see [2, 3, 4, 5, 9, 26, 27, 28, 29] and the references therein) and these solutions are important because of the applications to some applied areas (for example, see [5, 7, 8, 18, 19, 20, 24]).

In [22], geometric properties of the non-unique fixed points of a self-mapping have been investigated via simulation functions on metric (resp. S-metric and b-metric) spaces. The auxiliary numbers $M(x, y)$, $M_S(x, y)$ and $M_{d^S}(x, y)$ defined by

$$M(x, y) = \max \left\{ \begin{array}{c} ad(x, fx) + (1 - a)d(y, fy), \\ (1 - a)d(x, fx) + ad(y, fy), \frac{d(x, fy) + d(y, fx)}{2} \end{array} \right\},\ (0 \leq a < 1) \tag{1.3}$$

$$M_S(x, y) = \max \left\{ \begin{array}{c} a\mathcal{S}(x, x, fx) + (1 - a)\mathcal{S}(y, y, fy), \\ (1 - a)\mathcal{S}(x, x, fx) + a\mathcal{S}(y, y, fy), \frac{\mathcal{S}(x, x, fy) + \mathcal{S}(y, y, fx)}{4} \end{array} \right\},\ (0 \leq a < 1) \tag{1.4}$$

and

(1.5)

$$M_{d^S}(x,y) = \max\left\{ \begin{array}{c} ad^S(x,fx) + (1-a)d^S(y,fy), \\ (1-a)d^S(x,fx) + ad^S(y,fy), \frac{d^S(x,fy)+d^S(y,fx)}{2} \end{array} \right\}, \quad (0 \le a < 1)$$

had an efficient role for investigation of the geometric properties of the non-unique fixed points and for the studies on the Rhoades' discontinuity problem (see, [29] and [22] for more details). Here we will make use of the numbers $M_S(x,y)$ and $M_{d^S}(x,y)$. The following symmetry property

$$\mathcal{S}(x,x,y) = \mathcal{S}(y,y,x), \tag{1.6}$$

for all $x, y \in X$ on an S-metric space $(X, \mathcal{S})$, has also a key role in the studies for self-mappings of an S-metric space [33].

2. Simulation Functions and the Discontinuity Problem on S-metric spaces

In this section, we present new solutions to the Rhoades' problem on discontinuity at fixed point using the properties of simulation functions and the number $M_S(x,y)$ defined in (1.4) with the theory of an S-metric space. To do this, we use the Jachymski technique (see, [12] and [13]).

Theorem 2.1. *Let $(X,\mathcal{S})$ be a complete S-metric space and $f: X \to X$ a self-mapping satisfying the following conditions*

(i) Given $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\varepsilon \le M_S(x,y) < \varepsilon + \delta \Longrightarrow \mathcal{S}(fx,fx,fy) < \varepsilon,$$

(ii) $\zeta(\mathcal{S}(fx,fx,fy), M_S(x,y)) \ge 0$,

for all $x, y \in X$. Then f has a unique fixed point z and the sequence $\{f^n x\}$ for each $x \in X$ converges to the fixed point z. If $0 < a < 1$ then f is continuous at z, and if $a = 0$ then f is discontinuous at z if and only if $\lim\limits_{x \to z} M_S(x,z) \neq 0$.

Proof. Let x_0 be any point in X. Let us define a sequence $\{x_n\}$ in X as $x_n = fx_{n-1}$, that is, $x_n = f^n x_0$. If $x_n = x_{n+1}$ for some n then $x_n = x_{n+1} = x_{n+2} \ldots$. Hence $\{x_n\}$ is a Cauchy sequence and x_n is a fixed point of f. Therefore, we assume that $x_n \neq x_{n+1}$ for each n. Hence we obtain

$$M_S(x_{n-1},x_n) = \max\left\{ \begin{array}{c} a\mathcal{S}(x_{n-1},x_{n-1},x_n) + (1-a)\mathcal{S}(x_n,x_n,x_{n+1}), \\ (1-a)\mathcal{S}(x_{n-1},x_{n-1},x_n) + a\mathcal{S}(x_n,x_n,x_{n+1}), \frac{\mathcal{S}(x_{n-1},x_{n-1},x_{n+1})}{4} \end{array} \right\}.$$

Using the condition (ii), we get

(2.1)

$$0 \le \zeta(\mathcal{S}(fx_{n-1},fx_{n-1},fx_n), M_S(x_{n-1},x_n)) = \zeta(\mathcal{S}(x_n,x_n,x_{n+1}), M_S(x_{n-1},x_n)).$$

Suppose that $a \neq 0$. Let us consider the following cases:

Case 1. Let $M_S(x_{n-1},x_n) = a\mathcal{S}(x_{n-1},x_{n-1},x_n) + (1-a)\mathcal{S}(x_n,x_n,x_{n+1})$. From the inequality (2.1) and the condition (ζ_2), we have

$$\begin{aligned} 0 &\le \zeta(\mathcal{S}(x_n,x_n,x_{n+1}), a\mathcal{S}(x_{n-1},x_{n-1},x_n) + (1-a)\mathcal{S}(x_n,x_n,x_{n+1})) \\ &< a\mathcal{S}(x_{n-1},x_{n-1},x_n) + (1-a)\mathcal{S}(x_n,x_n,x_{n+1}) - \mathcal{S}(x_n,x_n,x_{n+1}) \\ &= a\mathcal{S}(x_{n-1},x_{n-1},x_n) - a\mathcal{S}(x_n,x_n,x_{n+1}) \end{aligned}$$

and so

$$\mathcal{S}(x_n, x_n, x_{n+1}) < \mathcal{S}(x_{n-1}, x_{n-1}, x_n).$$

Case 2. Let $M_S(x_{n-1}, x_n) = (1-a)\mathcal{S}(x_{n-1}, x_{n-1}, x_n) + a\mathcal{S}(x_n, x_n, x_{n+1})$. From the inequality (2.1) and the condition (ζ_2), we get

$$\begin{aligned} 0 &\leq \zeta\left(\mathcal{S}(x_n, x_n, x_{n+1}), (1-a)\mathcal{S}(x_{n-1}, x_{n-1}, x_n) + a\mathcal{S}(x_n, x_n, x_{n+1})\right) \\ &< (1-a)\mathcal{S}(x_{n-1}, x_{n-1}, x_n) + a\mathcal{S}(x_n, x_n, x_{n+1}) - \mathcal{S}(x_n, x_n, x_{n+1}) \\ &= (1-a)\mathcal{S}(x_{n-1}, x_{n-1}, x_n) - (1-a)\mathcal{S}(x_n, x_n, x_{n+1}) \end{aligned}$$

and so

$$\mathcal{S}(x_n, x_n, x_{n+1}) < \mathcal{S}(x_{n-1}, x_{n-1}, x_n).$$

Case 3. Let $M_S(x_{n-1}, x_n) = \frac{\mathcal{S}(x_{n-1}, x_{n-1}, x_{n+1})}{4}$. From the inequality (2.1), the symmetry property (1.6) and the condition (ζ_2), we obtain

$$\begin{aligned} 0 &\leq \zeta\left(\mathcal{S}(x_n, x_n, x_{n+1}), \frac{\mathcal{S}(x_{n-1}, x_{n-1}, x_{n+1})}{4}\right) \\ &< \frac{\mathcal{S}(x_{n-1}, x_{n-1}, x_{n+1})}{4} - \mathcal{S}(x_n, x_n, x_{n+1}) \end{aligned}$$

and so

$$\mathcal{S}(x_n, x_n, x_{n+1}) < \mathcal{S}(x_{n-1}, x_{n-1}, x_n).$$

Now assume that $a = 0$ and we consider the following cases:

Case 1$'$. Let $M_S(x_{n-1}, x_n) = \mathcal{S}(x_{n-1}, x_{n-1}, x_n)$. From the inequality (2.1) and the condition (ζ_2), we get

$$0 \leq \zeta\left(\mathcal{S}(x_n, x_n, x_{n+1}), \mathcal{S}(x_{n-1}, x_{n-1}, x_n)\right) < \mathcal{S}(x_{n-1}, x_{n-1}, x_n) - \mathcal{S}(x_n, x_n, x_{n+1})$$

and so

$$\mathcal{S}(x_n, x_n, x_{n+1}) < \mathcal{S}(x_{n-1}, x_{n-1}, x_n).$$

Case 2$'$. Let $M_S(x_{n-1}, x_n) = \mathcal{S}(x_n, x_n, x_{n+1})$. From the inequality (2.1) and the condition (ζ_2), we obtain

$$0 \leq \zeta\left(\mathcal{S}(x_n, x_n, x_{n+1}), \mathcal{S}(x_n, x_n, x_{n+1})\right) < \mathcal{S}(x_n, x_n, x_{n+1}) - \mathcal{S}(x_n, x_n, x_{n+1}) = 0,$$

a contradiction.

Case 3$'$. Let $M_S(x_{n-1}, x_n) = \frac{\mathcal{S}(x_{n-1}, x_{n-1}, x_n)}{4}$. By the similar arguments used in Case 3, we find

$$\mathcal{S}(x_n, x_n, x_{n+1}) < \mathcal{S}(x_{n-1}, x_{n-1}, x_n).$$

From the above cases, $\{\mathcal{S}(x_n, x_n, x_{n+1})\}$ is a strictly decreasing sequence of positive real numbers and it tends to a limit $\alpha \geq 0$. Let $\alpha > 0$. Then there exists a positive integer N such that

$$n \geq N \Longrightarrow \alpha < \mathcal{S}(x_n, x_n, x_{n+1}) < \alpha + \delta(\alpha), \tag{2.2}$$

or equivalently

$$\alpha < \max\left\{\begin{array}{c} a\mathcal{S}(x_n, x_n, fx_n) + (1-a)\mathcal{S}(x_{n+1}, x_{n+1}, fx_{n+1}), \\ (1-a)\mathcal{S}(x_n, x_n, fx_n) + a\mathcal{S}(x_{n+1}, x_{n+1}, fx_{n+1}), \\ \frac{\mathcal{S}(x_n, x_n, fx_{n+1})}{4} \end{array}\right\} < \alpha + \delta(\alpha).$$

From the condition (i), we get

$$\mathcal{S}(fx_n, fx_n, fx_{n+1}) = \mathcal{S}(x_{n+1}, x_{n+1}, x_{n+2}) < \alpha,$$

contradicts with (2.2). So $\mathcal{S}(x_n, x_n, x_{n+1}) \to 0$ as $n \to \infty$.

Now we show that $\{x_n\}$ is a Cauchy sequence. On the contrary, we suppose that it is not Cauchy. Then there exist an $\varepsilon > 0$ and a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\mathcal{S}(x_{n_i}, x_{n_i}, x_{n_{i+1}}) > 2\varepsilon. \tag{2.3}$$

Let us select δ in the condition (i) such that $0 < \delta \leq \varepsilon$. Since $\lim_{n\to\infty} \mathcal{S}(x_n, x_n, x_{n+1}) = 0$, then there exists an integer $N > 0$ such that

$$\mathcal{S}(x_n, x_n, x_{n+1}) < \frac{\delta}{12}, \tag{2.4}$$

whenever $n \geq N$. Let $n_i > N$. Then there exist integers m_i such that $n_i < m_i < n_{i+1}$ and

$$\mathcal{S}(x_{n_i}, x_{n_i}, x_{m_i}) \geq \varepsilon + \frac{\delta}{3}. \tag{2.5}$$

If not, using the inequalities (2.4), (2.5) and the symmetry property (1.6), we obtain

$$\begin{aligned}
\mathcal{S}(x_{n_i}, x_{n_i}, x_{n_{i+1}}) &= \mathcal{S}(x_{n_{i+1}}, x_{n_{i+1}}, x_{n_i}) \\
&\leq 2\mathcal{S}(x_{n_{i+1}}, x_{n_{i+1}}, x_{n_{i+1}-1}) + \mathcal{S}(x_{n_i}, x_{n_i}, x_{n_{i+1}-1}) \\
&< \frac{2\delta}{12} + \varepsilon + \frac{\delta}{3} = \varepsilon + \frac{\delta}{2} < 2\varepsilon,
\end{aligned}$$

a contradiction with the inequality (2.3). Let m_i^* be the smallest of m_i satisfying $n_i < m_i < n_{i+1}$ and

$$\mathcal{S}(x_{n_i}, x_{n_i}, x_{m_i^*}) \geq \varepsilon + \frac{\delta}{3}. \tag{2.6}$$

Then we have

$$\mathcal{S}(x_{n_i}, x_{n_i}, x_{m_i^*-1}) < \varepsilon + \frac{\delta}{3}.$$

From the inequality (2.6), the condition (ii) and the condition (ζ_2), we get

$$\begin{aligned}
\varepsilon &< \varepsilon + \frac{\delta}{3} \leq \mathcal{S}(x_{n_i}, x_{n_i}, x_{m_i^*}) = \mathcal{S}(fx_{n_i-1}, fx_{n_i-1}, fx_{m_i^*-1}) < M_S(x_{n_i-1}, x_{m_i^*-1}) \\
&= \max\left\{\begin{array}{c} a\mathcal{S}(x_{n_i-1}, x_{n_i-1}, x_{n_i}) + (1-a)\mathcal{S}(x_{m_i^*-1}, x_{m_i^*-1}, x_{m_i^*}), \\ (1-a)\mathcal{S}(x_{n_i-1}, x_{n_i-1}, x_{n_i}) + a\mathcal{S}(x_{m_i^*-1}, x_{m_i^*-1}, x_{m_i^*}), \\ \frac{\mathcal{S}(x_{n_i-1}, x_{n_i-1}, x_{m_i^*}) + \mathcal{S}(x_{m_i^*-1}, x_{m_i^*-1}, x_{n_i})}{4} \end{array}\right\} \\
&\leq \max\left\{\begin{array}{c} \mathcal{S}(x_{n_i-1}, x_{n_i-1}, x_{n_i}) + \mathcal{S}(x_{m_i^*-1}, x_{m_i^*-1}, x_{m_i^*}), \\ \mathcal{S}(x_{n_i-1}, x_{n_i-1}, x_{m_i^*}) + \mathcal{S}(x_{m_i^*-1}, x_{m_i^*-1}, x_{n_i}) \end{array}\right\} \\
&\leq \max\left\{\begin{array}{c} \mathcal{S}(x_{n_i-1}, x_{n_i-1}, x_{n_i}) + \mathcal{S}(x_{m_i^*-1}, x_{m_i^*-1}, x_{m_i^*}), \\ \mathcal{S}(x_{n_i-1}, x_{n_i-1}, x_{n_i}) + \mathcal{S}(x_{m_i^*-1}, x_{m_i^*-1}, x_{n_i}) \end{array}\right\} \\
&= \mathcal{S}(x_{n_i-1}, x_{n_i-1}, x_{n_i}) + \mathcal{S}(x_{m_i^*-1}, x_{m_i^*-1}, x_{n_i}) \\
(2.7)\quad & \frac{\delta}{12} + \varepsilon + \frac{\delta}{3} = \varepsilon + \frac{5\delta}{12} < \varepsilon + \delta.
\end{aligned}$$

By the inequality (2.7) and the condition (i), we get

$$\mathcal{S}(fx_{n_i-1}, fx_{n_i-1}, fx_{m_i^*-1}) \leq \varepsilon,$$

that is,

$$\mathcal{S}(x_{n_i}, x_{n_i}, x_{m_i^*}) \leq \varepsilon$$

which contradicts with the inequality (2.6). Hence $\{x_n\}$ is Cauchy. Using the completeness hypothesis, there exists a point $z \in X$ such that

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} f^n x_0 = z.$$

Now we prove that z is a fixed point of f, that is, $fz = z$. Conversely, we assume that $fz \neq z$. Let us consider the following two cases:

Case (i)**.** Let $a = 0$. For sufficiently large n, we obtain

$$\begin{aligned} M_S(x_n, z) &= \max\left\{\mathcal{S}(z,z,fz), \mathcal{S}(x_n,x_n,fx_n), \frac{\mathcal{S}(x_n,x_n,fz)+\mathcal{S}(z,z,fx_n)}{4}\right\} \\ &= \max\left\{\mathcal{S}(z,z,fz), \alpha_1, \frac{\mathcal{S}(x_n,x_n,fz)+\alpha_2}{4}\right\} = \mathcal{S}(z,z,fz), \end{aligned}$$

where $\lim\limits_{n\to\infty} \alpha_i = 0 (i \in \{1,2\})$. By the conditions (ii) and (ζ_2), we get

$$0 \leq \zeta\left(\mathcal{S}(fx_n, fx_n, fz), M_S(x_n, z)\right) = \zeta\left(\mathcal{S}(fx_n, fx_n, fz), \mathcal{S}(z,z,fz)\right)$$

and so taking a limit for $n \to \infty$, we obtain

$$0 \leq \lim_{n\to\infty} \zeta\left(\mathcal{S}(fx_n, fx_n, fz), \mathcal{S}(z,z,fz)\right) < 0,$$

a contradiction. Hence it should be $fz = z$.

Case (ii)**.** Let $a \neq 0$. For sufficiently large n, we find

$$\begin{aligned} M_S(x_n, z) &= \max\left\{\begin{array}{c} a\mathcal{S}(x_n,x_n,fx_n) + (1-a)\mathcal{S}(z,z,fz), \\ (1-a)\mathcal{S}(x_n,x_n,fx_n) + a\mathcal{S}(z,z,fz), \frac{\mathcal{S}(x_n,x_n,fz)+\mathcal{S}(z,z,fx_n)}{4} \end{array}\right\} \\ &= \max\left\{\alpha_1 + (1-a)\mathcal{S}(z,z,fz), \alpha_2 + a\mathcal{S}(z,z,fz), \frac{\mathcal{S}(x_n,x_n,fz)+\alpha_3}{4}\right\} \\ &= \alpha, \end{aligned}$$

where $\alpha < \mathcal{S}(z,z,fz)$ and $\lim\limits_{n\to\infty} \alpha_i = 0$ $(i \in \{1,2,3\})$. Using the conditions (ii) and (ζ_2), we get

$$\begin{aligned} 0 &\leq \zeta\left(\mathcal{S}(fx_n, fx_n, fz), M_S(x_n, z)\right) = \zeta\left(\mathcal{S}(fx_n, fx_n, fz), \alpha\right) \\ &< \alpha - \mathcal{S}(fx_n, fx_n, fz) \end{aligned}$$

and so taking a limit for $n \to \infty$, we obtain

$$0 \leq \alpha - \mathcal{S}(z,z,fz) < 0,$$

a contradiction. So it should be $fz = z$.

Under the above cases, z is a fixed point of f. We show that the fixed point z is a unique fixed point. On the contrary, suppose w is another fixed point of f such that $z \neq w$. Therefore, we find

$$\begin{aligned} M_S(z, w) &= \max\left\{\begin{array}{c} a\mathcal{S}(z,z,fz) + (1-a)\mathcal{S}(w,w,fw), \\ (1-a)\mathcal{S}(z,z,fz) + a\mathcal{S}(w,w,fw), \frac{\mathcal{S}(z,z,fw)+\mathcal{S}(w,w,fz)}{4} \end{array}\right\} \\ &= \frac{\mathcal{S}(z,z,w)}{2} \end{aligned}$$

and using the conditions (ii) and (ζ_2), we have

$$\begin{aligned} 0 &\leq \zeta\left(\mathcal{S}(fz, fz, fw), M_S(z, w)\right) = \zeta\left(\mathcal{S}(z,z,w), \frac{\mathcal{S}(z,z,w)}{2}\right) \\ &< \frac{\mathcal{S}(z,z,w)}{2} - \mathcal{S}(z,z,w) = -\frac{\mathcal{S}(z,z,w)}{2}, \end{aligned}$$

a contradiction. It should be $z = w$. Hence z is a unique fixed point of f. To prove the last part of this theorem, we show that f is continuous at z if and only if $\lim\limits_{x \to z} M_S(x, z) = 0$. Assume that f is continuous at the fixed point z and $x_n \to z$ as $n \to \infty$. Using the continuity of f, we have $fx_n \to fz = z$ and by the condition (ζ_2), we obtain

$$\mathcal{S}(x_n, x_n, fx_n) \leq 2\mathcal{S}(x_n, x_n, z) + \mathcal{S}(fx_n, fx_n, z) \to 0,$$

as $n \to \infty$. So we get $\lim\limits_{n \to \infty} M_S(x_n, z) = 0$. Conversely, if $\lim\limits_{n \to \infty} M_S(x_n, z) = 0$ then $\mathcal{S}(x_n, x_n, fx_n) \to 0$ as $x_n \to z$, which implies $fx_n \to z = fz$, that is, f is continuous at the fixed point z. □

We give the following corollaries. To do this, we suppose that the self-mapping f satisfies the condition (i) of Theorem 2.1 in all of the following corollaries.

If the following condition holds

$$\mathcal{S}(fx, fx, fy) \leq \lambda M_s(x, y),\ \lambda \in [0, 1)$$

for all $x, y \in X$, then f has a unique fixed point z and the sequence $\{f^n x\}$ for each $x \in X$ converges to the fixed point z. If $0 < a < 1$ then f is continuous at z, and if $a = 0$ then f is discontinuous at z if and only if $\lim\limits_{x \to z} M_S(x, z) \neq 0$.

Proof. Let us consider the mapping $\zeta_1 : [0, \infty)^2 \to \mathbb{R}$ defined by

$$\zeta_1(t, s) = \lambda s - t,$$

for all $s, t \in [0, \infty)$ [15]. Then f satisfies the condition (ii) of Theorem 2.1 with respect to $\zeta_1 \in \mathcal{Z}$. Hence, the proof can be easily obtained by taking $\zeta = \zeta_1$ in Theorem 2.1. □

If the following condition holds

$$\mathcal{S}(fx, fx, fy) \leq M_s(x, y) - \varphi\left(M_s(x, y)\right),$$

for all $x, y \in X$, where $\varphi : [0, \infty) \to [0, \infty)$ is lower semi continuous function and $\varphi^{-1}(0) = \{0\}$, then f has a unique fixed point z and the sequence $\{f^n x\}$ for each $x \in X$ converges to the fixed point z. If $0 < a < 1$ then f is continuous at z, and if $a = 0$ then f is discontinuous at z if and only if $\lim\limits_{x \to z} M_S(x, z) \neq 0$.

Proof. Let us consider the mapping $\zeta_2 : [0, \infty)^2 \to \mathbb{R}$ defined by

$$\zeta_2(t, s) = s - \varphi(s) - t,$$

for all $s, t \in [0, \infty)$ [15]. Then f satisfies the condition (ii) of Theorem 2.1 with respect to $\zeta_2 \in \mathcal{Z}$. Therefore, the proof can be easily obtained by taking $\zeta = \zeta_2$ in Theorem 2.1. □

If the following condition holds

$$\mathcal{S}(fx, fx, fy) \leq \phi\left(M_s(x, y)\right) M_s(x, y),$$

for all $x, y \in X$, where $\phi : [0, \infty) \to [0, 1)$ is a mapping such that $\limsup\limits_{t \to r^+} \phi(t) < 1$ for all $r > 0$, then f has a unique fixed point z and the sequence $\{f^n x\}$ for each $x \in X$ converges to the fixed point z. If $0 < a < 1$ then f is continuous at z, and if $a = 0$ then f is discontinuous at z if and only if $\lim\limits_{x \to z} M_S(x, z) \neq 0$.

Proof. Let us consider the mapping $\zeta_3 : [0,\infty)^2 \to \mathbb{R}$ defined by

$$\zeta_3(t,s) = s\phi(s) - t,$$

for all $s,t \in [0,\infty)$ [15]. Then f satisfies the condition (ii) of Theorem 2.1 with respect to $\zeta_3 \in \mathcal{Z}$. Therefore, the proof can be easily obtained by taking $\zeta = \zeta_3$ in Theorem 2.1. $\square$

If the following condition holds

$$\mathcal{S}(fx,fx,fy) \leq \eta\left(M_s(x,y)\right),$$

for all $x,y \in X$, where $\eta : [0,\infty) \to [0,\infty)$ is an upper semi continuous mapping such that $\eta(t) < t$ for all $t > 0$ and $\eta(0) = 0$, then f has a unique fixed point z and the sequence $\{f^n x\}$ for each $x \in X$ converges to the fixed point z. If $0 < a < 1$ then f is continuous at z, and if $a = 0$ then f is discontinuous at z if and only if $\lim\limits_{x \to z} M_S(x,z) \neq 0$.

Proof. Let us consider the mapping $\zeta_4 : [0,\infty)^2 \to \mathbb{R}$ defined by

$$\zeta_4(t,s) = \eta(s) - t,$$

for all $s,t \in [0,\infty)$ [15]. Then f satisfies the condition (ii) of Theorem 2.1 with respect to $\zeta_4 \in \mathcal{Z}$. Therefore, the proof can be easily obtained by taking $\zeta = \zeta_4$ in Theorem 2.1. $\square$

If the following condition holds

$$\int\limits_0^{\mathcal{S}(fx,fx,fy)} \psi(u)du \leq M_s(x,y),$$

for all $x,y \in X$, where $\psi : [0,\infty) \to [0,\infty)$ is a function such that $\int\limits_0^{\varepsilon} \psi(u)du$ exists and $\int\limits_0^{\varepsilon} \psi(u)du > \varepsilon$ for each $\varepsilon > 0$, then f has a unique fixed point z and the sequence $\{f^n x\}$ for each $x \in X$ converges to the fixed point z. If $0 < a < 1$ then f is continuous at z, and if $a = 0$ then f is discontinuous at z if and only if $\lim\limits_{x \to z} M_S(x,z) \neq 0$.

Proof. Let us consider the mapping $\zeta_5 : [0,\infty)^2 \to \mathbb{R}$ defined by

$$\zeta_5(t,s) = s - \int\limits_0^t \psi(u)du,$$

for all $s,t \in [0,\infty)$ [15]. Then f satisfies the condition (ii) of Theorem 2.1 with respect to $\zeta_5 \in \mathcal{Z}$. Therefore, the proof can be easily obtained by taking $\zeta = \zeta_5$ in Theorem 2.1. $\square$

3. Discontinuity Problem on b-Metric Spaces

In this section, we give some solutions to the Rhoades' discontinuity problem on b-metric spaces using Theorem 1.1 and the number $M_{d^S}(x,y)$ defined in (1.5). The followings are natural consequences of the proved discontinuity results in the previous section and so the proofs of them are clear.

Theorem 3.1. *Let (X,d^S) be a complete b-metric space and $f : X \to X$ a self-mapping satisfying the following conditions*

(i) Given $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\varepsilon \leq M_{d^S}(x,y) < \varepsilon + \delta \Longrightarrow d^S(fx,fy) < \varepsilon,$$

(ii) $\zeta\left(d^S(fx,fy), M_{d^S}(x,y)\right) \geq 0$,

for all $x,y \in X$. Then f has a unique fixed point z and the sequence $\{f^n x\}$ for each $x \in X$ converges to the fixed point z. If $0 < a < 1$ then f is continuous at z, and if $a = 0$ then f is discontinuous at z if and only if $\lim\limits_{x\to z} M_{d^S}(x,z) \neq 0$.

In all of the following corollaries, suppose that the self-mapping f satisfies the condition (i) of Theorem 3.1.

If the following condition holds

$$d^S(fx,fy) \leq \lambda M_{d^S}(x,y),\ \lambda \in [0,1)$$

for all $x,y \in X$, then f has a unique fixed point z and the sequence $\{f^n x\}$ for each $x \in X$ converges to the fixed point z. If $0 < a < 1$ then f is continuous at z, and if $a = 0$ then f is discontinuous at z if and only if $\lim\limits_{x\to z} M_{d^S}(x,z) \neq 0$.

If the following condition holds

$$d^S(fx,fy) \leq M_{d^S}(x,y) - \varphi\left(M_{d^S}(x,y)\right),$$

for all $x,y \in X$, where $\varphi : [0,\infty) \to [0,\infty)$ is lower semi continuous function and $\varphi^{-1}(0) = \{0\}$, then f has a unique fixed point z and the sequence $\{f^n x\}$ for each $x \in X$ converges to the fixed point z. If $0 < a < 1$ then f is continuous at z, and if $a = 0$ then f is discontinuous at z if and only if $\lim\limits_{x\to z} M_{d^S}(x,z) \neq 0$.

If the following condition holds

$$d^S(fx,fy) \leq \phi\left(M_{d^S}(x,y)\right) M_{d^S}(x,y),$$

for all $x,y \in X$, where $\phi : [0,\infty) \to [0,1)$ is a mapping such that $\limsup\limits_{t\to r^+} \phi(t) < 1$ for all $r > 0$, then f has a unique fixed point z and the sequence $\{f^n x\}$ for each $x \in X$ converges to the fixed point z. If $0 < a < 1$ then f is continuous at z, and if $a = 0$ then f is discontinuous at z if and only if $\lim\limits_{x\to z} M_{d^S}(x,z) \neq 0$.

If the following condition holds

$$d^S(fx,fy) \leq \eta\left(M_{d^S}(x,y)\right),$$

for all $x,y \in X$, where $\eta : [0,\infty) \to [0,\infty)$ is an upper semi continuous mapping such that $\eta(t) < t$ for all $t > 0$ and $\eta(0) = 0$, then f has a unique fixed point z and the sequence $\{f^n x\}$ for each $x \in X$ converges to the fixed point z. If $0 < a < 1$ then f is continuous at z, and if $a = 0$ then f is discontinuous at z if and only if $\lim\limits_{x\to z} M_{d^S}(x,z) \neq 0$.

If the following condition holds

$$\int_{0}^{d^S(fx,fy)} \psi(u)du \leq M_{d^S}(x,y),$$

for all $x, y \in X$, where $\psi : [0,\infty) \to [0,\infty)$ is a function such that $\int_0^{\varepsilon} \psi(u)du$ exists and $\int_0^{\varepsilon} \psi(u)du > \varepsilon$ for each $\varepsilon > 0$, then f has a unique fixed point z and the sequence $\{f^n x\}$ for each $x \in X$ converges to the fixed point z. If $0 < a < 1$ then f is continuous at z, and if $a = 0$ then f is discontinuous at z if and only if $\lim\limits_{x \to z} M_{d^S}(x,z) \neq 0$.

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A COMMON FIXED POINT THEOREM OF COMPATIBLE MAPPINGS CONCERNING F^*-CONTRACTION IN MODULAR METRIC SPACES

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ABSTRACT. In this study, the existence and uniqueness of common fixed points of compatible mappings satisfying F*-contraction are proved in the modular metric spaces. Moreover, some corollaries related to the main theorem are given.

1. INTRODUCTION

In 2008, Chistyakov introduced the notion of modular metric spaces, which has a physical interpretation [1] and he gave the fundamental properties of modular metric spaces [2]. Some authors proved different type fixed point theorems in modular metric spaces [6, 7, 9, 10].

Jungck gave some common fixed point theorems for commuting mappings satisfying contractive type conditions in 1976 [4]. Afterwards, he introduced the more generalized concept compatibility than commutativity and weak commutativity in metric space and proved common fixed point theorems [5].

In 2012, Wardowski introduced the concept of F-contraction as follows:[12]

Definition 1.1. Let $F:\mathbb{R}^+\to\mathbb{R}$ be a mapping satisfying:

(F1) F is strictly increasing,
(F2) For each sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ of positive numbers $\lim_{n\to\infty}\alpha_n=0$ if and only if

$$\lim_{n\to\infty}F(\alpha_n)=-\infty,$$

(F3) There exists $k\in(0,1)$ such that $\lim_{\alpha\to 0^+}\alpha^k F(\alpha)=0$.

$\mathscr{F}$ is the family of all functions F that satisfy the conditions $(F1)$, $(F2)$ and $(F3)$.

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Definition 1.2. Let (X,d) be metric space. A self-mapping T on X is called an F-contraction if there exist $F \in \mathscr{F}$ and $\tau \in \mathbb{R}^+$ such that

$$d(Tx,Ty) > 0 \Rightarrow \tau + F(d(Tx,Ty)) \leq F(d(x,y)) \tag{1.1}$$

for each $x,y \in X$.

In 2016, Piri and Kumam described a large class of functions by replacing the condition $(F2)$ in the definition of F-contraction introduced by Wardowski with the following one:
$(F2')$ F is continuous on $\mathbb{R}^+$.
They denoted by $\mathscr{F}_G$ the family of all functions $F : \mathbb{R}^+ \to \mathbb{R}$ which satisfy conditions $(F1)$ and $(F2')$. Using these families, Piri and Kumam introduced the modified generalized F-contractions and gave some fixed point result for these type mappings on complete metric space[13].

In this study, we prove the existence and uniqueness of common fixed points of compatible mappings satisfying F^*-contraction in the modular metric spaces. Moreover, we give some corollaries related to the main theorem.

2. Preliminaries

Here, we express a series of definitions of some basic concepts related to modular metric spaces.

Definition 2.1. [11] Let X be a linear space on $\mathbb{R}$. If a functional $\rho : X \to [0,\infty]$ satisfies the following conditions, we call that ρ is a modular on X:
(1) $\rho(0) = 0$;
(2) If $x \in X$ and $\rho(\alpha x) = 0$ for all numbers $\alpha > 0$, then $x = 0$;
(3) $\rho(-x) = \rho(x)$, for all $x \in X$;
(4) $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$ for all $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$ and $x,y \in X$.

Let $X \neq \emptyset$ and $\lambda \in (0,\infty)$. Generally, a function $\omega : (0,\infty) \times X \times X \to [0,\infty]$ is denoted as $\omega_\lambda(x,y) = \omega(\lambda,x,y)$ for all $x,y \in X$ and $\lambda > 0$.

Definition 2.2. [2] Let $X \neq \emptyset$. A function $\omega : (0,\infty) \times X \times X \to [0,\infty]$, which satisfies the following conditions for all $x,y,z \in X$, is called a metric modular on X:
(m1) $\omega_\lambda(x,y) = 0$ for all $\lambda > 0 \Leftrightarrow x = y$;
(m2) $\omega_\lambda(x,y) = \omega_\lambda(y,x)$ for all $\lambda > 0$;
(m3) $\omega_{\lambda+\mu}(x,y) \leq \omega_\lambda(x,z) + \omega_\mu(z,y)$ for all $\lambda, \mu > 0$.

If $0 < \mu < \lambda$, from properties of metric modular, we obtain that

$$\omega_\lambda(x,y) \leq \omega_{\lambda-\mu}(x,x) + \omega_\mu(x,y) = \omega_\mu(x,y)$$

for all $x,y \in X$.
From [2, 3], we know that for a fixed $x_0 \in X$, the set

$$X_\omega = X_\omega(x_0) = \{x \in X : \omega_\lambda(x,x_0) \to 0 \text{ as } \lambda \to \infty\}$$

is said to be modular metric space.

Definition 2.3. [7] Let ω be a metric modular in X, X_w be a modular metric space induced by ω, $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in X_ω and $C \subseteq X_\omega$. Then
(1) $\{x_n\}_{n\in\mathbb{N}}$ is called a convergent sequence such that $x_n \to x,\ x \in X_\omega$, if for $\lambda > 0$

$$\omega_\lambda(x_n,x) \to 0 \text{ as } n \to \infty.$$

(2) $\{x_n\}_{n\in\mathbb{N}}$ is called a Cauchy sequence, if for $\lambda > 0$

$$\omega_\lambda(x_n, x_m) \to 0 \text{ as } m, n \to \infty.$$

(3) C is called closed, if the limit of a convergent sequence in C always belong to C.
(4) C is called complete modular, if every Cauchy sequence $\{x_n\}_{n\in\mathbb{N}}$ in C is convergent in C.

Definition 2.4. [4] Self maps f and g of a metric space (X,d) are said to be commuting if $fg(x) = gf(x)$ for all $x \in X$.

Definition 2.5. [5] Self maps f and g of a metric space (X,d) are said to be compatible if $\lim_{n\to\infty} d(fg(x_n), gf(x_n)) = 0$ whenever $\{x_n\}_{n\in\mathbb{N}}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$.

Lemma 2.6. [5] *If f and g are compatible self maps of a metric space (X,d) and $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ in X, then $\lim_{n\to\infty} gfx_n = ft$, if f is continuous.*

We can rewrite the above definitions and lemma for modular metric spaces.

Definition 2.7. Self maps f and g of a modular metric space X_w are said to be commuting if $fg(x) = gf(x)$ for all $x \in X_w$.

Definition 2.8. Self maps f and g of a modular metric space X_w are said to be compatible if $\lim_{n\to\infty} \omega_\lambda(fg(x_n), gf(x_n)) = 0$ whenever $\{x_n\}_{n\in\mathbb{N}}$ is a sequence in X_w such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$.

Lemma 2.9. *If f and g are compatible self maps of a modular metric space X_w and $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ in X_w, then $\lim_{n\to\infty} gfx_n = ft$, if f is continuous.*

3. Main Results

Definition 3.1. Let ω be a metric modular in X, X_w be a modular metric space induced by ω. A pair of self mappings (f,g) in X_w is said to be a F^*-contraction if there exists $F \in \mathscr{F}_G$ and $\tau > 0$ such that

$$\tau + F(\omega_\lambda(gx, gy)) \leq F(\omega_\lambda(fx, fy)) \tag{3.1}$$

for all $gx \neq gy$ in X_w and for all $\lambda > 0$, where $\mathscr{F}_G$ is the family of all functions $F : \mathbb{R}^+ \to \mathbb{R}$ such that

(1) F is strictly increasing.
(2) F is continuous on $\mathbb{R}^+$.

Remark 3.2. Let f and g be two self mappings in modular metric space X_w such that

$$\tau + F(\omega_\lambda(gx, gy)) \leq F(\omega_\lambda(fx, fy))$$

where $\tau > 0$ and $F \in \mathscr{F}_G$ for all $gx \neq gy$ and $\lambda > 0$. Then

(a) $\omega_\lambda(gx, gy) < \omega_\lambda(fx, fy)$ for all $gx \neq gy$ and $\lambda > 0$.
(b) g is continuous, whenever f is continuous.

Theorem 3.3. *Let ω be a metric modular in X, X_w be a complete modular metric space induced by ω and (f,g) be a pair of F^*-contraction, compatible self mappings in X_w. Let f be continuous and $g(X) \subseteq f(X)$. Then f and g have a unique common fixed point in X_w.*

Proof. Let $x_0 \in X_w$ be arbitrary. Since $g(X) \subseteq f(X)$, there exists $x_1 \in X_w$ such that $fx_1 = gx_0$. By continue this process, we get $fx_n = gx_{n-1}$ for all $n > 0$. We show that $\{fx_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. Then from (3.1) we get

$$\begin{aligned} F(\omega_\lambda(fx_{n+1}, fx_n)) &= F(\omega_\lambda(gx_n, gx_{n-1})) \\ &< \tau + F(\omega_\lambda(gx_n, gx_{n-1})) \\ &\leq F(\omega_\lambda(fx_n, fx_{n-1})) \end{aligned}$$

for all $n > 0$ and $\lambda > 0$. Since F is strictly increasing, we have

$$\omega_\lambda(fx_{n+1}, fx_n) < \omega_\lambda(fx_n, fx_{n-1})$$

for all $n > 0$ and $\lambda > 0$. Then we can say that $\{\omega_\lambda(fx_{n+1}, fx_n)\}_{n\in\mathbb{N}}$ is a positive decreasing sequence of real numbers. Hence this sequence converges to a limit $r \geq 0$. We show that $r = 0$. We assume that $r > 0$. Then we get $r \leq \omega_\lambda(fx_{n+1}, fx_n)$ for all $n > 0$ and $\lambda > 0$. Using equation (3.1), we have

$$\begin{aligned} F(r) &\leq F(\omega_\lambda(fx_{n+1}, fx_n)) \\ &= F(\omega_\lambda(gx_n, gx_{n-1})) \\ &\leq F(\omega_\lambda(fx_n, fx_{n-1})) - \tau \\ &= F(\omega_\lambda(gx_{n-1}, gx_{n-2})) - \tau \\ &\leq F(\omega_\lambda(fx_{n-1}, fx_{n-2})) - 2\tau \\ &\vdots \\ &\leq F(\omega_\lambda(fx_1, fx_0)) - n\tau \end{aligned}$$

for all $n > 0$ and $\lambda > 0$. Since $F(r) \in \mathbb{R}$ and $\lim_{n\to\infty}[F(\omega_\lambda(fx_1, fx_0)) - n\tau] = -\infty$, there exists $n_1 > 0$ such that

$$F(\omega_\lambda(fx_1, fx_0)) - n\tau < F(r)$$

for all $n > n_1$ and $\lambda > 0$. Thus we get

$$F(r) \leq F(\omega_\lambda(fx_1, fx_0)) - n\tau < F(r)$$

for all $n > n_1$ and $\lambda > 0$, which is a contradiction. Hence we have

$$\lim_{n\to\infty} \omega_\lambda(fx_{n+1}, fx_n) = 0$$

for all $\lambda > 0$. So for each $\lambda > 0$, we have for all $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $\omega_\lambda(fx_{n+1}, fx_n) < \varepsilon$ for all $n \in \mathbb{N}$ with $n \geq n_0$. Without loss of generality, we suppose $m, n \in \mathbb{N}$ and $m > n$. We can say that for $\frac{\lambda}{m-n} > 0$, there exists $n_{\frac{\lambda}{m-n}} \in \mathbb{N}$ such that

$$\omega_{\frac{\lambda}{m-n}}(fx_{n+1}, fx_n) < \frac{\varepsilon}{m-n}$$

for all $m, n \geq \frac{\lambda}{m-n} \in \mathbb{N}$. Then we have

$$\begin{aligned} \omega_\lambda(fx_n, fx_m) &< \omega_{\frac{\lambda}{m-n}}(fx_{n+1}, fx_n) + \omega_{\frac{\lambda}{m-n}}(fx_{n+2}, fx_{n+1}) + \cdots + \omega_{\frac{\lambda}{m-n}}(fx_m, fx_{m-1}) \\ &< \frac{\varepsilon}{m-n} + \frac{\varepsilon}{m-n} + \cdots + \frac{\varepsilon}{m-n} \\ &= \varepsilon \end{aligned}$$

for all $m, n \geq n_{\frac{\lambda}{m-n}} \in \mathbb{N}$. This is implies that $\{fx_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. Since X_w is complete, we have

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$$

for some $t \in X$. Now from (3.1), we get

$$F(\omega_\lambda(gx,gy)) < \tau + F(\omega_\lambda(gx,gy)) \leq F(\omega_\lambda(fx,fy))$$

for all $gx \neq gy$ and $\lambda > 0$. Then g is continuous, as f is continuous. Therefore, we say that $gfx_n \to gt$ and $fgx_n \to ft$ as $n \to \infty$. Since f and g compatible, by Lemma 2.9, $gfx_n \to ft$ and they commute at their coincidence points. So we get

$$f(ft) = f(gt) = g(ft) = g(gt). \tag{3.2}$$

We show that $g(gt) = gt$. We suppose that $g(gt) \neq gt$. Then we get

$$\begin{aligned} F(\omega_\lambda(gt,g(gt))) &< \tau + F(\omega_\lambda(gt,g(gt))) \\ &\leq F(\omega_\lambda(ft,f(gt))) \\ &= F(\omega_\lambda(gt,g(gt))) \end{aligned}$$

for all $\lambda > 0$. This is a contradiction. So we get $g(gt) = gt$. Using (3.2), we say that $f(gt) = g(gt) = gt$. Then gt is a common fixed point of f and g. Now we show the uniqueness of the common fixed point. We suppose that there exist $x \neq y$ such that $x = fx = gx$ and $y = fy = gy$, then we get from (3.1)

$$F(\omega_\lambda(gx,gy)) < \tau + F(\omega_\lambda(gx,gy)) \leq F(\omega_\lambda(fx,fy)) = F(\omega_\lambda(gx,gy))$$

for all $\lambda > 0$. This is a contradiction. Therefore, f and g have a unique common fixed point in X_w. □

Corollary 3.4. *Let ω be a metric modular in X, X_w be a complete modular metric space induced by ω. f and g be commuting self mappings in X_w such that f be continuous and $g(X) \subseteq f(X)$. If there exists $F \in \mathscr{F}_G$, $\tau > 0$ and a positive integer k such that*

$$\tau + F(\omega_\lambda(g^k x, g^k y)) \leq F(\omega_\lambda(fx,fy))$$

for all $gx \neq gy$ in X_w and for all $\lambda > 0$, then f and g have a unique common fixed point in X_w.

Proof. g^k commutes with f and $g^k(X) \subseteq g(X) \subseteq f(X)$. From Theorem 3.3, f and g^k have a unique common fixed point in X_w. Let z be this fixed point. Then

$$z = f(z) = g^k(z). \tag{3.3}$$

On the other hand, since f and g commute, using equality (3.3), we obtain that

$$g(z) = g(f(z)) = f(g(z)) = g^k(g(z)).$$

Then $g(z)$ is a common fixed point of f and g^k. That contradicts with uniqueness of the common fixed point z. Therefore, $z = g(z) = f(z)$. Then f and g have a unique common fixed point in X_w. □

Corollary 3.5. *Let ω be a metric modular in X, X_w be a complete modular metric space induced by ω. f and g be commuting self mappings in X_w such that f be continuous and $g(X) \subseteq f(X)$. If there exists $\alpha \in (0,1)$ and a positive integer k such that*

$$\omega_\lambda(g^k x, g^k y) \leq \omega_\lambda(fx,fy) \tag{3.4}$$

for all $x, y \in X_w$ and $\lambda > 0$, then f and g have a unique common fixed point in X_w.

Proof. If we take ln on both sides of equation 3.4, then we get

$$\ln(\omega_\lambda(g^k x, g^k y)) \leq \ln(\omega_\lambda(fx, fy))$$

for all $\lambda > 0$. So we get

$$\ln\frac{1}{\alpha} + \ln(\omega_\lambda(g^k x, g^k y)) \leq \ln(\omega_\lambda(fx, fy))$$

for all $\lambda > 0$. Since $\alpha \in (0,1)$, we can take as $\tau = \ln\frac{1}{\alpha} > 0$. And, let $F(\alpha) = \ln(\alpha)$. Then F is strictly increasing and continuous on $(0,\infty)$. From Corollary 3.4, f and g have a unique common fixed point in X_w. $\square$

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SOME NEW FIXED-CIRCLE THEOREMS ON METRIC SPACES

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Abstract. This work presents some new existence theorems for fixed-circles of self-mappings on metric spaces. To do this, we obtain new conditions using the Caristi type contractive condition. Also, we confirm our results by illustrative examples.

1. Introduction and Preliminaries

Fixed point theory on metric spaces has been widely studied in the literature. In the last years, fixed-circle results with a geometric interpretation of this theory have been obtained in the context of both metric spaces and generalized metric spaces [1]-[15]. In this paper, we introduce new fixed-circle theorems for self-mappings on metric spaces.

This section provides some necessary definitions and concepts related to metric spaces.

Definition 1.1. *A metric space is a set X together with a function d (called a metric or distance function) which assigns a real number $d(x,y)$ to every pair $x,y \in X$ satisfying the properties:*
$(M1)$ $d(x,y) \geq 0$,
$(M2)$ $d(x,y) = d(y,x)$,
$(M3)$ $d(x,y) = 0$ *if and only if* $x = y$,
$(M4)$ $d(x,y) \leq d(x,z) + d(z,y)$.

Definition 1.2. *Let X be a nonempty set and $T : X \to X$ be a self-mapping. The point $x \in X$ satisfying $Tx = x$ is called a fixed point of the self-mapping T.*

Özgür and Taş [6] first proposed the concept of a fixed-circle on metric spaces in the study carried out in 2019.

Definition 1.3. [6] *Let (X,d) be a metric space and $C_{x_0,r} = \{x \in X : d(x,x_0) = r\}$ be a circle. For a self-mapping $T : X \to X$, if $Tx = x$ for every $x \in C_{x_0,r}$, then the circle $C_{x_0,r}$ is called the fixed-circle of T.*

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2. Some New Existence Conditions for Fixed-Circles on Metric Spaces

Now, we give the following existence theorems for a fixed-circle using Caristi type contractive condition [16].

Theorem 2.1. *Let (X,d) be a metric space and $C_{x_0,r}$ be any circle on X. Define the mapping*

$$\phi : X \to [0,\infty), \phi(x) = d(x,x_0) \tag{2.1}$$

for all $x \in X$. In this case, if the self-mapping $T : X \to X$ satisfies the following conditions

(1) $d(x,Tx) \leq \frac{r^2}{r+\phi(x)} - \frac{1}{2}\phi(Tx)$,

(2) $d(Tx,x_0) \geq r$

for all $x \in C_{x_0,r}$, then the circle $C_{x_0,r}$ is a fixed-circle of the mapping T.

Proof. Let X be a nonempty set and d is a metric on X. Let $C_{x_0,r}$ be a circle in (X,d) metric space. Think the function $\phi : X \to [0,\infty)$ as defined in (2.1) and take $T : X \to X$. For any arbitrary $x \in C_{x_0,r}$, we claim that $x = Tx$, that is, x is a fixed-point of T. Together with the condition (1) and the definition of ϕ, we obtain

$$\begin{aligned} d(x,Tx) &\leq \frac{r^2}{r+\phi(x)} - \frac{1}{2}\phi(Tx) \\ &= \frac{r^2}{r+d(x,x_0)} - \frac{1}{2}d(Tx,x_0) \\ &= \frac{r^2}{r+r} - \frac{1}{2}d(Tx,x_0) \\ &= \frac{r}{2} - \frac{1}{2}d(Tx,x_0) \\ &= \frac{1}{2}(r - d(Tx,x_0)) \end{aligned} \tag{2.2}$$

Since the condition (2) is satisfied, the point Tx should be lies on or exterior of the circle $C_{x_0,r}$. Hence, there are two cases. If the point Tx is the exterior of the circle, that is, $d(Tx,x_0) > r$, we obtain a contraction because of the inequality (2.2). In this case, it should be $d(Tx,x_0) = r$, that is the point Tx should be lies on the circle. Thus, we obtain

$$\begin{aligned} d(x,Tx) &\leq \frac{1}{2}(r - d(Tx,x_0)) \\ &= \frac{1}{2}(r-r) \\ &= 0. \end{aligned}$$

This gives $Tx = x$. Consequently, the self-mapping T fixes the circle $C_{x_0,r}$. □

Next, we present a fixed-circle example.

Example 2.1. *Let $(\mathbb{R},d)$ be the usual metric space. Let us take the circle $C_{0,5}$. If we define $T : \mathbb{R} \to \mathbb{R}$ by*

$$Tx = \begin{cases} x & , \quad x \in C_{0,5} \\ 10 & , \quad \textit{otherwise} \end{cases}$$

for all $x \in \mathbb{R}$, *then* T *confirms that the conditions* (1) *and* (2) *in Theorem 2.1. Hence, the circle* $C_{0,5}$ *is a fixed-circle of* T.

Now, in the following examples, we provide some examples of self-mappings that satisfy the condition (1) and do not satisfy the condition (2) and right after it, which fulfil the condition (2) and do not fulfil the condition (1).

Example 2.2. *Let* (X, d) *be any metric and* $C_{x_0,r}$ *be any circle on* X. *Let* α *be chosen such that* $d(\alpha, x_0) = \rho < r$ *and consider the self-mapping* $T : X \to X$ *defined by*

$$Tx = \alpha$$

for all $x \in X$. *Since the self-mapping* T *does not fix the circle* $C_{x_0,r}$, *at least one of the conditions of Theorem 2.1 (the condition* (2)*) is not satisfied.*

Example 2.3. *Let* $(\mathbb{R}, d)$ *be the usual metric space. Let us consider the circle* $C_{1,2}$. *Define* $T : \mathbb{R} \to \mathbb{R}$ *by*

$$Tx = \begin{cases} 3 & , \quad x \in \{-1, 3\} \\ 5 & , \quad otherwise \end{cases}$$

for all $x \in \mathbb{R}$. *Since the self-mapping* T *does not fix the circle* $C_{x_0,r}$, *at least one of the conditions of Theorem 2.1 (the condition* (1)*) is not satisfied.*

In the following example does not meet either the condition(1) or the condition (2).

Example 2.4. *Let* (X, d) *be any metric and* $C_{x_0,r}$ *be any circle on* X. *Define* $T : X \to X$ *as* $Tx = x_0$ *for* $x \in X$. *Since the self-mapping* T *does not fix the circle* $C_{x_0,r}$, *at least one of the conditions of Theorem 2.1 (both the condition* (1) *and the condition* (2)*) is not satisfied.*

We give another existence theorem for fixed-circles.

Theorem 2.2. *Let* (X, d) *be a metric space and* $C_{x_0,r}$ *be any circle on* X. *Let the mapping* ϕ *be defined as* (2.1). *In this case, if the self-mapping* $T : X \to X$ *satisfies the following conditions*

(3) $d(x, Tx) \leq \frac{max\{\phi(x),\phi(Tx)\}-r}{min\{\phi(x),\phi(Tx)\}+r}$,

(4) $d(Tx, x_0) \leq r$

for all $x \in C_{x_0,r}$, *then the circle* $C_{x_0,r}$ *is a fixed-circle of the mapping* T.

Proof. Let us consider the mapping ϕ defined in (2.1). Let $x \in C_{x_0,r}$ be any arbitrary point. We show that $Tx = x$ whenever $x \in C_{x_0,r}$. Using the condition (3) and the definition of ϕ, we obtain

$$\begin{aligned} d(x, Tx) &\leq \frac{max\{\phi(x), \phi(Tx)\} - r}{min\{\phi(x), \phi(Tx)\} + r} \\ &= \frac{max\{d(x, x_0), d(Tx, x_0)\} - r}{min\{d(x, x_0), d(Tx, x_0)\} + r} \\ &= \frac{max\{r, d(Tx, x_0)\} - r}{min\{r, d(Tx, x_0)\} + r} \end{aligned}$$

Since the condition (4) is satisfied, the point Tx should be lies on or interior of the circle $C_{x_0,r}$. Therefore, there are two cases. That is, either $d(Tx, x_0) < r$ or

$d(Tx, x_0) = r$. In both cases, we obtain

$$d(x, Tx) \leq \frac{max\{r, d(Tx, x_0)\} - r}{min\{r, d(Tx, x_0)\} + r} = \frac{r - r}{d(Tx, x_0) + r} = 0$$

and so we find $Tx = x$. Consequently, $C_{x_0,r}$ is a fixed-circle of T. □

Next, we present a fixed-circle example.

Example 2.5. *Let $(\mathbb{R}, d)$ be the usual metric space. Let us take the circle $C_{0,2}$. If we define $T : \mathbb{R} \to \mathbb{R}$ by*

$$Tx = \frac{3x + 4\sqrt{2}}{\sqrt{2}x + 3}$$

for all $x \in \mathbb{R}$, then T confirms that the conditions (3) and (4) in Theorem 2.2. Hence, the circle $C_{0,2}$ is a fixed-circle of T.

In the next example, we present an example of a self-mapping which verifies the condition (3) and does not verify the condition (4).

Example 2.6. *Let (X, d) be any metric and $C_{x_0,r}$ be any circle on X. Let α be chosen such that $d(\alpha, x_0) = \rho > r$ and consider the self-mapping $T : X \to X$ defined by*

$$Tx = \alpha$$

for all $x \in X$. Since the self-mapping T does not fix the circle $C_{x_0,r}$, at least one of the conditions of Theorem 2.2 (the condition (4)) is not satisfied.

Example 2.7. *Let $(\mathbb{R}, d)$ be the usual metric space. Let us think the circle $C_{0,4}$ and define the self-mapping $T : \mathbb{R} \to \mathbb{R}$ as*

$$Tx = \begin{cases} 3 & , \quad x \in \{-4, 4\} \\ 7 & , \quad otherwise \end{cases}$$

for all $x \in \mathbb{R}$. Since the self-mapping T does not fix the circle $C_{x_0,r}$, at least one of the conditions of Theorem 2.2 (the condition (3)) is not satisfied.

Now, we prove another existence fixed-circle theorem on metric spaces.

Theorem 2.3. *Let (X, d) be a metric space and $C_{x_0,r}$ be any circle on X. Let the mapping ϕ be defined as (2.1). Suppose that the following conditions hold:*
(5) $d(x, Tx) \leq 1 - e^{\phi(Tx) - \phi(x)}$
(6) $d(Tx, x_0) \geq r$
for all $x \in C_{x_0,r}$ such that $T : X \to X$. Then, the circle $C_{x_0,r}$ is a fixed-circle of T.

Proof. Let $x \in C_{x_0,r}$ be any arbitrary point. Together with (5), we obtain

$$d(x, Tx) \leq 1 - e^{\phi(Tx) - \phi(x)} = 1 - e^{d(Tx, x_0) - r}. \tag{2.3}$$

Because of the condition (6), the point (Tx) should be lies on or exterior of the circle $C_{x_0,r}$. Then, we have two cases. If $d(Tx, x_0) > r$, then using (2.3) we obtain a contradiction. Therefore, it should be $d(Tx, x_0) = r$. If $d(Tx, x_0) = r$, then using (2.3) we get

$$d(x, Tx) \leq 1 - e^{d(Tx, x_0) - r} = 1 - e^{r - r} = 1 - 1 = 0$$

and so we find $Tx = x$. As a result, $C_{x_0,r}$ is a fixed-circle of T. □

Example 2.8. *Let* $(\mathbb{R}, d)$ *be the usual metric space. Let us consider the circle* $C_{0,1}$ *and define the self-mapping* $T : \mathbb{R} \to \mathbb{R}$ *as*

$$Tx = \begin{cases} \frac{1}{x} & , \quad x \in C_{0,1} \\ 3 & , \quad otherwise \end{cases}$$

for all $x \in \mathbb{R}$. *Then, the self-mapping* T *satisfy the conditions* (5) *and* (6). *Obviously,* T *fixs the circle* $C_{0,1}$.

Example 2.9. *Let* $X = \mathbb{R}$ *and the mapping* $d : X^2 \to \mathbb{R}$ *be defined by*

$$d(x, y) = |e^x - e^y|$$

for all $x, y \in X$. *Let us take the circle* $C_{0,3}$ *and define the self-mapping* $T : X \to X$ *by*

$$Tx = \begin{cases} ln5 & , \quad x \in C_{0,3} \\ ln2 & , \quad otherwise \end{cases}$$

for all $x \in X$. *Since the self-mapping* T *does not fix the circle* $C_{x_0,r}$, *at least one of the conditions of Theorem 2.3 (the condition* (5)*) is not satisfied.*

Remark 2.1. *The ones in the literature provide the uniqueness of the obtained existence theorems (see* [6] *).*

3. Conclusion

In this paper, we obtain some new existence conditions for fixed-circle theorems on metric spaces. The relevant researchers can investigate the uniqueness conditions different from those in the literature of these existence theorems.

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ON THE STABILITY RESULT FOR INTEGRAL-TYPE MAPPING USING A THREE-STEP ITERATIVE ALGORITHM

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Abstract. The fixed point theory is an important research area in various disciplines. Moreover, this theory has wide theoretical and application areas in mathematics. In this work, the convergence of an iteration method under integral type conditions for a given mapping class has been shown. In addition, by using the definition of stability given by Harder and Hicks, it has been shown that this iteration method is stable under integral type conditions.

1. Introduction and Preliminaries

Theorem 1.1. [1] *Let (X, d) be a complete metric space, $\lambda \in (0,1)$, and let $T : X \to X$ be a mapping such that for each $x, y \in X$,*

$$d(Tx, Ty) \leq \lambda d(x, y) \tag{1.1}$$

then T has a unique fixed point $p \in X$ such that for each $x \in X$, $\lim_{n\to\infty} T^n x = p$.

Definition 1.2. [2, 3] Let $T : X \to X$. Define a fixed point iteration scheme by $x_{n+1} = f(T, x_n)$ such that $\{x_n\}$ converges to a fixed point p of T. Let $\{y_n\}$ be an arbitrary sequence in X. Define

$$\epsilon_n = \|y_{n+1} - f(T, y_n)\|$$

for $n \geq 1$. A fixed point iteration scheme is said to be T-stable if the following condition is satisfied:

$$\lim_{n\to\infty} \epsilon_n = 0 \text{ if and only if } \lim_{n\to\infty} y_n = p.$$

Lemma 1.3. [4] *Let $\{b_n\}_{n=0}^{\infty}$ and $\{d_n\}_{n=0}^{\infty}$ be nonnegative real sequences satisfying the following inequality:*

$$b_{n+1} \leq (1 - r_n)\, b_n + d_n$$

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where $r_n \in (0,1)$ *for all* $n \in \mathbb{N}$, $\sum\limits_{n=0}^{\infty} r_n = \infty$ *and* $\lim_{n\to\infty} \frac{d_n}{r_n} = 0$. *Then* $b_n \to 0$ *as* $n \to \infty$.

Theorem 1.4. [5] *Let* (X, d) *be a complete metric space,* $\lambda \in (0,1)$, *and let* $T : X \to X$ *be a mapping such that for each* $x, y \in X$,

$$\int_0^{d(Tx,Ty)} \varphi(t)\mathrm{d}t \leq \lambda \int_0^{d(x,y)} \varphi(t)\mathrm{d}t \tag{1.2}$$

where $\varphi : [0,\infty) \to [0,\infty)$ *is Lebesgue-integrable mapping which is summable (i.e., with finite integral) on each compact subset of* $[0,\infty)$ *nonnegative, and such that for each* $\varepsilon > 0$, $\int_0^{\varepsilon} \varphi(t)\mathrm{d}t > 0$; *then* T *has a unique fixed point* $p \in X$ *such that for each* $p \in X$, $\lim_{n\to\infty} T^n x = p$.

Lemma 1.5. [6] *Let* (X, d) *be a complete metric space and* $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ *a Lebesgue-Stieltjes integrable mapping which is summable, nonnegative, and such that for each* $\varepsilon > 0$, $\int_0^{\varepsilon} \varphi(t)\mathrm{d}t$. *Suppose that* $\{u_n\}_{n=0}^{\infty}$, $\{v_n\}_{n=0}^{\infty} \subset X$ *and* $\{a_n\}_{n=0}^{\infty} \subset (0,1)$ *are sequances such that be nonnegative real sequences satisfying the following inequality:*

$$\left| d(u_n, v_n) - \int_0^{d(u_n,v_n)} \varphi(t)\mathrm{d}t \right| \leq a_n,$$

with $\lim_{n\to\infty} a_n = 0$. *Then,*

$$d(u_n, v_n) - a_n \leq \int_0^{d(u_n,v_n)} \varphi(t)\mathrm{d}t \leq d(u_n, v_n) + a_n.$$

2. Fixed Point Algorithms

The general form of the iterative algorithm is as follows:
Let (X, d) be a metric space and $T : X \to X$ is self mapping.

$$\begin{cases} x_0 \in X \\ x_{n+1} = f(x_n, T) \qquad n = 0, 1, 2, ... \end{cases} \tag{2.1}$$

in which X is an arbitrary space, x_0 is initial point, $T : X \to X$ is an operator, and f is some function.
In 1890, Picard [7] defined Picard iterative algorithm as follows:

Algorithm 2.1. *Let* (X, d) *be a metric space and* $T : X \to X$ *is self mapping.*

$$\begin{cases} x_0 \in C \\ x_{n+1} = Tx_n \qquad n = 0, 1, 2, ... \end{cases} \tag{2.2}$$

In 1953, Mann [8] introduced Mann iterative algorithm as follows:

Algorithm 2.2. *Let* (X, d) *be a metric space and* $T : X \to X$ *is self mapping.*

$$\begin{cases} x_0 \in C \\ x_{n+1} = (1 - \alpha_n)\, x_n + \alpha_n T x_n \qquad n = 0, 1, 2, ... \end{cases} \tag{2.3}$$

in which $\{\alpha_n\}_{n=0}^{\infty} \in [0,1]$, $\sum_{n=0}^{\infty} \alpha_n = \infty$.

The Ishikawa iterative algorithm given in [9] is defined by

Algorithm 2.3. *Let C be nonempty convex subset of a norm space $(X,\|.\|)$ and $T: C\to C$ is self mapping.*

$$\begin{cases} x_0 \in C \\ x_{n+1} = (1-\alpha_n)\, x_n + \alpha_n T y_n \\ y_n = (1-\beta)\, x_n + \beta_n T x_n \qquad n = 0,1,2,... \end{cases} \tag{2.4}$$

in which $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty} \in [0,1]$, $\sum_{n=0}^{\infty} \alpha_n = \infty$.

Noor [10] proposed an iterative algorithm as follows:

Algorithm 2.4. *Let C be nonempty convex subset of a norm space $(X,\|.\|)$ and $T: C\to C$ is self mapping.*

$$\begin{cases} x_0 \in C \\ x_{n+1} = (1-\alpha_n)\, x_n + \alpha_n T y_n \\ y_n = (1-\beta_n)\, x_n + \beta_n T z_n \\ z_n = (1-\gamma_n)\, x_n + \gamma_n T x_n \qquad n = 0,1,2,... \end{cases} \tag{2.5}$$

in which $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$, $\{\gamma_n\}_{n=0}^{\infty} \in [0,1]$, $\sum_{n=0}^{\infty} \alpha_n = \infty$.

Karakaya et al. established an iterative algorithm in [11] as follows:

Algorithm 2.5. *Let C be nonempty convex subset of a norm space $(X,\|.\|)$ and $T: C\to C$ is self mapping.*

$$\begin{cases} x_0 \in C \\ x_{n+1} = (1-\alpha_n-\beta_n)\, y_n + \alpha_n T y_n + \beta_n T z_n \\ y_n = (1-a_n-b_n)\, z_n + a_n T z_n + b_n T x_n \\ z_n = (1-c_n)\, x_n + c_n T x_n \qquad n = 0,1,2,... \end{cases} \tag{2.6}$$

in which $\{\alpha_n+\beta_n\}_{n=0}^{\infty}$, $\{a_n+b_n\}_{n=0}^{\infty} \in [0,1]$, $\sum_{n=0}^{\infty} (\alpha_n+\beta_n) = \infty$.

3. CONVERGENCE AND STABILITY

Theorem 3.1. [12] *Let C be nonempty closed convex subset of a Banach space $(X,\|.\|)$, $T: C\to C$ satisfy condition (1.2) with $\varphi: [0,\infty) \to [0,\infty)$ is Lebesgue-integrable mapping which is summable, nonnegative, and such that for each $\varepsilon > 0$, $\int_0^{\varepsilon} \varphi(t)\mathrm{d}t > 0$. Let $\{x_n\}_{n=0}^{\infty}$ be a sequence generated by Algorithm 2.5 with control sequences. Then, the iterative sequence $\{x_n\}_{n=0}^{\infty}$ converges to the fixed point of T.*

Proof. By using Algorithm 2.5, Lemma 1.5, and condition (1.2), we have

$$\begin{aligned} \int_0^{\|x_{n+1}-p\|} \varphi(t)\mathrm{d}t &= \|x_{n+1}-p\| + k_n \\ &\le (1-\alpha_n-\beta_n) \int_0^{\|y_n-p\|} \varphi(t)\mathrm{d}t \\ &\quad + \lambda\alpha_n \int_0^{\|y_n-p\|} \varphi(t)\mathrm{d}t + \lambda\beta_n \int_0^{\|z_n-p\|} \varphi(t)\mathrm{d}t + 2k_n \end{aligned} \tag{3.1}$$

and

$$\int_0^{\|z_n-p\|} \varphi(t)\mathrm{d}t \le (1-c_n) \int_0^{\|x_n-p\|} \varphi(t)\mathrm{d}t + \lambda c_n \int_0^{\|x_n-p\|} \varphi(t)\mathrm{d}t + 2k_n \tag{3.2}$$

and
(3.3)

$$\begin{aligned}\int_0^{\|y_n-p\|}\varphi(t)\mathrm{d}t &\leq (1-a_n-b_n)(1-c_n)\int_0^{\|x_n-p\|}\varphi(t)\mathrm{d}t \\ &+\lambda c_n(1-a_n-b_n)\int_0^{\|x_n-p\|}\varphi(t)\mathrm{d}t+\lambda b_n\int_0^{\|x_n-p\|}\varphi(t)\mathrm{d}t \\ &+\lambda a_n(1-c_n)\int_0^{\|x_n-p\|}\varphi(t)\mathrm{d}t+\lambda^2 a_n c_n\int_0^{\|x_n-p\|}\varphi(t)\mathrm{d}t+6k_n\end{aligned}$$

Substituting (3.2) and (3.3) in (3.1), we obtain

$$\int_0^{\|x_{n+1}-p\|}\varphi(t)\mathrm{d}t \leq (1-(\alpha_n+\beta_n)(1-\lambda))\int_0^{\|x_n-p\|}\varphi(t)\mathrm{d}t+16k_n \tag{3.4}$$

Taking the limit on both sides of (3.4) and using $\lim_{n\to\infty} k_n = 0$ and Lemma 1.3, we get $\lim_{n\to\infty}\|x_n-p\| = 0$.

□

Corollary 3.2. [12] *Let C be nonempty closed convex subset of a Banach space $(X,\|.\|)$, $T,S : C\to C$ satisfy condition (1.2) with $\varphi : [0,\infty) \to [0,\infty)$ is Lebesgue-integrable mapping which is summable, nonnegative, and such that for each $\varepsilon > 0$, $\int_0^\varepsilon \varphi(t)\mathrm{d}t > 0$. Let $\{x_n\}_{n=0}^\infty$ be a sequence generated by Algorithm 2.5 with the control sequences. Then, the iterative sequence $\{x_n\}_{n=0}^\infty$ converges to common fixed point of T and S.*

Theorem 3.3. [12] *Let C be nonempty closed convex subset of a Banach space $(X,\|.\|)$, $T,S : C\to C$ satisfy condition (1.2) with $\varphi : [0,\infty) \to [0,\infty)$ is Lebesgue-integrable mapping which is summable, nonnegative, and such that for each $\varepsilon > 0$, $\int_0^\varepsilon \varphi(t)\mathrm{d}t > 0$. Let $\{x_n\}_{n=0}^\infty$ be a sequence generated by Algorithm 2.5 with the control sequences. Then, the iterative sequence $\{x_n\}_{n=0}^\infty$ converges to common fixed point of T. Then, the Algorithm 2.5 is stable.*

Proof. Let $\{y_n\}_{n=0}^\infty$ be an arbitrary sequence, $\lim_{n\to\infty}\epsilon = 0$ such that
$\epsilon_n = \|y_{n+1} - f(T, y_n)\|$. We have

$$\begin{cases} y_{n+1} = (1-\alpha_n-\beta_n)\,s_n+\alpha_n Ts_n+\beta_n Tz_n \\ \quad s_n = (1-a_n-b_n)\,z_n+a_n Tz_n+b_n Ty_n \\ \quad z_n = (1-c_n)\,y_n+c_n Ty_n \qquad n=0,1,2,... \end{cases} \tag{3.5}$$

in which $\{\alpha_n+\beta_n\}_{n=0}^\infty$, $\{a_n+b_n\}_{n=0}^\infty \in [0,1]$, $\sum_{n=0}^\infty(\alpha_n+\beta_n)=\infty$.

$$\begin{aligned}\int_0^{\|y_{n+1}-p\|}\varphi(t)\mathrm{d}t &= \|y_{n+1}-p\|+k_n \\ &\leq \int_0^{\epsilon_n}\varphi(t)\mathrm{d}t+(1-\alpha_n-\beta_n)\int_0^{\|s_n-p\|}\varphi(t)\mathrm{d}t \\ &+\lambda\alpha_n\int_0^{\|s_n-p\|}\varphi(t)\mathrm{d}t+\lambda\beta_n\int_0^{\|z_n-p\|}\varphi(t)\mathrm{d}t+3k_n\end{aligned} \tag{3.6}$$

and

$$\int_0^{\|z_n-p\|}\varphi(t)\mathrm{d}t \leq (1-c_n)\int_0^{\|y_n-p\|}\varphi(t)\mathrm{d}t+\lambda c_n\int_0^{\|y_n-p\|}\varphi(t)\mathrm{d}t+2k_n \tag{3.7}$$

and

(3.8)

$$\int_0^{\|s_n-p\|}\varphi(t)\mathrm{d}t \le (1-a_n-b_n)(1-c_n)\int_0^{\|y_n-p\|}\varphi(t)\mathrm{d}t + \lambda c_n(1-a_n-b_n)\int_0^{\|y_n-p\|}\varphi(t)\mathrm{d}t + \lambda b_n\int_0^{\|y_n-p\|}\varphi(t)\mathrm{d}t + \lambda a_n(1-c_n)\int_0^{\|y_n-p\|}\varphi(t)\mathrm{d}t + \lambda^2 a_n c_n\int_0^{\|y_n-p\|}\varphi(t)\mathrm{d}t + 6k_n$$

Substituting (3.7) and (3.8) in (3.6), we obtain

(3.9)

$$\int_0^{\|y_{n+1}-p\|}\varphi(t)\mathrm{d}t \le (1-(\alpha_n+\beta_n)(1-\lambda))\int_0^{\|y_n-p\|}\varphi(t)\mathrm{d}t + \int_0^{\epsilon_n}\varphi(t)\mathrm{d}t + 15k_n$$

Taking the limit on both sides of (3.9) and using $\lim_{n\to\infty} k_n = 0$ and Lemma 1.3, we get $\lim_{n\to\infty}\|y_n-p\| = 0$.

Suppose that $\lim_{n\to\infty}\|y_n-p\| = 0$

(3.10)

$$\int_0^{\epsilon_n}\varphi(t)\mathrm{d}t \le \int_0^{\|y_{n+1}-p\|}\varphi(t)\mathrm{d}t + (1-(\alpha_n+\beta_n)(1-\lambda))\int_0^{\|y_n-p\|}\varphi(t)\mathrm{d}t + 15k_n.$$

Taking the limit both side of the above inequality, we have $\lim_{n\to\infty}\int_0^{\epsilon_n}\varphi(t)\mathrm{d}t = 0$. □

Corollary 3.4. [12] *Let C be nonempty closed convex subset of a Banach space $(X, \|.\|)$, $T, S : C \to C$ satisfy condition (1.2) with $\varphi : [0,\infty) \to [0,\infty)$ is Lebesgue-integrable mapping which is summable, nonnegative, and such that for each $\varepsilon > 0$, $\int_0^{\varepsilon}\varphi(t)\mathrm{d}t > 0$. Let $\{x_n\}_{n=0}^{\infty}$ be a sequence generated by Algorithm 2.5 with the control sequences. Then, the iterative sequence $\{x_n\}_{n=0}^{\infty}$ converges to common fixed point of T and S. Then, the Algorithm 2.5 is stable.*

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DISCRETIZATION AND STABILITY ANALYSIS OF A CONFORMABLE FRACTIONAL ORDER COVID-19 MODEL

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Abstract. This work aims to examine the complex behaviors of a conformable fractional order predator-prey model. For this purpose, two-dimensional discrete system of the model is created by using of a discretization process based on the use of piecewise constant arguments. Then, we use the Schur-Cohn criterion to obtain the necessary and sufficient conditions for the stability of the equilibrium points. Finally, numerical simulations are used to show that the analytical results are correct.

1. Introduction

In recent years, there has been a significant increase in the use of fractional derivatives in many disciplines such as biology, mathematics, chemistry and physics, as the fractional derivative is more convenient than the integer derivative in modeling [1, 2, 3, 7, 4, 5, **?**, 6]. The fractional derivative gives an excellent instrument for the description of memory and hereditary properties of various materials and processes. Mohammed et al. [7] have investigated Lotka-Volterra based of COVID-19 model as follows:

$$\begin{cases} D^{\alpha}x(t) &= ax(t) - bx(t)y(t) + ey(t) \\ D^{\alpha}y(t) &= bx(t)y(t) + (c-d-e)\,y(t), \quad 0 < \alpha \leq 1 \end{cases} \tag{1.1}$$

where the healthy individual population by $x(t)$ at time t and the infected individual population is given by $y(t)$ at time t. Let $b > 0$ represents the infection rate (1-protection rate), the immigration rate of healthy individuals is given by $a > 0$, and $c > 0$ will introduce the immigration rate of infected individuals. Finally, the death rate is given by $d > 0$ and the cure rate is given by $e > 0$.

There are many definitions of the fractional derivative and one of them is conformable derivative in [8] which is introduced by Khalil et al. According to this definition, the left conformable fractional derivative starting from a of the function $f : [a, \infty) \to \infty$ of order $0 < \alpha \leq 1$ is given by

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$$(T^a_\alpha f)(t) = \lim_{\epsilon\to 0} \frac{f(t+\epsilon(t-a)^{1-\alpha}) - f(t)}{\epsilon} \tag{1.2}$$

and the right conformable fractional derivative of order $0 < \alpha \leq 1$ terminating at b of f is defined by

$$({}^b_\alpha T f)(t) = -\lim_{\epsilon\to 0} \frac{f(t+\epsilon(b-t)^{1-\alpha}) - f(t)}{\epsilon}. \tag{1.3}$$

Note that if f is differentiable then

$$(T^a_\alpha f)(t) = (t-a)^{1-\alpha} f'(t) \quad , \quad ({}^b_\alpha T f)(t) = -(b-t)^{1-\alpha} f'(t). \tag{1.4}$$

The aim of this study is to investigate dynamic behavior of a conformable fractional order COVID-19 model with piecewise constant arguments that is given as follows ;

$$\begin{cases} T_\alpha x(t) & = ax(t) - bx(t)y\left([\frac{t}{h}]h\right) + ey\left([\frac{t}{h}]h\right) \\ T_\alpha y(t) & = bx\left([\frac{t}{h}]h\right) y(t) + (c-d-e)\, y(t) \end{cases} \tag{1.5}$$

where $[t]$ denotes the integer part of $t \in [0,\infty)$ and $h > 0$ is a discretization parameter.

2. Discretization process

In here, we discretize the model (1.5) using the discretization method introduced in [9]. Using the left conformable fractional derivative (1.5), one gets

$$(t-nh)^{1-\alpha} \frac{dx(t)}{dt} + (by(nh) - a)x(t) = ey(nh). \tag{2.1}$$

From (2.1),

$$x'(t) + \frac{(by(nh)-a)}{(t-nh)^{1-\alpha}} x(t) = \frac{ey(nh)}{(t-nh)^{1-\alpha}}. \tag{2.2}$$

The equation 2.2 is multiplied by $e^{(by(nh)-a)\frac{(t-nh)^\alpha}{\alpha}}$ gives

$$\frac{d}{dt}\left(x(t)e^{(by(nh)-a)\frac{(t-nh)^\alpha}{\alpha}}\right) = \frac{ey(nh)}{(t-nh)^{1-\alpha}} e^{(by(nh)-a)\frac{(t-nh)^\alpha}{\alpha}}, \tag{2.3}$$

where $t \in [nh, (n+1)h)$. Integrating both sides of 2.3 with respect to t on $[nh, t)$ we obtain

$$x(t)e^{(by(nh)-a)\frac{(t-nh)^\alpha}{\alpha}} - x(nh) = \frac{ey(nh)}{(by(nh)-a)} e^{(by(nh)-a)\frac{(t-nh)^\alpha}{\alpha}} - 1] \tag{2.4}$$

Let $t \to (n+1)h$ in equation 2.4 and replacing $x(nh)$ and $y(nh)$ by $x(n)$ and $y(n)$ yields

$$x(n+1) = (x(n) - \frac{ey(n)}{(by(n)-a)})e^{(a-by(n))\frac{h^\alpha}{\alpha}} + \frac{ey(n)}{by(n)-a}. \tag{2.5}$$

Using the same procedures, discretizing the second equation of the system (1.5)

$$T_\alpha y(t) = bx\left([\frac{t}{h}]h\right) y(t) + (c-d-e)\, y(t) \tag{2.6}$$

obtains to the following difference equation

$$y(n+1) = y(n)e^{(bx(n)+c-d-e)\frac{h^\alpha}{\alpha}}. \tag{2.7}$$

From the equation (2.5) and (2.7), we obtain the two-dimensional discrete system as follows:

$$\begin{cases} x(n+1) & = (x(n) - \frac{ey(n)}{(by(n)-a)})e^{(a-by(n))\frac{h^\alpha}{\alpha}} + \frac{ey(n)}{by(n)-a} \\ y(n+1) & = y(n)e^{(bx(n)+c-d-e)\frac{h^\alpha}{\alpha}} \end{cases}. \tag{2.8}$$

3. Stability Analysis

In this section,, we examine local asymptotic stability of the system 2.8. As can be easily seen, the system 2.8 and the system 1.1 have the same equilibrium points and those are $E_0 = (0,0)$ and $E_1 = \left(\frac{e+d-c}{b}, \frac{a(e+d-c)}{b(d-c)}\right)$ where $d \neq c$. From these, E_0 is the trivial and E_1 is coexistence equilibrium points.

Theorem 3.1. *The trivial equilibrium point of the system 2.8 is saddle when the immigration rate of infected individuals is less than the summation of the death rate and the cure rate ($c < d+e$), otherwise it is unstable node ($c > d+e$).*

Proof. The Jacobian matrix of the system 2.8 about $E_0 = (0,0)$ is

$$J = \begin{pmatrix} e^{\frac{ah^\alpha}{\alpha}} & (e^{\frac{ah^\alpha}{\alpha}} - 1)\frac{e}{a} \\ 0 & e^{(c-d-e)\frac{h^\alpha}{\alpha}} \end{pmatrix}$$

and has eigenvalues $\lambda_1 = e^{\frac{ah^\alpha}{\alpha}} > 1$ and $\lambda_2 = e^{(c-d-e)\frac{h^\alpha}{\alpha}}$ which imply that $|\lambda_2| < 1$ for $c < d+e$. Therefore, the equilibrium point E_0 is a saddle point. □

Theorem 3.2. *Let the death rate be bigger than the immigration rate of infected individuals (i.e. $d > c$). The coexistence equilibrium point of the system 2.8 is local asymptotically stable if and only if $e > \frac{(c+d)^2h^\alpha}{\alpha - dh^\alpha + ch^\alpha}$.*

Proof. The Jacobian of the system 2.8 at E_1 is

$$J = \begin{pmatrix} e^{\frac{aeh^\alpha}{(c-d)\alpha}} & \frac{(c-d)^2(e^{\frac{aeh^\alpha}{(c-d)\alpha}}-1)}{ae} \\ \dfrac{a(c-d-e)h^\alpha}{(c-d)\alpha} & 1 \end{pmatrix}$$

and its characteristic equation is

$$\lambda^2 + p_1\lambda + p_0 = 0 \tag{3.1}$$

where

$$p_0 = e^{\frac{aeh^\alpha}{(c-d)\alpha}} + \frac{(d-c)(d-c+e)(1-e^{\frac{aeh^\alpha}{(c-d)\alpha}})}{e\alpha}$$

and

$$p_1 = -1 - e^{\frac{aeh^\alpha}{(c-d)\alpha}}.$$

For the stability conditions of E_1, we consider the Jury conditions which are defined as below $1 + p_1 + p_0 > 0$, $1 - p_1 + p_0 > 0$ and $1 - p_0 > 0$.

Putting values $d > c$, the condition $1 + p_1 + p_0 > 0$ yields

$$\frac{(d-c)(1-e^{\frac{aeh^\alpha}{(c-d)\alpha}})(d-c+e)h^\alpha}{e\alpha} > 0$$

and from $1 - p_1 + p_0 > 0$, we get

$$2 + 2e^{\frac{aeh^\alpha}{(c-d)\alpha}} + \frac{(d-c)(1-e^{\frac{aeh^\alpha}{(c-d)\alpha}})(d-c+e)h^\alpha}{e\alpha} > 0$$

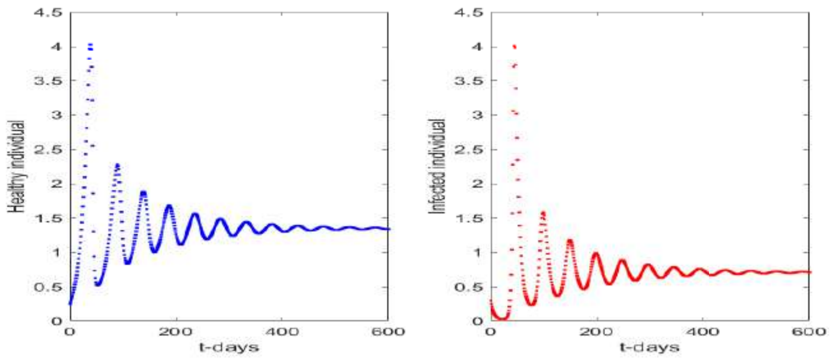

Figure 1. Stable equilibrium point of the system (2.8) for pram-eter values $a = 0.1$, $b = 0.2$, $c = 0.01$, $d = 0.2$, $h = 0.5$, $\alpha = 0.75$, $e = 0.08$ and $(x_0, y_0) = (0.25, 0.3)$.

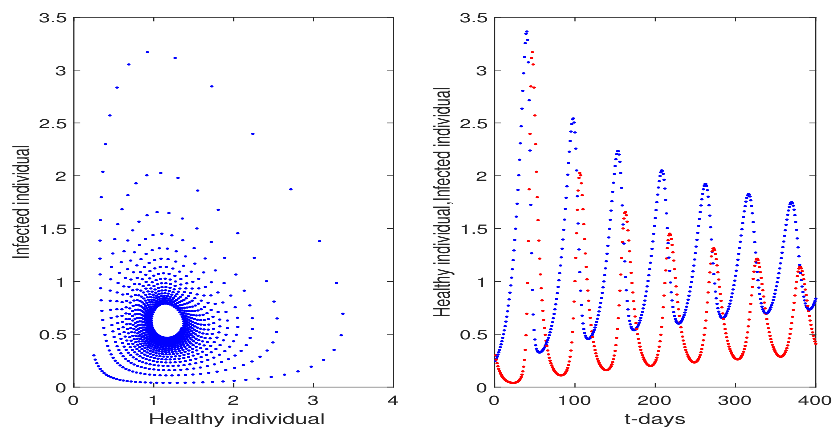

Figure 2. Graph of iteration solution of the system (2.8) for $e = 0.0411632$. The other parameters and initial conditions are the same as Figure (1).

those always hold. Furthermore, if $e > \frac{(c+d)^2 h^\alpha}{\alpha - dh^\alpha + ch^\alpha}$, then we have

$$\frac{(e^{\frac{aeh^\alpha}{(c-d)\alpha}} - 1)(c^2h^\alpha + d^2h^\alpha + deh^\alpha - c(2d+e)h^\alpha - e\alpha)}{e\alpha} > 0$$

which implies that $1 - p_0 > 0$.

Thus, the proof is completed. $\square$

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BILINEAR MULTIPLIERS OF SOME VARIABLE EXPONENT FUNCTION SPACES

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ABSTRACT. Assume that ω_1, ω_2, ω_3 are any weight functions on $\mathbb{R}^n$. Let $m(\xi,\eta)$ be a bounded, measurable function on $\mathbb{R}^n \times \mathbb{R}^n$. One defines

$$B_m(f,g)(x) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \hat{f}(\xi)\,\hat{g}(\eta)\,m(\xi,\eta)\,e^{2\pi i\langle \xi+\eta,x\rangle}d\xi d\eta$$

for all $f,g \in C_c^\infty(\mathbb{R}^n)$. We say that $m(\xi,\eta)$ is a bilinear multiplier on $\mathbb{R}^n$ of type $(WL(p_1(.),q_1(.),r_1,\omega_1;p_2(.),q_2(.),r_2,\omega_2;p_3(.),q_3(.),r_3,\omega_3))$ if B_m is bounded operator from
$W\left(L^{p_1(.),q_1(.)},L^{r_1}_{\omega_1}\right) \times W\left(L^{p_2(.),q_2(.)},L^{r_2}_{\omega_2}\right)$ to $W\left(L^{p_3(.),q_3(.)},L^{r_3}_{\omega_3}\right)$ where $p_i(.),q_i(.),r_i \in \wp_1([0,l]),(i=1,2,3)$. We denote by
$BM(WL(p_1(.),q_1(.),r_1,\omega_1;p_2(.),q_2(.),r_2,\omega_2;p_3(.),q_3(.),r_3,\omega_3))$ the vector space of bilinear multipliers of type
$(WL(p_1(.),q_1(.),r_1,\omega_1;p_2(.),q_2(.),r_2,\omega_2;p_3(.),q_3(.),r_3,\omega_3))$. In this work, we consider some properties of this space and we find examples of these bilinear multipliers.

1. INTRODUCTION

We denote by $C_c^\infty(\mathbb{R}^n)$ the space of infinitely differentiable complex-valued functions with compact support on $\mathbb{R}^n$. For $1 \le p \le \infty$, $L^p(\mathbb{R}^n)$ denotes the usual Lebesque space. Assume that f is a complex valued measurable function on $\mathbb{R}^n$. For $f \in L^1(\mathbb{R}^n)$, the Fourier transform of f is denoted by $\hat{f}$. It is known that $\hat{f}$ is a continuous function on $\mathbb{R}^n$, which vanishes at infinity and it has the inequality $\left\|\hat{f}\right\|_\infty \le \|f\|_1$ [6]. The translation and character operators T_x, M_x are defined by $T_xf(y) = f(y-x)$ and $M_xf(y) = e^{2\pi i\langle x,y\rangle}f(y)$, respectively for $x,y \in \mathbb{R}^n$. For $1 \le p \le \infty$ and $\frac{1}{p}+\frac{1}{p'}=1$, one gets

$$(T_xf)\,\hat{}\,(\xi) = M_{-x}\hat{f}(\xi),\ (M_xf)\,\hat{}\,(\xi) = T_x\hat{f}(\xi).$$

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A continuous function ω satisfying $1 \leq \omega(x)$ and $\omega(x+y) \leq \omega(x)\omega(y)$ for $x, y \in \mathbb{R}^n$ will be called a weight function on $\mathbb{R}^n$. If $\omega_1(x) \leq \omega_2(x)$ for all $x \in \mathbb{R}^n$, we say that $\omega_1 \leq \omega_2$. For $1 \leq p \leq \infty$, we set $L^p_\omega(\mathbb{R}^n) = \{f : f\omega \in L^p(\mathbb{R}^n)\}$ [6]. The distribution function of f is given by

$$\lambda_f(y) = \mu(\{x \in \mathbb{R}^n : |f(x)| > y\}) = \int\limits_{\{x \in \mathbb{R} : |f(x)| > y\}} d\mu(x).$$

The rearrangement function of f is given by

$$f^*(t) = \inf\{y > 0 : \lambda_f(y) \leq t\} = \sup\{y > 0 : \lambda_f(y) > t\}, t \geq 0.$$

Also, the average function of f is defined to be

$$f^{**}(t) = \frac{1}{t}\int\limits_0^t f^*(s)\, ds$$

[8]. Assume that $0 < l \leq \infty$. We use the notations

$$p_- = \inf_{x \in [0,l]} p(x), \qquad p^+ = \sup_{x \in [0,l]} p(x).$$

Also assume that $P_a = \{p : a < p_- \leq p^+ < \infty\}$, $a \in \mathbb{R}$. In this work, we take the special cases of the P_a with $a = 0$ or $a = 1$. The set $\wp[0,l]$ is the family of $p \in L^\infty([0,l])$ such that there exist the limits $p(0) = \lim_{x \to 0} p(x)$, $p(\infty) = \lim_{x \to \infty} p(x)$ and we have

$$|p(x) - p(0)| \leq \frac{C}{\ln \frac{1}{|x|}}, \qquad |x| \leq \frac{1}{2} \quad (C > 0)$$

and

$$|p(x) - p(\infty)| \leq \frac{C}{\ln(e + |x|)}, \qquad (C > 0). \tag{1.1}$$

If $l = \infty$, then it's enough to the inequality (1.1) satisfies. We also denote $\wp_a([0,l]) = \wp([0,l]) \cap P_a([0,l])$ [4]. Let Ω be an open set in $\mathbb{R}^n$. We denote $l = \mu(\Omega)$. Assume that $p, q \in \wp_0([0,l])$. The variable exponent Lorentz space $L^{p(.),q(.)}(\Omega)$ is defined as the set of all (equivalence classes) measurable functions f on Ω such that $\rho_{p,q}(f) < \infty$ [4], where

$$\rho_{p,q}(f) = \int\limits_0^l t^{\frac{q(t)}{p(t)} - 1} (f^*(t))^{q(t)}\, dt. \tag{1.2}$$

We use the notation

$$\|f\|^1_{L^{p(.),q(.)}(\Omega)} = \inf\left\{\lambda > 0 : \rho_{p,q}\left(\frac{f}{\lambda}\right) \leq 1\right\}.$$

The space $L^{p(.),q(.)}(\Omega)$ is a normed vector space with norm

$$\|f\|_{L^{p(.),q(.)}(\Omega)} = \inf\left\{\lambda > 0 : \rho_{p,q}\left(\frac{f}{\lambda}\right) \leq 1\right\},$$

where

$$\rho_{p,q}(f)=\int_0^l t^{\frac{q(t)}{p(t)}-1}\left(f^{**}(t)\right)^{q(t)}dt.$$

Many researchers worked on Wiener amalgam spaces in [3], [5], [7], [12]. Some of characterization of these spaces has been given in [1]. In [11], a kind of generalization of $W\left(L^{p(.),q(.)},L^r_\omega\right)$ has been given. Let $p(.),q(.),r\in\wp_1([0,l])$. The space $\left(L^{p(.),q(.)}(\mathbb{R}^n)\right)_{loc}$ consists classes of measurable functions f on $\mathbb{R}^n$ such that $f\chi_K\in L^{p(.),q(.)}(\mathbb{R}^n)$ for any compact subset $K\subset\mathbb{R}^n$, where χ_K is the characteristic function of K. Fix compact set $Q\subset\mathbb{R}^n$ and $Q^o\neq\emptyset$. The weighted variable exponent Wiener amalgam space $W\left(L^{p(.),q(.)},L^r_\omega\right)$ consists of all elements $f\in\left(L^{p(.),q(.)}(\mathbb{R}^n)\right)_{loc}$ such that $F_f(z)=\|f\chi_{z+Q}\|_{L^{p(.),q(.)}}$ belongs to $L^r_\omega(\mathbb{R}^n)$; the norm of $W\left(L^{p(.),q(.)},L^r_\omega\right)$ is $\|f\|_{W\left(L^{p(.),q(.)},L^r_\omega\right)}=\|F_f\|_{r,\omega}$. Moreover it is known that the space $C_c^\infty(\mathbb{R}^n)$ is dense in weighted variable exponent Wiener amalgam space $W\left(L^{p(.),q(.)},L^r_\omega\right)$ [11].

In this work, we will investigate bilinear multipliers for weighted variable exponent Wiener amalgam space whose local compenent is variable exponet Lorentz space.

2. Main Results

Definition 2.1. Let $p_i(.),q_i(.),r_i\in\wp_1([0,l]),(i=1,2,3)$. Assume that ω_1, ω_2, ω_3 are any weight functions on $\mathbb{R}^n$. Let $m(\xi,\eta)$ be a bounded, measurable function on $\mathbb{R}^n\times\mathbb{R}^n$. One defines

$$B_m(f,g)(x)=\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\hat{f}(\xi)\,\hat{g}(\eta)\,m(\xi,\eta)\,e^{2\pi i\langle\xi+\eta,x\rangle}d\xi d\eta$$

for all $f,g\in C_c^\infty(\mathbb{R}^n)$. Then m is said to be a bilinear multiplier on $\mathbb{R}^n$ of type $(WL(p_1(.),q_1(.),r_1,\omega_1;p_2(.),q_2(.),r_2,\omega_2;p_3(.),q_3(.),r_3,\omega_3))$ (shortly $(WL(p_i(.),q_i(.),r_{i,}\omega_i))$), if there exists $C>0$ such that

$$\|B_m(f,g)\|_{W\left(L^{p_3(.),q_3(.)},L^{r_3}_{\omega_3}\right)}\le C\|f\|_{W\left(L^{p_1(.),q_1(.)},L^{r_1}_{\omega_1}\right)}\|g\|_{W\left(L^{p_2(.),q_2(.)},L^{r_2}_{\omega_2}\right)}$$

for all $f,g\in C_c^\infty(\mathbb{R}^n)$. In other words, B_m extends to a bounded bilinear operator from $W\left(L^{p_1(.),q_1(.)},L^{r_1}_{\omega_1}\right)\times W\left(L^{p_2(.),q_2(.)},L^{r_2}_{\omega_2}\right)$ to $W\left(L^{p_3(.),q_3(.)},L^{r_3}_{\omega_3}\right)$.

Furthermore, one denotes by $BM(WL(p_1(.),q_1(.),r_1,\omega_1;p_2(.),q_2(.),r_2,\omega_2;p_3(.),q_3(.),r_3,\omega_3))$ (shortly $BM(WL(p_i(.),q_i(.),r_{i,}\omega_i))$) the space of all bilinear multipliers of type $(WL(p_1(.),q_1(.),r_1,\omega_1;p_2(.),q_2(.),r_2,\omega_2;p_3(.),q_3(.),r_3,\omega_3))$ and the norm;

$$\|m\|_{WL(p_1(.),q_1(.),r_1,\omega_1;p_2(.),q_2(.),r_2,\omega_2;p_3(.),q_3(.),r_3,\omega_3)}=\|B_m\|,$$

or briefly

$$\|m\|_{WL(p_i(.),q_i(.),r_{i,}\omega_i)}=\|B_m\|.$$

Theorem 2.2. *Let* $\frac{1}{p_3(x)}+\frac{1}{p_3'(x)}=1$, $\frac{1}{q_3(x)}+\frac{1}{q_3'(x)}=1$, $\frac{1}{r_3}+\frac{1}{r_3'}=1$, $q_3(-x)=q_3(x)$, $p_3(-x)=p_3(x)$ *and* ω_3 *be symetric weight function. Then* $m\in BM(WL(p_i(.),q_i(.),r_{i,}\omega_i))$

if and only if there exists $C > 0$ such that

$$\left| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \hat{f}(\xi)\, \hat{g}(\eta)\, \hat{h}(\xi+\eta)\, m(\xi,\eta)\, d\xi d\eta \right|$$

$$\leq C \|f\|_{W\left(L^{p_1(.),q_1(.)},L^{r_1}_{\omega_1}\right)} \|g\|_{W\left(L^{p_2(.),q_2(.)},L^{r_2}_{\omega_2}\right)} \|h\|_{W\left(L^{p_3'(.),q_3'(.)},L^{r_3'}_{\omega_3^{-1}}\right)}$$

for all $f,\ g,\ h \in C_c^{\infty}(\mathbb{R}^n)$.

Proof. Suppose that $m \in BM\left(WL\left(p_i(.), q_i(.), r_i, \omega_i\right)\right)$. From Theorem 2.2 in [9], we have for all $f,\ g,\ h \in C_c^{\infty}(\mathbb{R}^n)$

$$\left| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \hat{f}(\xi)\, \hat{g}(\eta)\, \hat{h}(\xi+\eta)\, m(\xi,\eta)\, d\xi d\eta \right| = \left| \int_{\mathbb{R}^n} h(y)\, \tilde{B}_m(f,g)(y)\, dy \right|$$

$$\leq \int_{\mathbb{R}^n} |h(y)| \left| \tilde{B}_m(f,g)(y) \right| dy, \tag{2.1}$$

where $\tilde{B}_m(f,g)(y) = B_m(f,g)(-y)$. Change the variable $-t = u$. Then we get

$$\left\| \tilde{B}_m(f,g) \right\|_{W\left(L^{p_3(.),q_3(.)},L^{r_3}_{\omega_3}\right)} = \left\| F^{Q}_{\tilde{B}_m(f,g)} \right\|_{r_3,\omega_3} = \left\| \left\| B_m(f,g)(u)\, \chi_{Q+x}(-u) \right\|_{L^{p_3(.),q_3(.)}} \right\|_{r_3,\omega_3}$$

$$= \left\| \left\| B_m(f,g)(u)\, \chi_{-Q-x}(u) \right\|_{L^{p_3(.),q_3(.)}} \right\|_{r_3,\omega_3} = \left\| F^{-Q}_{B_m(f,g)}(-x) \right\|_{r_3,\omega_3}. \tag{2.2}$$

Using the fact that ω_3 is symetric weight function and taking $-x = y$, we find

$$\left\| \tilde{B}_m(f,g) \right\|_{W\left(L^{p_3(.),q_3(.)},L^{r_3}_{\omega_3}\right)} = \left\| F^{-Q}_{B_m(f,g)}(y) \right\|_{r_3,\omega_3}. \tag{2.3}$$

It is known that the definition of $W\left(L^{p_3(.),q_3(.)}, L^{r_3}_{\omega_3}\right)$ is independed of the choice of Q [11]. So there exists $C > 0$ such that

$$\left\| F^{-Q}_{B_m(f,g)}(y) \right\|_{r_3,\omega_3} \leq C_1 \left\| F^{Q}_{B_m(f,g)}(y) \right\|_{r_3,\omega_3}. \tag{2.4}$$

Hence, by (2.3) and (2.4), we have

$$\left\| \tilde{B}_m(f,g) \right\|_{W\left(L^{p_3(.),q_3(.)},L^{r_3}_{\omega_3}\right)} \leq C_1 \left\| F^{Q}_{B_m(f,g)}(y) \right\|_{r_3,\omega_3} = C_1 \left\| B_m(f,g) \right\|_{W\left(L^{p_3(.),q_3(.)},L^{r_3}_{\omega_3}\right)} \tag{2.5}$$

By the assumption $m \in BM\left(WL\left(p_i(.), q_i(.), r_i, \omega_i\right)\right)$, we write $\tilde{B}_m(f,g) \in W\left(L^{p_3(.),q_3(.)}, L^{r_3}_{\omega_3}\right)$. Also again, since $m \in BM\left(WL\left(p_i(.), q_i(.), r_i, \omega_i\right)\right)$, there exists $C_2 > 0$ such that

$$\left\| B_m(f,g) \right\|_{W\left(L^{p_3(.),q_3(.)},L^{r_3}_{\omega_3}\right)} \leq C_2 \|f\|_{W\left(L^{p_1(.),q_1(.)},L^{r_1}_{\omega_1}\right)} \|g\|_{W\left(L^{p_2(.),q_2(.)},L^{r_2}_{\omega_2}\right)}. \tag{2.6}$$

Using the inequalities (2.5) and (2.6) , we get
(2.7)
$$\left\|\tilde{B}_m(f,g)\right\|_{W\left(L^{p_3(\cdot),q_3(\cdot)},L^{r_3}_{\omega_3}\right)} \leq C_1C_2\left\|f\right\|_{W\left(L^{p_1(\cdot),q_1(\cdot)},L^{r_1}_{\omega_1}\right)}\left\|g\right\|_{W\left(L^{p_2(\cdot),q_2(\cdot)},L^{r_2}_{\omega_2}\right)}.$$

From the Theorem 3.4 in [11] and the inequality (2.7), we achieve

$$\left|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\hat{f}(\xi)\,\hat{g}(\eta)\,\hat{h}(\xi+\eta)\,m(\xi,\eta)\,d\xi d\eta\right|$$
$$\leq \left\|\tilde{B}_m(f,g)\right\|_{W\left(L^{p_1(\cdot),q_1(\cdot)},L^{r_1}_{\omega_1}\right)}\left\|h\right\|_{W\left(L^{p_3'(\cdot),q_3'(\cdot)},L^{r_3'}_{\omega_3^{-1}}\right)}$$
$$\leq C_1C_2\left\|f\right\|_{W\left(L^{p_1(\cdot),q_1(\cdot)},L^{r_1}_{\omega_1}\right)}\left\|g\right\|_{W\left(L^{p_2(\cdot),q_2(\cdot)},L^{r_2}_{\omega_2}\right)}\left\|h\right\|_{W\left(L^{p_3'},L^{q_3'}_{\omega_3^{-1}}\right)}.$$

Now assume that there exists a constant $C>0$ such that

$$\left|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\hat{f}(\xi)\,\hat{g}(\eta)\,\hat{h}(\xi+\eta)\,m(\xi,\eta)\,d\xi d\eta\right|$$
$$\leq C\left\|f\right\|_{W\left(L^{p_1(\cdot),q_1(\cdot)},L^{r_1}_{\omega_1}\right)}\left\|g\right\|_{W\left(L^{p_2(\cdot),q_2(\cdot)},L^{r_2}_{\omega_2}\right)}\left\|h\right\|_{W\left(L^{p_3'(\cdot),q_3'(\cdot)},L^{r_3'}_{\omega_3^{-1}}\right)} \tag{2.8}$$

for all $f,\,g,\,h\in C_c^{\infty}(\mathbb{R}^n)$. So by (2.8), we rewrite

$$\left|\int_{\mathbb{R}^n}h(y)\,\tilde{B}_m(f,g)(y)\,dy\right|$$
$$\leq C\left\|f\right\|_{W\left(L^{p_1(\cdot),q_1(\cdot)},L^{r_1}_{\omega_1}\right)}\left\|g\right\|_{W\left(L^{p_2(\cdot),q_2(\cdot)},L^{r_2}_{\omega_2}\right)}\left\|h\right\|_{W\left(L^{p_3'(\cdot),q_3'(\cdot)},L^{r_3'}_{\omega_3^{-1}}\right)}. \tag{2.9}$$

Define a function l from $C_c^{\infty}(\mathbb{R}^n)\subset W\left(L^{p_3'(\cdot),q_3'(\cdot)},L^{r_3'}_{\omega_3^{-1}}\right)$ to $\mathbb{C}$ such that

$$\ell(h)=\int_{\mathbb{R}^n}h(y)\,\tilde{B}_m(f,g)(y)\,dy.$$

By (2.9), we can easily say that ℓ is linear and bounded. Using the density $\overline{C_c^{\infty}(\mathbb{R}^n)}=W\left(L^{p_3'(\cdot),q_3'(\cdot)},L^{r_3'}_{\omega_3^{-1}}\right)$, we find that ℓ extends to a bounded function from $W\left(L^{p_3'(\cdot),q_3'(\cdot)},L^{r_3'}_{\omega_3^{-1}}\right)$ to $\mathbb{C}$. Thus we get $\ell\in\left(W\left(L^{p_3'(\cdot),q_3'(\cdot)},L^{r_3'}_{\omega_3^{-1}}\right)\right)^*=W\left(L^{p_3(\cdot),q_3(\cdot)},L^{r_3}_{\omega_3}\right)$. From the definition of weighted variable exponent Wiener amlgam space, there exists $C_3>0$ such that

$$\left\|B_m(f,g)\right\|_{W\left(L^{p_3(\cdot),q_3(\cdot)},L^{r_3}_{\omega_3}\right)}\leq C_3\left\|\tilde{B}_m(f,g)\right\|_{W\left(L^{p_3(\cdot),q_3(\cdot)},L^{r_3}_{\omega_3}\right)}. \tag{2.10}$$

From (2.9) and (2.10), we achieve

$$\left\|B_m(f,g)\right\|_{W\left(L^{p_3(\cdot),q_3(\cdot)},L^{r_3}_{\omega_3}\right)}\leq C_3\left\|\tilde{B}_m(f,g)\right\|_{W\left(L^{p_3(\cdot),q_3(\cdot)},L^{r_3}_{\omega_3}\right)}$$

$$= C_3 \|\ell\| = C_3 \sup_{\|h\|_{W\left(L^{p_3'(.),q_3'(.)},L^{r_3'}_{\omega_3^{-1}}\right)}\leq 1} \frac{|l(h)|}{\|h\|_{W\left(L^{p_3'(.),q_3'(.)},L^{r_3'}_{\omega_3^{-1}}\right)}}$$

$$\leq C_3 C \|f\|_{W\left(L^{p_1(.),q_1(.)},L^{r_1}_{\omega_1}\right)} \|g\|_{W\left(L^{p_2(.),q_2(.)},L^{r_2}_{\omega_2}\right)}.$$

□

Theorem 2.3. *Let ω_i $(i = 1,2,3,4)$ be weight functions. If $m \in BM\left(WL\left(p_i(.), q_i(.), r_{i,}\omega_i\right)\right)$, then $T_{(\xi_0,\eta_0)}m \in BM\left(WL\left(p_i(.), q_i(.), r_{i,}\omega_i\right)\right)$ and*

$$\left\|T_{(\xi_0,\eta_0)}m\right\|_{WL(p_i(.),q_i(.),r_i,\omega_i)} = \|m\|_{WL(p_i(.),q_i(.),r_i,\omega_i)}$$

for all $(\xi_0,\eta_0) \in \mathbb{R}^{2n}$.

Proof. Take any $f, g \in C_c^\infty(\mathbb{R}^n)$. So, we write

$$\|M_{-\xi_0} f\|_{W\left(L^{p_1(.),q_1(.)},L^{r_1}_{\omega_1}\right)} = \left\|\left\|e^{2\pi i\langle -\xi_0,.\rangle} f(.)\,\chi_{z+Q}(.)\right\|_{L^{p_1(.),q_1(.)}}\right\|_{r_1,\omega_1} = \|f\|_{W\left(L^{p_1(.),q_1(.)},L^{r_1}_{\omega_1}\right)}.$$

Using the same method, we can easily write the equality

$$\|M_{-\eta_0} g\|_{W\left(L^{p_2(.),q_2(.)},L^{r_2}_{\omega_2}\right)} = \|g\|_{W\left(L^{p_2(.),q_2(.)},L^{r_2}_{\omega_2}\right)}.$$

Also we have the following equality by [9].

$$B_{T_{(\xi_0,\eta_0)}m}(f,g)(x) = B_m\left(M_{-\xi_0} f, M_{-\eta_0} g\right)(x).$$

From these equalities and the assumption $m \in BM\left(WL\left(p_i(.), q_i(.), r_{i,}\omega_i\right)\right)$, we get

$$\left\|B_{T_{(\xi_0,\eta_0)}m}(f,g)\right\|_{W\left(L^{p_3(.),q_3(.)},L^{r_3}_{\omega_3}\right)} = \left\|B_m\left(M_{-\xi_0} f, M_{-\eta_0} g\right)\right\|_{W\left(L^{p_3(.),q_3(.)},L^{r_3}_{\omega_3}\right)}$$

$$\leq \|B_m\| \|M_{-\xi_0} f\|_{W\left(L^{p_1(.),q_1(.)},L^{r_1}_{\omega_1}\right)} \|M_{-\eta_0} g\|_{W\left(L^{p_2(.),q_2(.)},L^{r_2}_{\omega_2}\right)}$$

$$= \|B_m\| \|f\|_{W\left(L^{p_1(.),q_1(.)},L^{r_1}_{\omega_1}\right)} \|g\|_{W\left(L^{p_2(.),q_2(.)},L^{r_2}_{\omega_2}\right)}.$$

Thus $T_{(\xi_0,\eta_0)}m \in BM\left(WL\left(p_i(.), q_i(.), r_{i,}\omega_i\right)\right)$. Hence, we obtain

$$\left\|T_{(\xi_0,\eta_0)}m\right\|_{WL(p_i(.),q_i(.),r_i,\omega_i)} = \|m\|_{WL(p_i(.),q_i(.),r_i,\omega_i)}.$$

□

Theorem 2.4. *Assume that $m \in BM\left(WL\left(p_i(.), q_i(.), r_{i,}\omega_i\right)\right)$. Then $\Phi * m \in BM\left(WL\left(p_i(.), q_i(.), r_{i,}\omega_i\right)\right)$ and there exists $C > 0$ such that*

$$\|\Phi * m\|_{WL(p_i(.),q_i(.),r_i,\omega_i)} \leq C \|\Phi\|_1 \|m\|_{WL(p_i(.),q_i(.),r_i,\omega_i)}$$

for all $\Phi \in L^1\left(\mathbb{R}^{2n}\right)$.

Proof. Given $f, g \in C_c^\infty(\mathbb{R}^n)$. By Proposition 2.5 in [2]. The following equality is written

$$B_{\Phi * m}(f,g)(x) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \Phi(u,v)\, B_{T_{(\xi_u,\eta_v)}m}(f,g)(x)\, du dv. \tag{2.11}$$

Since $m \in BM\left(WL\left(p_i(.), q_i(.), r_{i,}\omega_i\right)\right)$, from Theorem 2.1, we write that $T_{(u,v)}m$ in the space $BM\left(WL\left(p_i(.), q_i(.), r_{i,}\omega_i\right)\right)$ and we have

$$\left\|T_{(u,v)}m\right\|_{WL(p_i(.),q_i(.),r_i,\omega_i)} = \|m\|_{WL(p_i(.),q_i(.),r_i,\omega_i)}.$$

By the equation (2.11) and the last equality, there exists $C>0$ such that

$$\|B_{\Phi*m}(f,g)\|_{W\left(L^{p_3(.),q_3(.)},L^{r_3}_{\omega_3}\right)}=\left\|\left\|\left(\int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n}\Phi(u,v)\,B_{T_{(u,v)}m}(f,g)\,dudv\right)\chi_{z+Q}\right\|_{L^{p_3(.),q_3(.)}}\right\|_{r_3,\omega_3}$$

$$\leq C\int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n}|\Phi(u,v)|\left\|\left\|B_{T_{(u,v)}m}(f,g)\,\chi_{z+Q}\right\|_{L^{p_3(.),q_3(.)}}\right\|_{r_3,\omega_3}dudv$$

$$=C\int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n}|\Phi(u,v)|\left\|B_{T_{(u,v)}m}(f,g)\right\|_{W\left(L^{p_3(.),q_3(.)},L^{r_3}_{\omega_3}\right)}dudv$$

$$\leq C\int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n}|\Phi(u,v)|\left\|T_{(u,v)}m\right\|_{WL(p_i(.),q_i(.),r_i,\omega_i)}\|f\|_{W\left(L^{p_1(.),q_1(.)},L^{r_1}_{\omega_1}\right)}\|g\|_{W\left(L^{p_2(.),q_2(.)},L^{r_2}_{\omega_2}\right)}dudv$$

$$=C\|m\|_{WL(p_i(.),q_i(.),r_i,\omega_i)}\|f\|_{W\left(L^{p_1(.),q_1(.)},L^{r_1}_{\omega_1}\right)}\|g\|_{W\left(L^{p_2(.),q_2(.)},L^{r_2}_{\omega_2}\right)}\|\Phi\|_1\,. \tag{2.12}$$

Thus we achieve $\Phi*m\in BM\left(WL\left(p_i(.),q_i(.),r_i,\omega_i\right)\right)$. Finally from (2.12), we obtain that

$$\|\Phi*m\|_{WL(p_i(.),q_i(.),r_i,\omega_i)}\leq C\|\Phi\|_1\|m\|_{WL(p_i(.),q_i(.),r_i,\omega_i)}\,.$$

□

Now we will give examples for bilinear multipliers in the following theorems.

Theorem 2.5. *Let* $\frac{1}{p_3(x)}+\frac{1}{p_3'(x)}=1$, $\frac{1}{q_3(x)}+\frac{1}{q_3'(x)}=1$, $\frac{1}{r_3}+\frac{1}{r_3'}=1$, $q_3(-x)=q_3(x)$, $p_3(-x)=p_3(x)$ *and* ω_3 *be symetric weight function. If* $\Psi_1\in L^1_{\omega_1}(\mathbb{R}^n)$, $\Psi_2\in L^1_{\omega_2}(\mathbb{R}^n)$ *and* $m\in BM\left(WL\left(p_1,q_1,r_1,\omega_1;p_2,q_2,r_2,\omega_2;p_3(.),q_3(.),r_3,\omega_3\right)\right)$, *then* $\hat{\Psi}_1(\xi)\,m(\xi,\eta)\,\hat{\Psi}_2(\eta)\in BM\left(WL\left(p_1,q_1,r_1,\omega_1;p_2,q_2,r_2,\omega_2;p_3(.),q_3(.),r_3,\omega_3\right)\right)$.

Proof. Assume that $f,g,h\in C_c^\infty(\mathbb{R}^n)$. From Theorem 2.10 in [9], [10], we know that

$$\left|\int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n}\hat{f}(\xi)\,\hat{g}(\eta)\,\hat{h}(\xi+\eta)\,m(\xi,\eta)\,d\xi d\eta\right|\leq\int\limits_{\mathbb{R}^n}\left|h(y)\,\widetilde{B}_m(f*\Psi_1,g*\Psi_2)(y)\right|dy.$$

If we use the method in the inequalities (2.1), (2.5) and take the last inequality, then there exists $C>0$ such that

$$\left|\int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n}\hat{f}(\xi)\,\hat{g}(\eta)\,\hat{h}(\xi+\eta)\,\hat{\Psi}_1(\xi)\,m(\xi,\eta)\,\hat{\Psi}_2(\eta)\,d\xi d\eta\right|$$

$$\leq\|h\|_{W\left(L^{p_3'(.),q_3'(.)},L^{r_3'}_{\omega_3^{-1}}\right)}\left\|\tilde{B}_m(f*\Psi_1,g*\Psi_2)\right\|_{W\left(L^{p_3(.),q_3(.)},L^{r_3}_{\omega_3}\right)}$$

$$\leq C\|h\|_{W\left(L^{p_3'(.),q_3'(.)},L^{r_3'}_{\omega_3^{-1}}\right)}\|B_m(f*\Psi_1,g*\Psi_2)\|_{W\left(L^{p_3(.),q_3(.)},L^{r_3}_{\omega_3}\right)}$$

$$\leq C\|h\|_{W\left(L^{p_3'(.),q_3'(.)},L^{r_3'}_{\omega_3^{-1}}\right)}\|B_m\|\|f*\Psi_1\|_{W\left(L^{p_1,q_1},L^{r_1}_{\omega_1}\right)}\|g*\Psi_2\|_{W\left(L^{p_2,q_2},L^{r_2}_{\omega_2}\right)}$$

$$\leq C\|B_m\|\|\Psi_1\|_{1,\omega_1}\|\Psi_2\|_{1,\omega_2}\|f\|_{W\left(L^{p_1,q_1},L^{r_1}_{\omega_1}\right)}\|g\|_{W\left(L^{p_2,q_2},L^{r_2}_{\omega_2}\right)}\|h\|_{W\left(L^{p_3'(.),q_3'(.)},L^{r_3'}_{\omega_3^{-1}}\right)}\,.$$

Finally, we conclude that
$\hat{\Psi}_1(\xi)\, m(\xi,\eta)\, \hat{\Psi}_2(\eta) \in BM\left(WL\left(p_1, q_1, r_1,\omega_1; p_2, q_2, r_2, \omega_2; p_3(.), q_3(.), r_3, \omega_3\right)\right)$ by Theorem 2.1. □

Theorem 2.6. *Suppose that* $m \in BM\left(WL\left(p_i(.), q_i(.), r_i, \omega_i\right)\right)$. *If* $Q_1,\ Q_2 \subset \mathbb{R}^n$ *are bounded sets, then*

$$h(\xi,\eta) = \frac{1}{\mu(Q_1 \times Q_2)} \int\int_{Q_1\times Q_2} m(\xi+u, \eta+v)\, dudv \in BM\left(WL\left(p_i(.), q_i(.), r_i, \omega_i\right)\right).$$

Proof. Take $f, g \in C_c^{\infty}(\mathbb{R}^n)$. The following equality is known by Theorem 2.9 in [9], [10].

$$B_h(f,g)(x) = \frac{1}{\mu(Q_1\times Q_2)} \int\int_{Q_1\times Q_2} B_{T_{(-u,-v)}m}(f,g)(x)\, dudv$$

By Theorem 2.3, we achieve

$$\|B_h(f,g)\|_{W\left(L^{p_3(x)}, L^{q_3}_{\omega_3}\right)} = \left\| \frac{1}{\mu(Q_1\times Q_2)} \int\int_{Q_1\times Q_2} B_{T_{(-u,-v)}m}(f,g)\, dudv \right\|_{W\left(L^{p_3(.),q_3(.)}, L^{r_3}_{\omega_3}\right)}$$

$$\le \frac{1}{\mu(Q_1\times Q_2)} \left\| \int\int_{Q_1\times Q_2} \left\|B_{T_{(-u,-v)}m}(f,g)\right\|_{L^{p_3(.),q_3(.)}} \right\|_{r_3,\omega_3} dudv$$

$$\le \frac{1}{\mu(Q_1\times Q_2)} \int\int_{Q_1\times Q_2} \left\| \left\|B_{T_{(-u,-v)}m}(f,g)\right\|_{L^{p_3(.),q_3(.)}} \right\|_{r_3,\omega_3} dudv$$

$$= \frac{1}{\mu(Q_1\times Q_2)} \int\int_{Q_1\times Q_2} \left\|B_{T_{(-u,-v)m}}(f,g)\right\|_{W\left(L^{p_3(.),q_3(.)}, L^{r_3}_{\omega_3}\right)} dudv$$

$$\le \frac{1}{\mu(Q_1\times Q_2)} \int\int_{Q_1\times Q_2} \left\|T_{(-u,-v)}m\right\|_{WL(p_i(.),q_i(.),r_i,\omega_i)} \|f\|_{W\left(L^{p_1(.),q_1(.)}, L^{r_1}_{\omega_1}\right)} \|g\|_{W\left(L^{p_2(x)}, L^{q_2}_{\omega 2}\right)} dudv$$

$$= \|m\|_{WL(p_i(.),q_i(.),r_i,\omega_i)} \|f\|_{W\left(L^{p_1(.),q_1(.)}, L^{r_1}_{\omega_1}\right)} \|g\|_{W\left(L^{p_2(x)}, L^{q_2}_{\omega 2}\right)}.$$

That means $h(\xi,\eta) \in m \in BM\left(WL\left(p_i(.), q_i(.), r_i, \omega_i\right)\right)$. □

The following theorem gives us inclusion relation between bilinear multipliers spaces under some conditions.

Theorem 2.7. *Let* $s(0) \ge o(0),\ p(0) \ge t(0),\ q(0) \ge u(0),\ v(0) \le o(0), k(0) \le t(0), l(0) \le u(0), v(0) \ge r(0), k(0) \ge p(0), l(0) \ge m(0),\ s \le r,\ q \le m,\ p \le n,\ \omega_3 \le \upsilon_3,\ \upsilon_2 \le \omega_2,\ \upsilon_1 \le \omega_1$. *Then*

$$BM[WL(n(.), k(.), n, \omega_1; m(.), l(.), m, \omega_2; s(.), o(.), s, \omega_3)]$$
$$\subset BM[W(p(.), t(.), p, \upsilon_1; q(.), u(.), q, \upsilon_2; r(.), v(.), r, \upsilon_3)].$$

Proof. One takes any $m \in BM[WL(n(.), k(.), n, \omega_1; m(.), l(.), m, \omega_2; s(.), o(.), s, \omega_3)]$. So, there exists $C_1 > 0$ such that

$$\|B_m(f,g)\|_{W\left(L^{s(.),o(.)}, L^{s}_{\omega_3}\right)} \le C_1 \|f\|_{W\left(L^{n(.),k(.)}, L^{n}_{\omega_1}\right)} \|g\|_{W\left(L^{m(.),l(.)}, L^{m}_{\omega 2}\right)}. \tag{2.13}$$

Moreover by the Theorem 3.5 in [11], we know that $W\left(L^{s(.),o(.)}, L^{s}_{\omega_3}\right) \subset W\left(L^{r(.),v(.)}, L^{r}_{\upsilon_3}\right)$, $W\left(L^{p(.),t(.)}, L^{p}_{\upsilon_1}\right) \subset W\left(L^{n(.),k(.)}, L^{n}_{\omega_1}\right)$ and $W\left(L^{q(.),u(.)}, L^{q}_{\upsilon_2}\right) \subset W\left(L^{m(.),l(.)}, L^{m}_{\omega 2}\right)$. Thus, there exist $C_2 > 0$, $C_3 > 0$ and $C_4 > 0$ such that

$$\|B_m(f,g)\|_{W\left(L^{r(.),v(.)},L^{r}_{\upsilon_3}\right)} \leq C_2 \|B_m(f,g)\|_{W\left(L^{s(.),o(.)},L^{s}_{\omega_3}\right)}, \tag{2.14}$$

$$\|f\|_{W\left(L^{n(.),k(.)},L^{n}_{\omega_1}\right)} \leq C_3 \|f\|_{W\left(L^{p(.),t(.)},L^{p}_{\upsilon_1}\right)}, \tag{2.15}$$

and

$$\|g\|_{W\left(L^{m(.),l(.)},L^{m}_{\omega 2}\right)} \leq C_4 \|g\|_{W\left(L^{q(.),u(.)},L^{q}_{\upsilon_2}\right)}. \tag{2.16}$$

If we combine (2.13), (2.14), (2.15) and (2.16), we have

$$\|B_m(f,g)\|_{W\left(L^{r(.),v(.)},L^{r}_{\upsilon_3}\right)} \leq C_1C_2C_3C_4 \|f\|_{W\left(L^{p(.),t(.)},L^{p}_{\upsilon_1}\right)} \|g\|_{W\left(L^{q(.),u(.)},L^{q}_{\upsilon_2}\right)}.$$

Finally we obtain that $m \in BM[W(p(.), t(.), p, \upsilon_1; q(.), u(.), q, \upsilon_2; r(.), v(.), r, \upsilon_3)]$ and we conclude $BM[WL(n(.), k(.), n, \omega_1; m(.), l(.), m, \omega_2; s(.), o(.), s, \omega_3)] \subset BM[W(p(.), t(.), p, \upsilon_1; q(.), u(.), q, \upsilon_2; r(.), v(.), r, \upsilon_3)]$. □

3. Conclusion

In the literature, bilinear multipliers theory is considered for Lebesgue spaces, weighted Lebesgue spaces, variable Lebesgue spaces, weighted Wiener amalgam spaces and variable exponent Wiener amalgam spaces etc. [2] [9], [10]. In these spaces, some properties of the spaces of bilinear multipliers are investigated and some examples of these bilinear multipliers are given. In this paper, we study properties of bilinear multipliers for the weighted variable exponent Wiener amalgam space whose local compenent is variable exponent Lorentz space and we give examples of these bilinear multipliers.

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SIEVE METHOD TOWARD THE SOLUTION OF THE GOLDBACH CONJECTURE

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ABSTRACT
Goldbach's conjecture is one of the oldest problems in the Number theory field. This problem states that for any even number greater than 2, at least one prime binary can be found in which the sum of primes is equal to the initial even number. Here in this paper, a potential solution is proposed which employs the sieve method to eliminate unacceptable binaries. And as the result, it proves that for even numbers greater than 2810, at least one pair of primes remains in which the sum of prime numbers is equal to the initial even number.

1. INTRODUCTION

The Goldbach conjecture was first expressed by Christian Goldbach via a letter to Leonhard Euler in 1742 [1]. Since then, this problem has occupied the mind of many researchers in Number theory. This problem states that for every even number greater than 2, at least one pair of primary numbers can be found in which the sum of them is equal to the initial even number.

From this theorem, another immediate theorem can be extracted that states every odd integer greater than 5 can be written as a sum of 3 prime numbers. A potential proof for the later problem is being proposed by Harald Helfgott in 2013 [2]–[4]. Also, the binary Goldbach Conjecture is computationally checked for numbers up to 4×10^{14} [5], however, this problem officially is being considered not proven.

Here in this paper, a potential solution based on the sieve method for the binary Goldbach problem is proposed. the sieve strategy is being utilised to check all the potential pairs for a given

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2. METHODOLOGY

The Goldbach theorem states that for every even number such as 2A, there is at least one pair of numbers as p and q which p and q are both are prime numbers and

$$2A=p+q$$

Firstly, it is aimed to calculate the numbers of pairs as (a, b) which a+b=2A. For this reason, the combination formula is being used:

$$\frac{2A+2-1}{2-1}=2A+1$$

2A+1 is the number of possible pairs that can be written for 2A. However, among these answers, there are two answers such as (0, 2A) and (2A, 0) which are not acceptable pairs. Hence the whole number is subtracted by 2 to eliminate these two unacceptable answers:

$$2A+1-2=2A-1$$

If the pair answers are being written sequentially they would sort out as:

$$(1,2A-1),(2,2A-2),(3,2A-3),\ldots.,(2A-1,1)$$

Among these answers there are pairs as (a, b) and (b, a) which have been counted twice, however, in this problem, sortation is not important and to resolve this issue every twice counted answer should be eliminated.

Before dividing the final number by 2, it should be considered that there is one answer as (A, A) which is already counted once. So, this answer should be separated the rest can be halved:

$$2A-1\rightarrow(2A-2)+1\rightarrow$$

$$\frac{2A-2}{2}+1=A$$

Till here "A" number of the paired answers as (a, b) which "a" and "b" are not zero and also are not sorted, is isolated. It is possible to write these pairs as two (one ascending and one descending) sequences:

$$\begin{matrix}1\\+\\2A-1\end{matrix},\begin{matrix}2\\+\\2A-2\end{matrix},\begin{matrix}3\\+\\2A-3\end{matrix},\ldots,\begin{matrix}A\\+\\A\end{matrix}$$

These two sequences as can be observed are sorted with an equal arithmetic progression of 1. What is necessary next is to prove another theorem that is being used throughout the process:

Theorem: In each D sequence of numbers with the arithmetic progression of d, if the greatest common divisor of GCD(d, p)=1, then there is at least one number in this sequence that is divisible by p.

Proof: If the sequence is sorted, this series would be achieved:

$$c + d, \qquad c + 2d, \qquad c + 3d, \qquad \ldots \qquad c + (p - 1)d, \qquad c + pd$$

The remaining division of these numbers by "p" has to be all a different amounts. This is a fact, because if there are at least two numbers with the same remainings after dividing by p (let's say "c+md" and "c+nd"), then:

$$-\frac{\begin{array}{c}\boldsymbol{c + nd = pk + r} \\ \boldsymbol{c + md = pk' + r}\end{array}}{\boldsymbol{(n - m)d = p(k - k')}}$$

In the resulting equation, p and d share no common factor, so it has to be this:

$$CGD(p, n - m) \neq 1$$

But it is known that p is a prime, and (n-m)<p. So this equation can not be true as there is no factor of p on the left side of the equation.

End of proof.

So, until here binary pairs are being isolated and sequenced with the first pair of (1, 2A-1) and the last pair of (A, A). The first pair here is always unacceptable since it includes 1, and 1 is not a prime number. So this pair is eliminated and then so this sequence results into:

$$(2, 2A - 2), (3, 2A - 3), (4, 2A - 4) \ \ldots., (A, A)$$

In this sequence, the pairs that contain even numbers are not acceptable, as they have an element of 2 and so they cannot be a prime number (except 2 itself, but it is clear that 2 is always paired with another even number as all even numbers are paired and also all odd numbers are paired as well. This fact is known because if an odd number pair with an even number such as:

$$(2k, 2k' + 1)$$

The sum will result in an odd number:

$$2k + 2k' = 2(k + k') + 1$$

Which cannot be correct as the starting number was an even number.)

Hence, the result is being divided by two to get the number of odd pairs:

$$\frac{A-1}{2}$$

It is possible for (A-1) to be an odd number and not be dividable by 2. This happens when A is an even number and sequencing pairs will result in an odd number of pairs. In this situation the number of the pairs would be calculated as:

$$\frac{A-2}{2} = \frac{A}{2} - 1$$

But to have a consistent solution for both odd and even "A"s, the absolute of this division is calculated:

$$\left[(A-1) \times \frac{1}{2}\right]$$

The remaining pairs are pairs with odd numbers and with an arithmetic progression of 2 (in fact, the initial sequence here is being divided into two sequences of 2k+0 and 2k+1, in which the 2k+0 sequence is being eliminated).

This action resulted in paired sequences with odd numbers and a progression space of 2. Based on the discussed theorem, for the ascending sequence (sequence of the first numbers in each pair), in every 3 numbers 1 is dividable by 3, and in descending sequence (sequence of the second numbers in each pair), in every 3 sequential numbers 1 has a factor of 3. So for the total pairs, in every 3 pairs, the maximum of 2 pairs is not acceptable as they have a factor of 3 in at least one of the numbers (the term "maximum" is being used because there can be a situation where there is a sequence with both numbers in one pair are dividable by 3 and this happens when A has a factor of 3. In this situation, in every 3 pairs, one is unacceptable. However, since it is aimed to find the minimum number of answers, this extra elimination to simplify the solution does not falsify the general trend.)

So the result is divided by 3:

$$\left[(A-1) \times \frac{1}{2}\right] \times \frac{1}{3}$$

Utilising this operation, the current sequence of pairs is being divided into 2 sequences of:

$$\begin{array}{ccc} 3k+r & 3k+r' & 3k \\ + & + & + \\ 3k' & 3k'+r & 3k'+r \end{array}$$

Which 2 sequences (that one of the numbers has a factor of 3) are being eliminated and one sequence (in which both parameters in each pair are odd) is being kept.

As a potential result of this division, some remaining out of this operation can be produced. However, since the minimum number of answers is being aimed, again absolute of this result will be calculated.

$$\left[\left[(A-1)\times\frac{1}{2}\right]\times\frac{1}{3}\right]$$

Till here, pairs that are odd and are not dividable by 3 are being sifted. Now the numbers with the factor of 5 should be calculated and removed.

As per proved theorem, in every 5 sequential arguments, one in the ascending and one in the descending sequences has a factor of 5 and is needed to be eliminated. Assuming that a maximum of 2 pairs are not acceptable and has to be eliminated, 3 pairs will be remaining in our sequences:

$$\left[\left[(A-1)\times\frac{1}{2}\right]\times\frac{1}{3}\right]\times\frac{3}{5}$$

Again, the minimum amount is intended, so the absolute of this argument is calculated:

$$\left[\left[\left[(A-1)\times\frac{1}{2}\right]\times\frac{1}{3}\right]\times\frac{3}{5}\right]$$

Until this stage, 3 consecutively ordered sequences are being calculated that have an arithmetic progression of 30 (2x3x5). For these sequences, in each 7 pairs maximum of 2 has to be eliminated and a minimum of 5 would be acceptable (as a result of having the factor of 7):

$$\left[\left[\left[(A-1)\times\frac{1}{2}\right]\times\frac{1}{3}\right]\times\frac{3}{5}\right]\times\frac{5}{7}$$

And the remaining can be ignored:

$$\left[\left[\left[\left[(A-1)\times\frac{1}{2}\right]\times\frac{1}{3}\right]\times\frac{3}{5}\right]\times\frac{5}{7}\right]$$

If this elimination process is continued till the last prime number before $\sqrt{2A}$, then the remaining pairs are the prime numbers that the sum results in 2A. This is a fact since all numbers with factors less than $\sqrt{2A}$ are being eliminated, and if there is a compound number in between, it has to have a factor of a prime number that is greater than $\sqrt{2A}$. If that prime number is labelled as "q", then the minimum amount for compounded number would be "q^2" which is bigger than 2A:

$$q > \sqrt{2A} \rightarrow q^2 > 2A$$

Hence, if this elimination for all the prime numbers till the $\sqrt{2A}$ is continued (here the last prime number is being labelled as "p") the result is:

$$\left\lceil\left\lceil\left\lceil\left\lceil(A-1)\times\frac{1}{2}\right\rceil\times\frac{1}{3}\right\rceil\times\frac{3}{5}\right\rceil\times\frac{5}{7}\right\rceil\times\ldots\times\frac{p-2}{p}\right\rceil$$

Now it is needed to prove for this argument to be bigger or equal to 1. It means that if it is proven that this argument for a given number of (2A) is greater or equal to 1, then at least one pair of prime numbers with their sum of (2A)does exist:

$$\left\lceil\left\lceil\left\lceil\left\lceil(A-1)\times\frac{1}{2}\right\rceil\times\frac{1}{3}\right\rceil\times\frac{3}{5}\right\rceil\times\frac{5}{7}\right\rceil\times\ldots\times\frac{p-2}{p}\right\rceil \geq 1$$

The brackets can be removed and instead, for each bracket, the amount is subtracted by 1 (this operation is legal as the minimum amount is aimed):

$$\left(\left(\left((A-1)\times\frac{1}{2}-1\right)\times\frac{1}{3}-1\right)\times\frac{3}{5}-1\right)\times\frac{5}{7}-1\times\ldots\times\frac{p-2}{p}-1\geq 1$$

The above unequal equation is expanded which results to:

$$\left(A\times\frac{1}{2}\times\frac{1}{3}\times\frac{3}{5}\times\ldots\times\frac{p-2}{p}\right)-\left(\frac{1}{2}\times\frac{1}{3}\times\frac{3}{5}\times\ldots\times\frac{p-2}{p}\right)-\left(\frac{1}{3}\times\frac{3}{5}\times\ldots\times\frac{p-2}{p}\right)-\cdots-\left(\frac{p-2}{p}\right)-1\geq 1$$

İt can be seen that in the above argument all the parentheses except the first parentheses are less than 1, as they are made up of multiplication of numbers less than 1. So again it is legal to assume them as 1 and so it results to:

$$\left(A\times\frac{1}{2}\times\frac{1}{3}\times\frac{3}{5}\times\ldots\times\frac{p-2}{p}\right)-1-1-\cdots-1-1\geq 1$$

The number of these subtracted "ones" is the number of prime numbers before $\sqrt{2A}$, and it is known that the amount is always less than $\frac{\sqrt{2A}}{2}$ (since the number of prime numbers is always less than the number of odd numbers and $\frac{\sqrt{2A}}{2}$ gives us the number of odd numbers before $\sqrt{2A}$). So this argument can be expressed instead:

$$\left(A \times \frac{1}{2} \times \frac{1}{3} \times \frac{3}{5} \times \ldots \times \frac{p-2}{p}\right) - \frac{\sqrt{2A}}{2} \geq 1$$

Some small displacement in this unequal equation is implemented for simplification:

$$\frac{A}{2} \times \frac{3}{3} \times \frac{5}{5} \times \frac{9}{7} \times \ldots \times \frac{p-2}{p'} \times \frac{1}{p} \geq 1 + \frac{\sqrt{2A}}{2}$$

On the left side of the above equation all the fractions except the first and last fraction, are greater than 1. And also, from the previous assumption, it is obvious that p is the latest prime before $\sqrt{2A}$. Hence always $p < \sqrt{2A}$.

The minimum amount is aimed and it is legal to replace $\frac{1}{p}$ by $\frac{1}{\sqrt{2A}}$:

$$\frac{A}{2} \times \frac{3}{3} \times \frac{5}{5} \times \frac{9}{7} \times \ldots \times \frac{p-2}{p'} \times \frac{1}{\sqrt{2A}} \geq 1 + \frac{\sqrt{2A}}{2} \rightarrow$$

$$\left(\frac{3}{3} \times \frac{5}{5} \times \frac{9}{7} \times \ldots \times \frac{p-2}{p'}\right) \times \frac{2A}{4} \times \frac{1}{\sqrt{2A}} \geq 1 + \frac{\sqrt{2A}}{2} \rightarrow$$

$$\left(\frac{3}{3} \times \frac{5}{5} \times \frac{9}{7} \times \ldots \times \frac{p-2}{p'}\right) \times \frac{\sqrt{2A}}{4} \geq 1 + \frac{\sqrt{2A}}{2} \rightarrow$$

$$\left(\frac{3}{3} \times \frac{5}{5} \times \frac{9}{7} \times \ldots \times \frac{p-2}{p'}\right) \geq \frac{(1 + \frac{\sqrt{2A}}{2})}{\frac{\sqrt{2A}}{4}} \rightarrow$$

$$\left(\frac{3}{3} \times \frac{5}{5} \times \frac{9}{7} \times \ldots \times \frac{p-2}{p'}\right) \geq \frac{4}{\sqrt{2A}} + \frac{1}{2} \rightarrow$$

The left-hand side of the last equation is increasing, and the right-hand side is decreasing arguments with a maximum value of 2.5. At this point, the only remaining requirement is to find the smallest even number for which the left argument becomes greater than the right amount and after that number, all greater numbers will naturally meet this condition:

$$p(2810) = \frac{3}{3} \times \frac{5}{5} \times \frac{9}{7} \times \frac{11}{11} \times \frac{15}{13} \times \frac{17}{17} \times \frac{21}{19} \times \frac{27}{23} \times \frac{29}{29} \times \frac{35}{31} \times \frac{39}{37} \times \frac{41}{41} \times \frac{45}{43} \times \frac{51}{47} \approx 2.6 > 2.5$$

End of the solution.

Here through the sieve method, we proved that for all even numbers greater than 2810, there is at least one pair of primes in which the sum of primes results in the initial even number.

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ON SPHERICAL INVERSIONS IN THREE DIMENSIONAL DD - SPACE

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ABSTRACT

In this paper, we introduce inversion with respect to a sphere in Disdyakis Dodecahedron space and we study on general properties and basic concepts of this transformation. Additionally we investigate some properties such as cross ratio and harmonic conjugates and inverses of lines, planes and Disdyakis Dodecahedron spheres in $\mathbb{R}^3_{DD}$ under an inversion with respect to a Disdyakis Dodecahedron sphere.

1. INTRODUCTION

Inversion reveals difficult questions and many challenging problems thus it is one of the most interesting transformation in the plane. Also in geometry many problems become much manageable when an inversion is applied. Probably the first to reveal this transformation was Apollonious of Perga as it has been stated in [1]. He stated this transformation in his last book Plane Loci. Jakob Steiner investigated inversion systematically in 1820s. Inversion would be used to examine some problems and theorems in geometry as Pappus chain theorem, Feuerbach's theorem, Ptolemy's theorem, Steiner porism, the problem of Apollonius, etc. [2]. Inversion is classically determined with respect to a circle, but some authors studied on different inversion maps by using other objects, see [3, 4, 5, 6, 7, 8] and some authors defined new inversion maps by using different distance functions, see [9, 10, 11]. Furthermore inversion has been studied in higher dimensions in Euclidean and non-Euclidean spaces, see [12, 13, 14, 15, 16].

As it has stated in [17] Minkowski geometry is a non-Euclidean geometry in a finite number of dimensions and only because the distance is not uniform in all directions it is a non-Euclidean geometry. The unit ball of a Minkowski geometry is a general symmetric convex set. Metric geometry has been studied and improved by some mathematicians and throughout these studies and studies on polyhedra it has seen that unit balls of some Minkowski geometries are convex solids, some of these studies are [18, 19, 20, 21]. In [22, 23, 24, 25, 26, 27, 28, 29] some metrics are given which are induced by some convex polyhedra such that their unit spheres are corresponding convex solids. Since the only difference of a Minkowski geometry and the Euclidean geometry is the distance, it is interesting to study on the problems of the Euclidean geometry that include the distance concept in different Minkowski geometries. By these motivations in this study first we define the inversion with respect to a sphere in Disdyakis Dodecahedron space. Then we investigate general properties and basic concepts of this new inversion. Furthermore we give some properties related with spherical inversion in Disdyakis dodecahedron space such as cross-ratio and harmonic conjugates.

2. BASIC DEFINITIONS AND THEOREMS

2.1 Some Basics of Disdyakis Dodecahedron Space

Here we give some basic definitions of tetrakis disdyakis dodecahedron space, for more detail see [23]. Geometrical construction of Disdyakis Dodecahedron space $\mathbb{R}^3_{DD}$ is similar to the well-known Euclidean space $\mathbb{R}^3$. Set of points and collection of lines are the same, the angles are measured by the same way. The only difference is the definition of the distance. Disdyakis Dodecahedron metric in $\mathbb{R}^3$ is defined by using the distance function

$$d_{DD}(P_1,P_2) = max\{|x_1-x_2|,|y_1-y_2|,|z_1-z_2|\} + (\sqrt{2}-1)min\{|x_1-x_2|+|y_1-y_2|,|x_1-x_2|+|z_1-z_2|,|y_1-y_2|+|z_1-z_2|\} + (\sqrt{3}-2\sqrt{2}+1)min\{|x_1-x_2|,|y_1-y_2|,|z_1-z_2|\} \quad (1)$$

or

$$d_{DD}(P_1,P_2) = max\{|x_1-x_2|,|y_1-y_2|,|z_1-z_2|\} + (\sqrt{2}-1)mid\{|x_1-x_2|,|y_1-y_2|,|z_1-z_2|\} + (\sqrt{3}-\sqrt{2})min\{|x_1-x_2|,|y_1-y_2|,|z_1-z_2|\}$$

where $P_1=(x_1,y_1,z_1)$, $P_2=(x_2,y_2,z_2)\in\mathbb{R}^3$. Thus the distance is sum of maximum, $(\sqrt{2}-1)$ times of middle of $\{|x_1-x_2|,|y_1-y_2|,|z_1-z_2|\}$ and $(\sqrt{3}-\sqrt{2})$ times of minimum of $\{|x_1-x_2|,|y_1-y_2|,|z_1-z_2|\}$. The unit ball in $\mathbb{R}^3_{DD}$ is the set of all points (x,y,z) satisfing the equation

$$max\{|x|,|y|,|z|\} + (\sqrt{2}-1)\min\{|x|+|y|,|x|+|z|,|y|+|z|\} + (\sqrt{3}-2\sqrt{2}+1)min\{|x|,|y|,|z|\} = 1$$

which is a Disdyakis Dodecahedron.

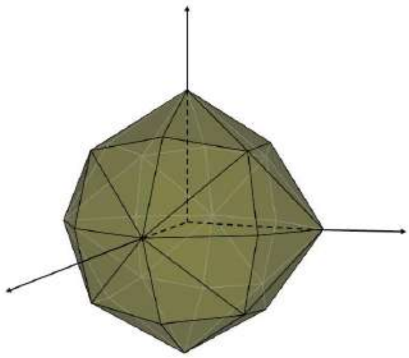

FIGURE 1. Unit ball in $\mathbb{R}^3_{DD}$

2.2 Preliminaries about Inversions in Disdyakis Dodecahedron Space

The circle inversion is one of the most important and interesting transformations in the geometry. The inversion of a point P in a given circle $\mathcal{D}$ is a point P′ taken from the ray OP such that OP $\cdot$OP′ = r^2, where r is the radius of $\mathcal{D}$.

Definition 2.4 Let $\mathcal{D}$ be a DD-sphere centered at the point O with radius r in $\mathbb{R}^3_{DD}$, and P_∞ be the ideal point adjoined to the Disdyakis Dodecahedron space. In $\mathbb{R}^3_{DD}$ the DD-spherical inversion with respect to $\mathcal{D}$ is the transformation

$$I_{\mathcal{D}(O,r)}: \mathbb{R}^3_{DD} \cup \{P_\infty\} \to \mathbb{R}^3_{DD} \cup \{P_\infty\}$$

defined by $I_{D(O,r)}(O) = P_\infty$, $I_{D(O,r)}(P_\infty) = O$, $I_{\mathcal{D}(O,r)}(P) = P'$ for $P \neq 0$ and P' lies on the ray $\overrightarrow{OP}$ and

$$d_{DD}(O,P).d_{DD}(O,P') = r^2 \qquad (2)$$

D is called the sphere of the inversion, O is called the center of inversion, the point P' is called the inverse of the point P with respect to the sphere D.

In Euclidean space, an inversion shifts the points outside to the inside of the sphere and vice versa. Now the following theorem states that this property is valid in the Disdyakis Dodecahedron space.

Lemma 2.5 Let D be the TH-sphere with center O and the radius r. If the point P is in the interior of D, the point P' is exterior to D, and viceversa.

Proof. Let us consider the inversion $I_{D(O,r)}$ with respect to the sphere D with center O and the radius r and the point P which is in the interior of D. Thus, $d_{DD}(O,P) < r$. Since $P' = I_{D(O,r)}(P)$ and by Eq. (3), $r^2 = d_{DD}(O,P).d_{DD}(O,P') < r.d_{DD}(O,P')$ then $d_{DD}(O,P') > r$. So the point P' is in the exterior of D.

Corollary 2.6 Under a spherical inversion $I_{D(O,r)}$ in $\mathbb{R}^3_{DD}$, $\mathcal{T}$ itself is left pointwise fixed.

Theorem 2.7 If P and P' is a pair of inverse points with respect to the tetrakis hexahedron spherical inversion $I_{D(O,r)}$ with center $O = (0,0,0)$ and radius r then

$$P' = \omega P \qquad (3)$$

where $\omega = r^2/\left(d_{DD}(O,P)\right)^2$

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$$P' = \omega P \qquad (3)$$

where $\omega = r^2/\left(d_{DD}(O,P)\right)^2$

Proof. Let $P = (x,y,z)$ and $P' = (x',y',z')$ be inverse pair with respect to the disdyakis dodecahedron spherical inversion $I_{D(O,r)}$ with center $O = (0,0,0)$ and radius r. Since the points P and P' are on the ray emanating from O

$$\overrightarrow{OP'} = \overrightarrow{\omega OP} \ , \omega \in \mathbb{R}^+$$

Thus $(x',y',z') = (\omega x, \omega y, \omega z)$. By the equation (2) we get that $\omega = r^2/\left(d_{DD}(O,P)\right)^2$ and by substituting the resulting value of ω the required result is obtained.

Note that since P and P' is a pair of inverse points with respect to the disdyakis dodecahedron spherical inversion $I_{D(O,r)}$ with center $O = (0,0,0)$ and radius r, the coordinates of P would be obtained by the coordinates of P' by the same way in the Theorem 2.7. Thus $P = \omega P'$ where $\omega = r^2/\left(d_{DD}(O,P')\right)^2$

Corollary 2.8 Let $P = (x, y, z)$ and $P' = (x', y', z')$ is an inverse pair under the disdyakis dodecahedron spherical inversion $I_{\mathcal{D}(O,r)}$ with center $O = (x_0, y_0, z_0)$ and radius r then

$$P' - O = \omega(P - O) \tag{4}$$

where $\omega = r^2/\left(d_{DD}(O,P)\right)^2$.

Proof. It is easy to see that translation preserves distances in $\mathbb{R}^3_{DD}$. Thus by translating $(0,0,0)$ to (x_0, y_0, z_0) in $\mathbb{R}^3_{DD}$ values of x', y', z' would easily be obtained as required.

Theorem 2.9 Let O, P and Q be any three collinear distinct points in $\mathbb{R}^3_{DD}$. If P, P' and Q, Q' are inverse pairs with respect to the disdyakis dodecahedron spherical inversion $I_{\mathcal{D}(O,r)}$ then

$$d_{DD}(P', Q') = \frac{r^2 .\ d_{DD}(P,Q)}{d_{DD}(O,P) .\ d_{DD}(O,Q)} \tag{5}$$

Proof. Let $I_{(O,r)}$ be the spherical inversion with center O and radius r in $\mathbb{R}^3_{DD}$. If P, P' and Q, Q' are inverse pairs with respect to $I_{\mathcal{D}(O,r)}$ then by equation (1), $d_{DD}(O,P).d_{DD}(O,P') = r^2 = d_{DD}(O,Q).\ d_{DD}(O,Q')$. Since O, P and Q are collinear points and ratios of Euclidean and Disdyakis Dodecahedron distances along a line are the same,

$$\begin{aligned} d_{DD}(P', Q') &= |d_{DD}(O,P') - d_{DD}(O,Q')| \\ &= \left| \frac{r^2}{d_{DD}(O,P)} - \frac{r^2}{d_{DD}(O,Q)} \right| \\ &= \frac{r^2 .\ d_{DD}(P,Q)}{d_{DD}(O,P) .\ d_{DD}(O,Q)} \end{aligned}$$

is obtained.

Note that converse statement of the theorem above is not true. Also the theorem is not valid for any three non-collinear points in $\mathbb{R}^3_{DD}$. But under some other conditions the equation (5) holds.

Now we give the following theorem that shows the equation (5) is satisfied under such conditions.

Theorem 2.10 Let O, P and Q be any three distinct points in $\mathbb{R}^3_{DD}$, P, P' and Q, Q' be inverse pairs with respect to the disdyakis dodecahedron spherical inversion $I_{\mathcal{T}(O,r)}$ with center O and radius r, and d and d' be direction vectors of the rays $\overrightarrow{OP}$ and $\overrightarrow{OQ}$, respectively. If $d \in \Delta_i$ and $d' \in \Delta_i \setminus \{d\}$ where

$\Delta_1 = \{(1,0,0), (0,1,0), (0,0,1), (-1,0,0), (0,-1,0), (0,0,-1)\}$
Δ_2
$= \{(1,1,0), (1,0,1), (0,1,1), (1,0,-1), (1,-1,0), (0,1,-1), (0,-1,1), (0,-1,-1), (-1,1,0), (-1,0,1), (-1,0,-1), (-1,\cdot$
$\Delta_3 = \{(1,1,1), (1,1,-1), (1,-1,1), (-1,1,1), (1,-1,-1), (-1,1,-1), (-1,-1,1), (-1,-1,-1)\}$

and $i = 1,2,3$, then

$$d_{DD}(P', Q') = \frac{r^2 .d_{DD}(P,Q)}{d_{DD}(O,P) .d_{DD}(O,Q)}$$

Proof. Since all translations are elements of the group of isometries of Disdyakis Docedehedron space it is convenient to consider O the center of inversion as origin. So let $I_{(O,r)}$ be the disdyakis docedehedron spherical inversion with center O and radius r in $\mathbb{R}^3_{DD}$. Suppose that $d \in \Delta_1$ and $d' \in \Delta_1 \setminus \{d\}$. Let us consider $P = (0, p, 0)$ and $Q = (0,0,-q)$ thus the inverses of P and Q with respect to $I_{D(O,r)}$ are $P' = \left(0, \frac{r^2}{p}, 0\right)$ and $Q' = \left(0,0, \frac{-r^2}{q}\right)$, respectively. Thus we obtain that

$$d_{DD}(P',Q') = max\left\{\left|\frac{r^2}{p}\right|,\left|\frac{r^2}{q}\right|,0\right\} + (\sqrt{2}-1)mid\left\{\left|\frac{r^2}{p}\right|,\left|\frac{r^2}{q}\right|,0\right\} + (\sqrt{3}-\sqrt{2})min\left\{\left|\frac{r^2}{p}\right|,\left|\frac{r^2}{q}\right|,0\right\}$$

Now there are two subcases;

Case 1 :If $|p| \geq |q|$,then

$$d_{DD}(P',Q') = \frac{r^2(|p|+(\sqrt{2}-1)|q|)}{|p||q|} = \frac{r^2 d_{DD}(P,Q)}{d_{DD}(O,P).d_{DD}(O,Q)}$$

Case 2: If $|p|<|q|$,then

$$d_{DD}(P',Q') = \frac{r^2(|q|+(\sqrt{2}-1)|p|)}{|p||q|} = \frac{r^2 d_{DD}(P,Q)}{d_{DD}(O,P).d_{DD}(O,Q)}$$

Consider $d \in \Delta_2$ and $d' \in \Delta_2 \setminus \{d\}$. Choose $P = (0,p,p)$ and $Q = (q,-q,0)$,so the inverses of P and Q with respect to $I_{\mathcal{D}(O,r)}$ are $P' = \left(0,\frac{r^2}{2p},\frac{r^2}{2p}\right)$ and $Q' = \left(\frac{r^2}{2q},\frac{-r^2}{2q},0\right)$, respectively. Thus we get that

$d_{DD}(P',Q') = max\left\{\left|\frac{r^2}{2q}\right|,\left|\frac{r^2}{2p}+\frac{r^2}{2q}\right|,\left|\frac{r^2}{2p}\right|\right\} + (\sqrt{2}-1)mid\left\{\left|\frac{r^2}{2q}\right|,\left|\frac{r^2}{2p}+\frac{r^2}{2q}\right|,\left|\frac{r^2}{2p}\right|\right\} + (\sqrt{3}-\sqrt{2})min\left\{\left|\frac{r^2}{2q}\right|,\left|\frac{r^2}{2p}+\frac{r^2}{2q}\right|,\left|\frac{r^2}{2p}\right|\right\}$. Here there are six subcases;

Case 1: If $|q| > |p| > |p+q|$, then

$$d_{DD}(P',Q') = \frac{r^2\left(|q| + (\sqrt{2}-1)|p| + (\sqrt{3}-\sqrt{2})|p+q|\right)}{2|p||q|} = \frac{r^2 d_{DD}(P,Q)}{d_{DD}(O,P).d_{DD}(O,Q)}.$$

Case 2: If $|q| > |p+q| > |p|$, then

$$d_{DD}(P',Q') = \frac{r^2(|q|+(\sqrt{2}-1)|p+q|+(\sqrt{3}-\sqrt{2})|p|)}{2|p||q|} = \frac{r^2 d_{DD}(P,Q)}{d_{DD}(O,P).d_{DD}(O,Q)}.$$

Case 3: If $|p|>|q|>|p+q|$,then

$$d_{DD}(P',Q') = \frac{r^2(|p|+(\sqrt{2}-1)|q|+(\sqrt{3}-\sqrt{2})|p+q|)}{2|p||q|} = \frac{r^2 d_{DD}(P,Q)}{d_{DD}(O,P).d_{DD}(O,Q)}$$

Case 4: If $|p|> |p+q| >|q|$ then

$$d_{DD}(P',Q') = \frac{r^2\left(|p| + (\sqrt{2}-1)|p+q| + (\sqrt{3}-\sqrt{2})|q|\right)}{2|p||q|} = \frac{r^2 d_{DD}(P,Q)}{d_{DD}(O,P).d_{DD}(O,Q)}$$

Case 5: If $|p+q|> |p| >|q|$ then

$$d_{DD}(P',Q') = \frac{r^2(|p+q|+(\sqrt{2}-1)|p|+(\sqrt{3}-\sqrt{2})|q|)}{2|p||q|} = \frac{r^2 d_{DD}(P,Q)}{d_{DD}(O,P).d_{DD}(O,Q)}$$

Case 6: If $|p+q|> |q| >|p|$ then

$$d_{DD}(P',Q') = \frac{r^2\left(|p+q| + (\sqrt{2}-1)|q| + (\sqrt{3}-\sqrt{2})|p|\right)}{2|p||q|} = \frac{r^2 d_{DD}(P,Q)}{d_{DD}(O,P).d_{DD}(O,Q)}$$

Suppose that $d \in \Delta_3$ and $d' \in \Delta_3 \setminus \{d\}$. If $P = (p,p,p)$ and $Q = (q,-q,q)$ then the inverses of P and Q with respect to $I_{\mathcal{D}(O,r)}$ are $P' = \left(\frac{r^2}{3p},\frac{r^2}{3p},\frac{r^2}{3p}\right)$ and $Q' = \left(\frac{r^2}{3q},\frac{-r^2}{3q},\frac{r^2}{3q}\right)$, respectively. Thus we obtain that

$d_{TH}(P',Q') = max\left\{\left|\frac{r^2}{3p}-\frac{r^2}{3q}\right|,\left|\frac{r^2}{3p}+\frac{r^2}{3q}\right|,\left|\frac{r^2}{3p}-\frac{r^2}{3q}\right|\right\} + (\sqrt{2}-1)mid\left\{\left|\frac{r^2}{3p}-\frac{r^2}{3q}\right|,\left|\frac{r^2}{3p}+\frac{r^2}{3q}\right|,\left|\frac{r^2}{3p}-\frac{r^2}{3q}\right|\right\} + (\sqrt{3}-\sqrt{2})min\left\{\left|\frac{r^2}{3p}-\frac{r^2}{3q}\right|,\left|\frac{r^2}{3p}+\frac{r^2}{3q}\right|,\left|\frac{r^2}{3p}-\frac{r^2}{3q}\right|\right\}$. Now there are two possible subcases;

Case 1: If $|p-q| > |p+q|$, then $d_{DD}(P',Q') = \frac{r^2(\sqrt{2}|p-q|+(\sqrt{3}-\sqrt{2})|p+q|)}{3|p||q|} = \frac{r^2 d_{DD}(P,Q)}{d_{DD}(O,P).d_{DD}(O,Q)}$.

Case 2: If $|p+q| > |p-q|$, then $d_{DD}(P',Q') = \frac{r^2(\sqrt{2}|p+q|+(\sqrt{3}-\sqrt{2})|p-q|)}{3|p||q|} = \frac{r^2 d_{DD}(P,Q)}{d_{DD}(O,P).d_{DD}(O,Q)}$.

For other possible choices of elements in $\Delta_i, i = 1,2,3$, by similar calculations it is easy to see that equality is valid.

3. Results

This section includes two subsections to investigate results and definitions obtained by spherical inversions in disdyakis dodecahedron space. We study on inverses of lines, planes and disdyakis dodecahedron spheres under an inversion $I_{\mathcal{D}(O,r)}$ as a comparison of inverses of lines and circles in Euclidean plane under a circular inversion. Also we investigate cross-ratio and harmonic conjugates in $\mathbb{R}^3_{DD}$.

3.1. Spherical Inversions of Lines, Planes and Disdyakis Dodecahedron Spheres in $\mathbb{R}^3_{DD}$

In Euclidean version inverse of a line is a circle and inverse of a circle is a line, only the lines passing through the inversion center is invariant. In this section, disdyakis dodecahedron spherical inversions of lines, planes and disdyakis dodecahedron spheres are studied according to their positions in $\mathbb{R}^3_{DD}$.

Theorem 3.11 Let $I_{\mathcal{D}(O,r)}$ be a disdyakis dodecahedron spherical inversion with center O and radius r. Any line and any plane containing O is invariant under $I_{\mathcal{D}(O,r)}$.

Proof. Consider the disdyakis dodecahedron spherical inversion $I_{\mathcal{D}(O,r)}$ with center O and radius r. By equation (2) it is obvious that a line passing through O is invariant under $I_{\mathcal{D}(O,r)}$. Let $Ax + By + Cz = 0$ be a plane containing O. Under $I_{\mathcal{D}(O,r)}$ we get the equation of the plane as;

$$A\frac{r^2x\prime}{(d_{DD}(O,P\prime))^2} + B\frac{r^2y\prime}{(d_{DD}(O,P\prime))^2} + C\frac{r^2z\prime}{(d_{DD}(O,P\prime))^2} = 0.$$

That is $Ax' + By' + Cz' = 0$ which completes the proof.

Theorem 3.12 Let $I_{\mathcal{D}(O,r)}$ be a tetrakis hexahedron spherical inversion with center O and radius r. The inverse of a disdyakis dodecahedron sphere with center O under $I_{\mathcal{D}(O,r)}$ is a disdyakis dodecahedron sphere with center O.

Proof. Since the translation preserves distance in $\mathbb{R}^3_{DD}$ we would take center of inversion $I_{\mathcal{D}(O,r)}$ as $O = (0,0,0)$, thus the disdyakis dodecahedron sphere $\mathcal{D}$ with center O and radius r is

$$\mathcal{D} = \{P = (x,y,z): d_{DD}(O,P) = r\}$$

Let $\mathcal{D}_1$ be the disdyakis dodecahedron sphere with center O and radius r_1, then

$$\mathcal{D}_1 = \{P = (x,y,z): d_{DD}(O,P) = r_1\}$$

Thus the inverse of $\mathcal{D}_1$ under $I_{\mathcal{D}(O,r)}$ is $\mathcal{D}'_1 = \left\{P' = (x',y',z'): d_{DD}(O,P') = \frac{r^2}{r_1}\right\}$ which is a disdyakis dodecahedron sphere.

Theorem 3.13 Let $I_{\mathcal{D}(O,r)}$ be a disdyakis dodecahedron spherical inversion with center O and radius r. The inverse of every edges, vertices and faces of $\mathcal{D}$ is itself.

Proof. By Corollary 2.6, $\mathcal{D}$ is pointwise fixed under $I_{\mathcal{D}(O,r)}$. Thus every edges, vertices and faces of $\mathcal{D}$ is invariant under $I_{\mathcal{D}(O,r)}$.

3.2. The Cross Ratio and Harmonic Conjugates in $\mathbb{R}^3_{DD}$

The distance is not invariant under disdyakis dodecahedron spherical inversion. Thus, the inversion in disdyakis dodecahedron space is not an isometry. However, the fact that the cross-ratio is preserved under inversion reveals the necessity of focusing on the cross-ratio by means of the distance. Therefore, in this section, we investigate the cross ratio and harmonic conjugates in $\mathbb{R}^3_{DD}$ under a spherical inversion.

The following definition will be given in a similar sense of the definition given in [29].

Definition 3.13 For any two points X and Y on a directed line l, the directed disdyakis dodecahedron length of the line segment $\overline{XY}$ is denoted by $d_{DD}[X,Y]$. If the line segment $\overline{XY}$ and l have the same direction, then $d_{DD}[X,Y] = d_{DD}(X,Y)$ and if have the opposite direction, then $d_{DD}[X,Y] = -d_{DD}(X,Y)$.

Definition 3.14 Let P, Q, R and S are four distinct points on an oriented line in $\mathbb{R}^3_{DD}$. The disdyakis dodecahedron cross-ratio $(PQ,RS)_{DD}$ is defined by

$$(PQ,RS)_{DD} = \frac{d_{DD}[P,R]d_{DD}[Q,S]}{d_{DD}[P,S]d_{DD}[Q,R]} \tag{6}$$

Corollary 3.15 Let P, Q, R and S are four distinct points on an oriented line in $\mathbb{R}^3_{DD}$. The disdyakis dodecahedron cross-ratio $(PQ,RS)_{DD}$ is positive if both R and S are between P and Q or if neither R nor S are between P and Q.

Proof. Let both R and S points be between P and Q points. For the directed line PQ the tetrakis hexahedron cross-ratio is

$$\begin{aligned}(PQ,RS)_{DD} &= \frac{d_{DD}[PR]d_{DD}[QS]}{d_{DD}[PS]d_{DD}[QR]} \\ &= \frac{d_{DD}(P,R).\left(-d_{TH}(Q,S)\right)}{d_{DD}(P,S).\left(-d_{DD}(Q,R)\right)} = \frac{d_{DD}(P,R).d_{DD}(Q,S)}{d_{DD}(P,S).d_{DD}(Q,R)}\end{aligned}$$

and thus $(PQ,RS)_{DD}$ is positive.

If neither R nor S are between P and Q, then there are six arrays for R and S. Since it is similar to prove for all possible combinations we give the proof for the orientation $R - P - Q - S$. Thus the disdyakis dodecahedron cross-ratio is

$$\begin{aligned}(PQ,RS)_{DD} &= \frac{d_{DD}[PR]d_{DD}[QS]}{d_{DD}[PS]d_{DD}[QR]} \\ &= \frac{\left(-d_{DD}(P,R)\right).d_{DD}(Q,S)}{d_{DD}(P,S).\left(-d_{DD}(Q,R)\right)} = \frac{d_{DD}(P,R).d_{DD}(Q,S)}{d_{DD}(P,S).d_{DD}(Q,R)}\end{aligned}$$

and thus $(PQ,RS)_{DD}$ is positive.

Corollary 3.16 Let P, Q, R and S are four distinct points on an oriented line in $\mathbb{R}^3_{DD}$. If the pairs $\{P,Q\}$ and $\{R,S\}$ seperate each other, then the disdyakis dodecahedron cross-ratio $(PQ,RS)_{DD}$ is negative.

Proof. If the pairs $\{P,Q\}$ and $\{R,S\}$ seperate each other, then there are four arrays for R and S. For the orientation $R - P - S - Q$ the disdyakis dodecahedron cross-ratio is

$$\begin{aligned}(PQ,RS)_{DD} &= \frac{d_{DD}[PR]d_{DD}[QS]}{d_{DD}[PS]d_{DD}[QR]} \\ &= \frac{\left(-d_{DD}(P,R)\right).\left(-d_{DD}(Q,S)\right)}{d_{DD}(P,S).\left(-d_{DD}(Q,R)\right)} = -\frac{d_{DD}(P,R).d_{DD}(Q,S)}{d_{DD}(P,S).d_{DD}(Q,R)}\end{aligned}$$

and since for other possible arrays, by similar calculations, same results are obtained, thus $(PQ,RS)_{DD}$ is negative.

Theorem 3.17 The disdyakis dodecahedron cross-ratio is invariant under disdyakis dodecahedron spherical inversion in $\mathbb{R}^3_{DD}$.

Proof. Let $I_{\mathcal{D}(O,r)}$ be a disdyakis dodecahedron spherical inversion with center O and radius r, and P, Q, R and S be four points on an oriented line l passing through O. Let P', Q', R' and S' be inverse points of P, Q, R and S respectively under $I_{\mathcal{D}(O,r)}$. Observe that the disdyakis dodecahedron spherical inversion preserves the seperation or non-seperation of the pairs $\{P,Q\}$ and $\{R,S\}$ and also it reverses the disdyakis dodecahedron - directed distance from the point P to the point Q along a line l to disdyakis dodecahedron -directed distance from the point Q' to the point P'. The required result follows from Theorem 2.9;

$$(P'Q',R'S')_{DD} = \frac{d_{DD}(P',R').d_{DD}(Q'S')}{d_{DD}(P'S').d_{DD}(Q'R')}$$

$$=\frac{\frac{r^2.d_{DD}(P,R)}{d_{DD}(O,P).d_{DD}(O,R)}\cdot\frac{r^2.d_{DD}(Q,S)}{d_{DD}(O,Q).d_{DD}(O,S)}}{\frac{r^2.d_{DD}(P,S)}{d_{DD}(O,P).d_{DD}(O,S)}\cdot\frac{r^2.d_{DD}(Q,R)}{d_{DD}(O,Q).d_{DD}(O,R)}}$$

$$=\frac{d_{DD}(P,R)d_{DD}(Q,S)}{d_{DD}(P,S).d_{DD}(Q,R)}$$

$$=(PQ,RS)_{DD}$$

Definition 3.18 Let l be a line in $\mathbb{R}^3_{DD}$. Suppose that P,Q,R and S are four points on l. It is called that P,Q,R and S form a harmonic set if $(PQ,RS)_{DD}=-1$ and it is denoted by $H(PQ,RS)_{DD}$. That is, any pair R and S on l for which

$$\frac{d_{DD}[P,R]d_{DD}[Q,S]}{d_{DD}[P,S]d_{DD}[Q,R]}=-1 \tag{7}$$

is said to divide P and Q harmonically. The points R and S are called disdyakis dodecahedron harmonic conjugates with respect to P and Q.

Theorem 3.19 Let T be a disdyakis dodecahedron sphere with center O, and line segment $[PQ]$ be a diameter of T in $\mathbb{R}^3_{DD}$. Let R and S be distinct points of the ray $\overrightarrow{OP}$, which divide the segment $[PQ]$ internally and externally. Then R and S are disdyakis dodecahedron harmonic conjugates with respect to P and Q if and only if R and S are inverse points with respect to the disdyakis dodecahedron spherical inversion $I_{\mathcal{D}(O,r)}$.
Proof. Let R and S are disdyakis dodecahedron harmonic conjugates with respect to P and Q. Then

$$(PQ,RS)_{DD}=-1\Rightarrow\frac{d_{DD}[P,R].d_{DD}[Q,S]}{d_{DD}[P,S].d_{DD}[Q,R]}=-1$$

Since R divides the line segment $[PQ]$ internally and R is on the ray $\overrightarrow{OQ}$,

$$d_{DD}(R,Q)=r-d_{DD}(O,R)\ \text{ and }\ d_{DD}(P,R)=r+d_{DD}(O,R).$$

Since S divides the line segment $[PQ]$ externally and S is on the ray $\overrightarrow{OQ}$,

$$d_{DD}(P,S)=r+d_{DD}(O,S)\text{ and }d_{DD}(Q,S)=d_{DD}(O,S)-r.$$

Thus

$$\frac{\left(r+d_{DD}(O,R)\right).\left(d_{DD}(O,S)-r\right)}{\left(r+d_{DD}(O,S)\right).\left(r-d_{DD}(O,R)\right)}=-1$$

$$\Rightarrow\left(r+d_{DD}(O,R)\right).\left(d_{DD}(O,S)-r\right)=\left(r+d_{DD}(O,S)\right).\left(d_{DD}(O,R)-r\right).$$

Simplifying the last equality $d_{DD}(O,R).d_{DD}(O,S)=r^2$ is obtained. So R and S are disdyakis dodecahedron spherical inverse points with respect to the disdyakis dodecahedron spherical inversion $I_{\mathcal{D}(O,r)}$. For the other condition (S and R are on the ray $\overrightarrow{OP}$) by similar calculations the same conclusion is obtained.
Conversely, if R and S are disdyakis dodecahedron spherical inverse points with respect to the disdyakis dodecahedron spherical inversion $I_{\mathcal{D}(O,r)}$ the proof is similar.

4. Discussion and Conclusion

Inversion theory is of interest to geometers today, as it used to be, since it suggests challenging problems and when it is applied many problems in geometry became much manageable. Classical inversion is defined with respect to a circle but there are many different definitions of inversion in the literature by using other objects or using different distance functions or expanding dimension. In this study inversion is defined in a three dimensional non-Euclidean geometry and by using obtained results in this space some properties of this inversion is investigated. We hope that this topic would provoke further researches by interested readers or their students.

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ON FIXED POINTS OF $d_{\mathbb{D}}^{b}-$CYCLICAL CONTRACTIONS

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ABSTRACT

In the current study, we give some new fixed point results by using different types of cyclic contractions in the setting of hyperbolic valued $b-$metric space. Also, we establish several illustrative examples to verify accuracy of our findings.

1. INTRODUCTION AND PRELIMINARIES

Fixed point theory is an important and popular topic in applications to various problems in nonlinear analysis, differential equations, approximation theory and control systems etc. Obtaining fixed points of different type contraction mappings has been the main goal of researchers. Many researchers have published various articles on fixed point theorems to extend the famous Banach contraction principle [1] with new and different contractive and expansive conditions in metric spaces or generalizations of metric spaces.

Cyclic contractive conditions has recently been the focus of many authors. Kirk et al. [2] furnished some fixed point results with cyclical contractive conditions in 2003. Hereupon, Karapınar and Erhan [3] obtained some proximity points by utilizing various types of cyclic contractions.

In 1989, Bakhtin [4] initiated a generalization of metric spaces called $b-$metric spaces and stated some fixed point results in $b-$metric spaces that are generalizations of the Banach's fixed point theorem (see also Czerwik [5]). Several authors have reported some fixed point theorems for cyclic contractions in $b-$metric spaces (see, e.g., [6,7,8]).

In [9], we define the notion of a hyperbolic valued $b-$metric space as a generalization of a hyperbolic valued metric space introduced by Kumar and Saini [10] in 2016 and give Zamfirescu type fixed point results.

In this study, we transform some known fixed point results and contractions such as Banach's in [1], Kannan's in [11], Chatterjea's in [12], Zamfirescu's in [13], Reich's in [14] and Ćirić's in [15] for classical metric spaces to the cyclic case in hyperbolic valued $b-$metric spaces by supporting the obtained theorems by some concrete examples.

Before starting our main results, we recall some known facts which will be used in next sections.

A bicomplex number is defined as $z = z_1 + jz_2$ where $j^2 = -1, ij = ji$, z_1 and z_2 are complex numbers, and i and j are independent imaginary units. The set of bicomplex numbers is denoted by $\mathbb{BC}$ and the set forms a Banach space with the operations $+, \cdot$ and the norm $|.|$

$$z + w = (z_1 + jz_2) + (w_1 + jw_2) = (z_1 + w_1) + j(z_2 + w_2),$$
$$\lambda.z = \lambda.(z_1 + jz_2) = \lambda z_1 + j\lambda z_2,$$
$$|.| : \mathbb{BC} \to R, z \to |z| = \sqrt{|z_1|^2 + |z_2|^2}$$

for all $z = z_1 + jz_2, w = w_1 + jw_2 \in \mathbb{BC}$ and for all $\lambda \in R$. Also, $\mathbb{D} = \{a + kb \; : \; k = ij, \; a, b \in R\} \subset \mathbb{BC}$ is the set of hyperbolic numbers.

Three types of conjugates and moduli of $z = z_1 + jz_2 \in \mathbb{BC}$ are as follows:

$$z^{\dagger_1} = \overline{z_1} + j\overline{z_2},$$
$$z^{\dagger_2} = z_1 - jz_2,$$
$$z^{\dagger_3} = \overline{z_1} - j\overline{z_2},$$
$$|z|_i^2 = zz^{\dagger_2} = z_1^2 + z_2^2 \in C,$$
$$|z|_j^2 = zz^{\dagger_1} = \left(|z_1|^2 - |z_2|^2\right) + j\left(2\operatorname{Re}\left(z_1.\overline{z_2}\right)\right),$$
$$|z|_k^2 = zz^{\dagger_3} = \left(|z_1|^2 + |z_2|^2\right) + k\left(-\operatorname{Im}\left(z_1.\overline{z_2}\right)\right) \in \mathbb{D}.$$

The set $\left\{e_1 = \frac{1+ij}{2}, e_2 = \frac{1-ij}{2}\right\}$ is idempotent basis of the set of bicomplex numbers and so idempotent representation of $z = z_1 + jz_2$ is uniquely written as $z = e_1\beta_1 + e_2\beta_2$ where $\beta_1 = z_1 - iz_2,\ \ \beta_2 = z_1 + iz_2 \in C$ [16].

Let $\alpha = x + ky \in \mathbb{D}$. Then, $\alpha = e_1\alpha_1 + e_2\alpha_2 = e_1(x+y) + e_2(x-y)$ is the idempotent representation of α. Also, $\mathbb{D}^+ = \{\alpha = e_1\alpha_1 + e_2\alpha_2 \ :\ \alpha_1 \geq 0, \alpha_2 \geq 0\}$ is the set of positive hyperbolic numbers.

For $\alpha, \beta \in \mathbb{D}$, if $\beta - \alpha \in \mathbb{D}^+$ (or $\beta - \alpha \in \mathbb{D}^+ - \{0\}$), then we write $\alpha \precsim \beta$ (or $\alpha \precnsim \beta$). and also we have that

$$\alpha \precsim \beta \text{ if and only if } \alpha_1 \leq \beta_1 \text{ and } \alpha_2 \leq \beta_2,$$
$$\alpha \precnsim \beta \text{ if and only if } \alpha \neq \beta,\ \alpha_1 \leq \beta_1 \text{ and } \alpha_2 \leq \beta_2,$$
$$\alpha \prec \beta \text{ if and only if } \alpha_1 < \beta_1 \text{ and } \alpha_2 < \beta_2$$

for $\alpha = e_1\alpha_1 + e_2\alpha_2,\ \beta = \beta_1 e_1 + \beta_2 e_2 \in \mathbb{D}$ where $\alpha_1, \alpha_2, \beta_1, \beta_2 \in R$.

The followings hold for any $\alpha, \beta, \gamma, \delta \in \mathbb{D}$ and $\theta, \rho \in \mathbb{BC}$:

(i) $|\theta + \rho|_k \precsim |\theta|_k + |\rho|_k$, $|\theta\rho|_k = |\theta|_k\,|\rho|_k$ and $\left|\frac{\theta}{\rho}\right|_k = \frac{|\theta|_k}{|\rho|_k}$ where ρ is invertible.

(ii) If $\alpha \in \mathbb{D}^+$, then $|\alpha|_k = \alpha$ and $|\alpha\theta|_k = \alpha|\theta|_k$.

(iii) $|\theta|_k = |\beta_1|e_1 + |\beta_2|e_2$ for $\theta = \beta_1 e_1 + \beta_2 e_2$.

(iv) If $\alpha \precsim \beta$ and $\gamma \precsim \delta$, then $\alpha + \gamma \precsim \beta + \delta$.

(v) If $\alpha \precsim \beta$ and $0 \precsim \gamma$, then $\alpha\gamma \precsim \beta\gamma$.

(vi) If $\alpha, \beta \in \mathbb{D}^+$, then $\alpha \precsim \beta$ implies that $|\alpha| \leq |\beta|$.

(vii) If $\alpha \precsim \beta$ and $\beta \precsim \gamma$, then $\alpha \precsim \gamma$ [17].

(viii) If $\alpha \in \mathbb{D}^+$, $\alpha \neq 1$ and $1-\alpha$ is invertible, then

$$1 + \alpha + \alpha^2 + \ldots + \alpha^n = \frac{1-\alpha^{n+1}}{1-\alpha}$$

for all $n \in N$.

(ix) If $\alpha \in \mathbb{D}^+$ and $\alpha \prec 1$, then $0 \precsim \alpha^n \prec 1$ for all $n \in N$ and $\alpha^n \to 0$ [18].

Definition 1.1. [9] Let $X \neq \emptyset$, $s \succsim 1$ be a given hyperbolic number and $d_{\mathbb{D}}^{b}: X \times X \to \mathbb{D}^{+}$ be a function such that the following properties hold:

(i) $d_{\mathbb{D}}^{b}(x,y) = 0$ iff $x = y$,

(ii) $d_{\mathbb{D}}^{b}(x,y) = d_{\mathbb{D}}^{b}(y,x)$,

(iii) $d_{\mathbb{D}}^{b}(x,z) \precsim s\left[d_{\mathbb{D}}^{b}(x,y) + d_{\mathbb{D}}^{b}(y,z)\right]$

for any $x,y,z \in X$. Then, $d_{\mathbb{D}}^{b}$ is called a hyperbolic valued $b-$metric on X and $\left(X, d_{\mathbb{D}}^{b}\right)$ is called a hyperbolic valued $b-$metric space.

Definition 1.2. [9] Let $\left(X, d_{\mathbb{D}}^{b}\right)$ be a hyperbolic valued $b-$metric space, $(x_n) \subset X$ and $x \in X$. If for every $0 \prec \varepsilon \in \mathbb{D}$ there exists $n_0(\varepsilon) \in N$ such that $d_{\mathbb{D}}^{b}(x_n, x) \prec \varepsilon$ for all $n \geq n_0$ then we say that (x_n) is $d_{\mathbb{D}}^{b}-$convergent, and denoted by $x_n \xrightarrow{d_{\mathbb{D}}^{b}} x$ as $n \to \infty$.

If for every $0 \prec \varepsilon \in \mathbb{D}$ there exists $n_0(\varepsilon) \in N$ such that $d_{\mathbb{D}}^{b}(x_n, x_m) \prec \varepsilon$ for all $n, m \geq n_0$ then we say that (x_n) is a $d_{\mathbb{D}}^{b}-$Cauchy sequence.

If every $d_{\mathbb{D}}^{b}-$Cauchy sequence $d_{\mathbb{D}}^{b}-$converges to a point in $\left(X, d_{\mathbb{D}}^{b}\right)$, then we say that $\left(X, d_{\mathbb{D}}^{b}\right)$ is a complete hyperbolic valued $b-$metric space.

All the way through this paper, we will use the abbreviations "hvbms" and "chvbms" for "hyperbolic valued $b-$metric spaces" and "complete hyperbolic valued $b-$metric space" when needed.

Proposition 1.3. [9] Let $\left(X, d_{\mathbb{D}}^{b}\right)$ be a hvbms, $(x_n) \subset X$ and $x \in X$. Then,

(i) $x_n \xrightarrow{d_{\mathbb{D}}^{b}} x$ as $n \to \infty$ iff $\left|d_{\mathbb{D}}^{b}(x_n, x)\right| \to 0$ as $n \to \infty$.

(ii) (x_n) is a $d_{\mathbb{D}}^{b}-$Cauchy sequence iff for all $m \in N$, $\left|d_{\mathbb{D}}^{b}(x_n, x_{n+m})\right| \to 0$ as $n \to \infty$.

(iii) x is unique.

(iv) (x_n) is a $d_{\mathbb{D}}^{b}-$Cauchy sequence.

(v) All subsequences of (x_n) are $d_{\mathbb{D}}^{b}-$convergent to x.

In the closing of this part, we recall the following concepts that form the basis of this article given by Kirk et al. [2].

Definition 1.4. Let A and B be nonempty subsets of a metric space (X,d). A mapping $f: A \cup B \to A \cup B$ is said to be cyclic if $f(A) \subset B$ and $f(B) \subset A$.

Definition 1.5. Let A and B be nonempty subsets of a metric space (X,d). A mapping $f: A \cup B \to A \cup B$ is called a cyclic contraction if there exists $k \in [0,1)$ such that

$$d(fx, fy) \leq k d(x,y)$$

for all $x \in A$ and $y \in B$.

2. MAIN RESULTS

In this part, we define some new types of cyclic contractions defined on a hvbms and we achieve several fixed point results for them.

Definition 2.1. Let $\left(X, d_{\mathbb{D}}^{b}\right)$ be a hvbms, $E, F \subset X$, $E \neq \varnothing$, $F \neq \varnothing$ and $T : E \cup F \to E \cup F$ be a mapping. If $T(E) \subset F$ and $T(F) \subset E$, then T is called a $d_{\mathbb{D}}^{b}-$cyclic map.

Definition 2.2. Let $\left(X, d_{\mathbb{D}}^{b}\right)$ be a hvbms and $E \subset X$. If $(x_n) \subset E$ and $x_n \xrightarrow{d_{\mathbb{D}}^{b}} x$ imply $x \in E$, the subset E is called a $d_{\mathbb{D}}^{b}-$closed set in X.

Our first definition in the cyclic case is a new version of Banach's contraction [1] and Corollary 3.7 in [9], as follows:

Definiton 2.3. Let $\left(X, d_{\mathbb{D}}^{b}\right)$ be a hvbms with $s \succsim 1$, $E, F \subset X$, $E \neq \varnothing$, $F \neq \varnothing$ and $T : E \cup F \to E \cup F$ be a $d_{\mathbb{D}}^{b}-$cyclic map. If there exists $0 \prec \alpha \prec \dfrac{1}{s}$ such that $1 - s\alpha$ is invertible and

$$d_{\mathbb{D}}^{b}(Tx, Ty) \precsim \alpha d_{\mathbb{D}}^{b}(x, y)$$

for all $x \in E$ and $y \in F$, then we say that T is a Banach type $d_{\mathbb{D}}^{b}-$cyclic contraction.

Our first result is the analogue of the celebrated Banach contraction principle [1] for cyclic contraction in hyperbolic valued $b-$metric spaces, as follows:

Theorem 2.4. Let $\left(X, d_{\mathbb{D}}^{b}\right)$ be a chvbms with $s \succsim 1$, $E, F \subset X$ be $d_{\mathbb{D}}^{b}-$closed subsets, $E \neq \varnothing$, $F \neq \varnothing$ and $T : E \cup F \to E \cup F$ be a Banach type $d_{\mathbb{D}}^{b}-$cyclic contraction. Then T has a unique fixed point in $E \cap F$.

Proof. Let $x \in E$. Then we write $d_{\mathbb{D}}^{b}\left(T^2x, Tx\right) \precsim \alpha d_{\mathbb{D}}^{b}(Tx, x)$, and repeating this process, we obtain $d_{\mathbb{D}}^{b}\left(T^{n+1}x, T^n x\right) \precsim \alpha^n d_{\mathbb{D}}^{b}(Tx, x)$ for any $n \in N$. Letting $n \to \infty$, we get $d_{\mathbb{D}}^{b}\left(T^{n+1}x, T^n x\right) \to 0.$ Considering that $\left(T^n x\right)$ is a $d_{\mathbb{D}}^{b}-$Cauchy sequence, there exists $u \in E \cup F$ such that $T^n x \xrightarrow{d_{\mathbb{D}}^{b}} u.$ Since $x \in E,$ $\left(T^{2n}x\right)$ is a sequence in E and $\left(T^{2n-1}x\right)$ is a sequence in F. These sequences converge to the same limit u. Because of the $d_{\mathbb{D}}^{b}-$closedness of the sets E and F, we obtain that $u \in E \cap F$, so $E \cap F \neq \varnothing$.

We assert that u is a fixed point of T. Indeed, for any $n \in N$ we have

$$d_{\mathbb{D}}^{b}(Tu, u) = \lim_{n\to\infty} d_{\mathbb{D}}^{b}\left(Tu, T^{2n}x\right) \precsim \alpha \lim_{n\to\infty} d_{\mathbb{D}}^{b}\left(u, T^{2n-1}x\right) = \alpha d_{\mathbb{D}}^{b}(u, u) = 0$$

which purports that $Tu = u.$

For uniqueness, let $v \in E \cup F$ be another fixed point of T. Since T is a $d_{\mathbb{D}}^{b}-$cyclic map, we get $v \in E \cap F$. Then it follows from Definition 2.3 that

$$d_{\mathbb{D}}^{b}(u, v) = d_{\mathbb{D}}^{b}(Tu, Tv) \precsim \alpha d_{\mathbb{D}}^{b}(u, v),$$

so $\left(1-\alpha\right)d_{\mathbb{D}}^{b}\left(u,v\right)=0$. Therefore, we derive that $u=v$ and so u is a unique fixed point of T.

Remark 2.5. 1) Note that Theorem 2.4 corresponds to Corollary 3.7 in [9] for $E=F=X$.

2) We emphasize that Theorem 2.4 is a generalization of Theorem 3.1 in [18] for $E=F=X$ and $s=1$.

Example 2.6. Consider the chvbms $\left(\left[0,1\right],d_{\mathbb{D}}^{b}\right)$ where the function $d_{\mathbb{D}}^{b}:\left[0,1\right]\times\left[0,1\right]\to\mathbb{D}^{+}$ is defined as

$$d_{\mathbb{D}}^{b}\left(x,y\right)=\frac{1}{3}\left|x-y\right|^{2}e_{1}+\frac{2}{3}\left|x-y\right|^{2}e_{2}.$$

given in [9] with $s=2$. Let $E=\left[0,\frac{1}{2}\right]$ and $F=\left[0,\frac{1}{3}\right]$. Define a self-mapping $T:E\cup F=\left[0,\frac{1}{2}\right]\to\left[0,\frac{1}{2}\right]$ as $Tx=\frac{x}{2}-\frac{x^{2}}{4}$ for all $x\in\left[0,1\right]$. It can be simply obtained that $T\left(E\right)\subset F$ and $T\left(F\right)\subset E$. Also, we get

$$\begin{aligned}d_{\mathbb{D}}^{b}\left(Tx,Ty\right)&=\frac{1}{3}\left|\left(\frac{x}{2}-\frac{x^{2}}{4}\right)-\left(\frac{y}{2}-\frac{y^{2}}{4}\right)\right|^{2}e_{1}+\frac{2}{3}\left|\left(\frac{x}{2}-\frac{x^{2}}{4}\right)-\left(\frac{y}{2}-\frac{y^{2}}{4}\right)\right|^{2}e_{2}\\&=\frac{1}{3}\left|\frac{x-y}{2}\right|^{2}\left|1-\frac{x-y}{2}\right|^{2}e_{1}+\frac{2}{3}\left|\frac{x-y}{2}\right|^{2}\left|1-\frac{x-y}{2}\right|^{2}e_{2}\\&\prec\left(\frac{1}{4}e_{1}+\frac{1}{4}e_{2}\right)\left[\frac{1}{3}\left|x-y\right|^{2}e_{1}+\frac{2}{3}\left|x-y\right|^{2}e_{2}\right]=\left(\frac{1}{4}e_{1}+\frac{1}{4}e_{2}\right)d_{\mathbb{D}}^{b}\left(x,y\right)\end{aligned}$$

If we choose $\alpha=\frac{1}{4}e_{1}+\frac{1}{4}e_{2}$, all requirements of Theorem 2.4 are satisfied to get a unique fixed point $0\in E\cap F=\left[0,\frac{1}{3}\right]$ of T.

We continue this section with a new definition generalizes the Kannan's contraction [11] and Corollary 3.9 in [9].

Definiton 2.7. Let $\left(X,d_{\mathbb{D}}^{b}\right)$ be a hvbms with $s\succsim 1$, $E,F\subset X$, $E\neq\varnothing$, $F\neq\varnothing$ and $T:E\cup F\to E\cup F$ be a $d_{\mathbb{D}}^{b}-$cyclic map. If there exists $0\prec\beta\prec\frac{1}{2}$ such that $1-\beta$ is invertible and

$$d_{\mathbb{D}}^{b}\left(Tx,Ty\right)\precsim\beta\left[d_{\mathbb{D}}^{b}\left(x,Tx\right)+d_{\mathbb{D}}^{b}\left(y,Ty\right)\right]$$

for all $x\in E$ and $y\in F$, then we say that T is a Kannan type $d_{\mathbb{D}}^{b}-$cyclic contraction.

Inspired of the idea of the Kannan's fixed point theorem [11] in classical metric spaces we state the following theorem in hyperbolic valued $b-$metric spaces.

Theorem 2.8. Let $\left(X,d_{\mathbb{D}}^{b}\right)$ be a chvbms with $s\succsim 1$, $E,F\subset X$ be $d_{\mathbb{D}}^{b}-$closed subsets, $E\neq\varnothing$, $F\neq\varnothing$ and $T:E\cup F\to E\cup F$ be a Kannan type $d_{\mathbb{D}}^{b}-$cyclic contraction. Then T has a unique fixed point in $E \cap F$.

Proof. Let $x \in E$. Then we write $d_{\mathbb{D}}^{b}\left(T^{2}x, Tx\right) \precsim \beta\left[d_{\mathbb{D}}^{b}\left(T^{2}x, Tx\right)+d_{\mathbb{D}}^{b}\left(Tx, x\right)\right]$ and so $d_{\mathbb{D}}^{b}\left(T^{2}x, Tx\right) \precsim \frac{\beta}{1-\beta} d_{\mathbb{D}}^{b}\left(Tx, x\right)$ where $0 \prec \frac{\beta}{1-\beta} \prec 1$ Repeating this process, we derive $d_{\mathbb{D}}^{b}\left(T^{n+1}x, T^{n}x\right) \precsim \left(\frac{\beta}{1-\beta}\right)^{n} d_{\mathbb{D}}^{b}\left(Tx, x\right)$ for any $n \in N$. Take $n \to \infty$, we get $d_{\mathbb{D}}^{b}\left(T^{n+1}x, T^{n}x\right) \to 0.$ This guarantees that $\left(T^{n}x\right)$ is a $d_{\mathbb{D}}^{b}-$Cauchy sequence. Therefore there is an element $u \in E \cup F$ such that $T^{n}x \xrightarrow{d_{\mathbb{D}}^{b}} u.$ Since $x \in E,$ $\left(T^{2n}x\right)$ is a sequence in E and $\left(T^{2n-1}x\right)$ is a sequence in F. Both sequences are convergent to u and because of the $d_{\mathbb{D}}^{b}-$closedness of the sets E and F, we conclude that $u \in E \cap F$ and $E \cap F \neq \varnothing$.

Now, we indicate that u is a fixed point of T. Again, for any $n \in N$ we have

$$d_{\mathbb{D}}^{b}\left(Tu, u\right)=\lim_{n \to \infty} d_{\mathbb{D}}^{b}\left(Tu, T^{2n}x\right) \precsim \beta \lim_{n \to \infty}\left[d_{\mathbb{D}}^{b}\left(T^{2n}x, T^{2n-1}x\right)+d_{\mathbb{D}}^{b}\left(Tu, u\right)\right] \precsim \beta d_{\mathbb{D}}^{b}\left(Tu, u\right)$$

and so $\left(1-\beta\right) d_{\mathbb{D}}^{b}\left(Tu, u\right)=0$. Since $0 \prec \beta \prec \frac{1}{2}$, we get $d_{\mathbb{D}}^{b}\left(Tu, u\right)=0$ which implies that $Tu = u.$

For uniqueness, let $v \in E \cup F,$ $u \neq v$ and $Tv = v$. Since T is a $d_{\mathbb{D}}^{b}-$cyclic map, we get $v \in E \cap F$. Then it follows from Definiton 2.7 that

$$d_{\mathbb{D}}^{b}\left(u, v\right)=d_{\mathbb{D}}^{b}\left(Tu, Tv\right) \precsim \beta\left[d_{\mathbb{D}}^{b}\left(Tu, u\right)+d_{\mathbb{D}}^{b}\left(Tv, v\right)\right]=0.$$

Therefore, we conclude that $u = v$ and so u is a unique fixed point of T.

Remark 2.9. We draw attention to the fact that Theorem 2.8 is equivalent to Corollary 3.9 in [9] for $E = F = X$.

Example 2.10. Consider the chvbms $\left(\left[0, \infty\right), d_{\mathbb{D}}^{b}\right)$ where the function $d_{\mathbb{D}}^{b} : \left[0, \infty\right) \times \left[0, \infty\right) \to \mathbb{D}^{+}$ is defined as

$$d_{\mathbb{D}}^{b}\left(x, y\right)=\frac{1}{4}\left|x-y\right|^{2} e_{1}+\frac{3}{4}\left|x-y\right|^{2} e_{2}.$$

given in [9]. Suppose that $E = F = \left[0,1\right]$. Define a self-mapping $T : E \cup F = \left[0,1\right] \to \left[0,1\right]$ as $Tx=\begin{cases} \frac{1}{4}, & x=1 \\ \frac{1}{2}, & x \in \left[0,1\right) \end{cases}$ for all $x \in \left[0,1\right]$. It can be easily seen that $T\left(E\right) \subset F$ and $T\left(F\right) \subset E$.

Let $a = c = 1$. Then, we have

$$d_{\mathbb{D}}^{b}\left(Ta, Tc\right)=d_{\mathbb{D}}^{b}\left(\frac{1}{4}, \frac{1}{4}\right)=0,$$

$$d_{\mathbb{D}}^{b}\left(a, Ta\right)=d_{\mathbb{D}}^{b}\left(c, Tc\right)=d_{\mathbb{D}}^{b}\left(1, \frac{1}{4}\right)=\frac{1}{4}\left(\frac{3}{4}\right)^{2} e_{1}+\frac{3}{4}\left(\frac{3}{4}\right)^{2} e_{2},$$

$$d_{\mathbb{D}}^{b}\left(a, Ta\right)+d_{\mathbb{D}}^{b}\left(c, Tc\right)=\frac{1}{2}\left(\frac{3}{4}\right)^{2} e_{1}+\frac{3}{2}\left(\frac{3}{4}\right)^{2} e_{2}.$$

This implies that $d_{\mathbb{D}}^{b}(Ta,Tc) \precsim \beta\left[d_{\mathbb{D}}^{b}(a,Ta)+d_{\mathbb{D}}^{b}(c,Tc)\right]$ for all $\beta \in \mathbb{D}^{+}$.

Let $a,c \in [0,1)$. Then, we have

$$d_{\mathbb{D}}^{b}(Ta,Tc)=d_{\mathbb{D}}^{b}\left(\frac{1}{2},\frac{1}{2}\right)=0,$$

$$d_{\mathbb{D}}^{b}(a,Ta)==d_{\mathbb{D}}^{b}\left(a,\frac{1}{2}\right)=\frac{1}{4}\left|a-\frac{1}{2}\right|^{2}e_1+\frac{3}{4}\left|a-\frac{1}{2}\right|^{2}e_2,$$

$$d_{\mathbb{D}}^{b}(c,Tc)=d_{\mathbb{D}}^{b}\left(c,\frac{1}{2}\right)=\frac{1}{4}\left|c-\frac{1}{2}\right|^{2}e_1+\frac{3}{4}\left|c-\frac{1}{2}\right|^{2}e_2,$$

$$d_{\mathbb{D}}^{b}(a,Ta)+d_{\mathbb{D}}^{b}(c,Tc)=\frac{1}{4}\left[\left|a-\frac{1}{2}\right|^{2}+\left|c-\frac{1}{2}\right|^{2}\right]e_1+\frac{3}{4}\left[\left|a-\frac{1}{2}\right|^{2}+\left|c-\frac{1}{2}\right|^{2}\right]e_2.$$

This implies that $d_{\mathbb{D}}^{b}(Ta,Tc) \precsim \beta\left[d_{\mathbb{D}}^{b}(a,Ta)+d_{\mathbb{D}}^{b}(c,Tc)\right]$ for all $\beta \in \mathbb{D}^{+}$.

Let $a=1$ and $c \in [0,1)$. Then, we have

$$d_{\mathbb{D}}^{b}(Ta,Tc)=d_{\mathbb{D}}^{b}\left(\frac{1}{4},\frac{1}{2}\right)=\frac{1}{4}\left(\frac{1}{4}\right)^{2}e_1+\frac{3}{4}\left(\frac{1}{4}\right)^{2}e_2,$$

$$d_{\mathbb{D}}^{b}(a,Ta)=d_{\mathbb{D}}^{b}\left(1,\frac{1}{4}\right)=\frac{1}{4}\left(\frac{3}{4}\right)^{2}e_1+\frac{3}{4}\left(\frac{3}{4}\right)^{2}e_2,$$

$$d_{\mathbb{D}}^{b}(c,Tc)=d_{\mathbb{D}}^{b}\left(c,\frac{1}{2}\right)=\frac{1}{4}\left|c-\frac{1}{2}\right|^{2}e_1+\frac{3}{4}\left|c-\frac{1}{2}\right|^{2}e_2,$$

$$d_{\mathbb{D}}^{b}(a,Ta)+d_{\mathbb{D}}^{b}(c,Tc)=\frac{1}{4}\left[\left(\frac{3}{4}\right)^{2}+\left|c-\frac{1}{2}\right|^{2}\right]e_1+\frac{3}{4}\left[\left(\frac{3}{4}\right)^{2}+\left|c-\frac{1}{2}\right|^{2}\right]e_2.$$

If we choose $\beta=\frac{1}{9}e_1+\frac{1}{9}e_2$ we obtain $d_{\mathbb{D}}^{b}(Ta,Tc) \precsim \beta\left[d_{\mathbb{D}}^{b}(a,Ta)+d_{\mathbb{D}}^{b}(c,Tc)\right]$.

The case $c=1$ and $a \in [0,1)$ is seen similarly. Thus, all conditions of Theorem 2.8 hold. This means that $0 \in E \cap F=[0,1]$ is fixed point of T and is unique.

Next, we give the following definition as a generalized version of Chatterjea's contractive condition [12] and Corollary 3.10 in [9].

Definiton 2.11. Let $(X,d_{\mathbb{D}}^{b})$ be a hvbms with $s \succsim 1$, $E,F \subset X$, $E \neq \varnothing$, $F \neq \varnothing$ and $T: E \cup F \to E \cup F$ be a $d_{\mathbb{D}}^{b}-$cyclic map. If there exists $0 \prec \gamma \prec \frac{1}{2s}$ such that $1-\gamma s$ is invertible and

$$d_{\mathbb{D}}^{b}(Tx,Ty) \precsim \gamma\left[d_{\mathbb{D}}^{b}(Tx,y)+d_{\mathbb{D}}^{b}(Ty,x)\right]$$

for all $x \in E$ and $y \in F$, then we say that T is a Chatterjea type $d_{\mathbb{D}}^{b}-$cyclic contraction.

We now transform Chatterjea's fixed point result and contraction [12] for classical metric spaces to the cyclic case in hyperbolic valued $b-$metric spaces.

Theorem 2.12. Let (X, d_{D}^b) be a chvbms with $s \succsim 1$, $E, F \subset X$ be d_{D}^b-closed subsets, $E \neq \varnothing$, $F \neq \varnothing$ and $T : E \cup F \to E \cup F$ be a Chatterjea type d_{D}^b-cyclic contraction. Then T has a unique fixed point in $E \cap F$.

Proof. Let $x \in E$. Then it follows that

$$\begin{aligned} d_{\mathsf{D}}^b\left(T^2x, Tx\right) &\precsim \gamma\left[d_{\mathsf{D}}^b\left(T^2x, x\right) + d_{\mathsf{D}}^b\left(Tx, Tx\right)\right] \\ &= \gamma d_{\mathsf{D}}^b\left(T^2x, x\right) \\ &\precsim \gamma s\left[d_{\mathsf{D}}^b\left(T^2x, Tx\right) + d_{\mathsf{D}}^b\left(Tx, x\right)\right] \end{aligned}$$

and so $d_{\mathsf{D}}^b\left(T^2x, Tx\right) \precsim \dfrac{\gamma s}{1-\gamma s} d_{\mathsf{D}}^b\left(Tx, x\right)$ where $0 \prec \dfrac{\gamma s}{1-\gamma s} \prec 1$. Repeating this process, we obtain $d_{\mathsf{D}}^b\left(T^{n+1}x, T^n x\right) \precsim \left(\dfrac{\gamma s}{1-\gamma s}\right)^n d_{\mathsf{D}}^b\left(Tx, x\right)$ for any $n \in N$. Taking the limit as $n \to \infty$, we get $d_{\mathsf{D}}^b\left(T^{n+1}x, T^n x\right) \to 0$. This implies that $\left(T^n x\right)$ is a d_{D}^b-Cauchy sequence. Hence there exists $u \in E \cup F$ such that $T^n x \xrightarrow{d_{\mathsf{D}}^b} u$. Since $x \in E$, we have $\left(T^{2n}x\right) \subset E$ and $\left(T^{2n-1}x\right) \subset F$. These sequences converge to the same limit u. Because of the d_{D}^b-closedness of the sets E and F, we conclude that $E \cap F \neq \varnothing$.

On the other hand, for any $n \in N$ we have

$$d_{\mathsf{D}}^b\left(Tu, u\right) = \lim_{n\to\infty} d_{\mathsf{D}}^b\left(Tu, T^{2n}x\right) \precsim \gamma \lim_{n\to\infty}\left[d_{\mathsf{D}}^b\left(Tu, T^{2n-1}x\right) + d_{\mathsf{D}}^b\left(T^{2n}x, u\right)\right] \precsim \gamma d_{\mathsf{D}}^b\left(Tu, u\right)$$

and so $(1-\gamma) d_{\mathsf{D}}^b\left(Tu, u\right) = 0$. Since $0 \prec \gamma \prec \dfrac{1}{2s}$, we get $d_{\mathsf{D}}^b\left(Tu, u\right) = 0$ which implies that $Tu = u$.

Now we choose another fixed point $v \in E \cup F$ of T. Since T is d_{D}^b-cyclic map we get $v \in E \cap F$. Then taking Definiton 2.11 into account we obtain

$$d_{\mathsf{D}}^b\left(u, v\right) = d_{\mathsf{D}}^b\left(Tu, Tv\right) \precsim \gamma\left[d_{\mathsf{D}}^b\left(Tu, v\right) + d_{\mathsf{D}}^b\left(Tv, u\right)\right] = 2\gamma d_{\mathsf{D}}^b\left(u, v\right).$$

which is equivalent to $(1-2\gamma) d_{\mathsf{D}}^b\left(u, v\right) = 0$ and hence $u = v$. So, u is a unique fixed point of T.

Remark 2.13. We notice that Theorem 2.12 is analogous to Corollary 3.10 in [9] for $E = F = X$.

Example 2.14. Consider the chvbms $\left([0,\infty), d_{\mathsf{D}}^b\right)$ where the function $d_{\mathsf{D}}^b : [0,\infty) \times [0,\infty) \to \mathsf{D}^+$ is defined as

$$d_{\mathsf{D}}^b\left(x, y\right) = \frac{1}{4}|x-y|^2 e_1 + \frac{3}{4}|x-y|^2 e_2.$$

given in [9] with $s = 2$. Suppose that $E = F = [0,1]$. Define a self-mapping $T : E \cup F = [0,1] \to [0,1]$ as $Tx = \begin{cases} \dfrac{1}{4}, & x = 0 \\ \dfrac{1}{2}, & x \in (0,1] \end{cases}$ for all $x \in [0,1]$. It can be easily observed that $T(E) \subset F$ and $T(F) \subset E$.

Let $a=c=0$. Then, we have

$$d_{\mathbb{D}}^{b}\left(Ta,Tc\right)=d_{\mathbb{D}}^{b}\left(\frac{1}{4},\frac{1}{4}\right)=0,$$

$$d_{\mathbb{D}}^{b}\left(a,Tc\right)=d_{\mathbb{D}}^{b}\left(c,Ta\right)=d_{\mathbb{D}}^{b}\left(0,\frac{1}{4}\right)=\frac{1}{4}\left(\frac{1}{4}\right)^{2}e_{1}+\frac{3}{4}\left(\frac{1}{4}\right)^{2}e_{2},$$

$$d_{\mathbb{D}}^{b}\left(a,Tc\right)+d_{\mathbb{D}}^{b}\left(c,Ta\right)=\frac{1}{2}\left(\frac{3}{4}\right)^{2}e_{1}+\frac{3}{2}\left(\frac{3}{4}\right)^{2}e_{2}.$$

This implies that $d_{\mathbb{D}}^{b}\left(Ta,Tc\right)\precsim\gamma\left[d_{\mathbb{D}}^{b}\left(a,Tc\right)+d_{\mathbb{D}}^{b}\left(c,Ta\right)\right]$ for all $\gamma\in\mathbb{D}^{+}$.

Let $a,c\in\left(0,1\right]$. Then, we have

$$d_{\mathbb{D}}^{b}\left(Ta,Tc\right)=d_{\mathbb{D}}^{b}\left(\frac{1}{2},\frac{1}{2}\right)=0,$$

$$d_{\mathbb{D}}^{b}\left(a,Tc\right)==d_{\mathbb{D}}^{b}\left(a,\frac{1}{2}\right)=\frac{1}{4}\left|a-\frac{1}{2}\right|^{2}e_{1}+\frac{3}{4}\left|a-\frac{1}{2}\right|^{2}e_{2},$$

$$d_{\mathbb{D}}^{b}\left(c,Ta\right)=d_{\mathbb{D}}^{b}\left(c,\frac{1}{2}\right)=\frac{1}{4}\left|c-\frac{1}{2}\right|^{2}e_{1}+\frac{3}{4}\left|c-\frac{1}{2}\right|^{2}e_{2},$$

$$d_{\mathbb{D}}^{b}\left(a,Tc\right)+d_{\mathbb{D}}^{b}\left(c,Ta\right)=\frac{1}{4}\left[\left|a-\frac{1}{2}\right|^{2}+\left|c-\frac{1}{2}\right|^{2}\right]e_{1}+\frac{3}{4}\left[\left|a-\frac{1}{2}\right|^{2}+\left|c-\frac{1}{2}\right|^{2}\right]e_{2}.$$

This implies that $d_{\mathbb{D}}^{b}\left(Ta,Tc\right)\precsim\gamma\left[d_{\mathbb{D}}^{b}\left(a,Tc\right)+d_{\mathbb{D}}^{b}\left(c,Ta\right)\right]$ for all $\gamma\in\mathbb{D}^{+}$.

Let $a=0$ and $c\in\left(0,1\right]$. Then, we have

$$d_{\mathbb{D}}^{b}\left(Ta,Tc\right)=d_{\mathbb{D}}^{b}\left(\frac{1}{4},\frac{1}{2}\right)=\frac{1}{4}\left(\frac{1}{4}\right)^{2}e_{1}+\frac{3}{4}\left(\frac{1}{4}\right)^{2}e_{2},$$

$$d_{\mathbb{D}}^{b}\left(a,Tc\right)=d_{\mathbb{D}}^{b}\left(1,\frac{1}{2}\right)=\frac{1}{4}\left(\frac{1}{2}\right)^{2}e_{1}+\frac{3}{4}\left(\frac{1}{2}\right)^{2}e_{2},$$

$$d_{\mathbb{D}}^{b}\left(c,Ta\right)=d_{\mathbb{D}}^{b}\left(c,\frac{1}{4}\right)=\frac{1}{4}\left|c-\frac{1}{4}\right|^{2}e_{1}+\frac{3}{4}\left|c-\frac{1}{4}\right|^{2}e_{2},$$

$$d_{\mathbb{D}}^{b}\left(a,Ta\right)+d_{\mathbb{D}}^{b}\left(c,Tc\right)=\frac{1}{4}\left[\left(\frac{1}{2}\right)^{2}+\left|c-\frac{1}{4}\right|^{2}\right]e_{1}+\frac{3}{4}\left[\left(\frac{1}{2}\right)^{2}+\left|c-\frac{1}{4}\right|^{2}\right]e_{2}.$$

If we choose $\gamma=\frac{1}{4}e_{1}+\frac{1}{4}e_{2}$ we obtain $d_{\mathbb{D}}^{b}\left(Ta,Tc\right)\precsim\gamma\left[d_{\mathbb{D}}^{b}\left(a,Tc\right)+d_{\mathbb{D}}^{b}\left(c,Ta\right)\right]$.

The case $c=0$ and $a\in\left(0,1\right]$ is seen similarly. Thus, all conditions of Theorem 2.12 hold. This means that $0\in E\cap F=\left[0,1\right]$ is fixed point of T and is unique.

Now, we present a new $d_{\mathbb{D}}^{b}-$cyclic contraction in the following way:

Definiton 2.15. Let $\left(X,d_{\mathbb{D}}^{b}\right)$ be a hvbms with $s\succsim1$, $E,F\subset X$, $E\neq\varnothing$, $F\neq\varnothing$ and $T:E\cup F\to E\cup F$ be a $d_{\mathbb{D}}^{b}-$cyclic map. If there exists $0\prec\delta\prec\frac{1}{3}$ such that $1-\delta$ is invertible and

$$d_{\mathbb{D}}^{b}\left(Tx,Ty\right) \precsim \delta\left[d_{\mathbb{D}}^{b}\left(x,y\right)+d_{\mathbb{D}}^{b}\left(Tx,x\right)+d_{\mathbb{D}}^{b}\left(Ty,y\right)\right]$$

for all $x \in E$ and $y \in F$, then we say that T is a Reich type $d_{\mathbb{D}}^{b}-$cyclic contraction.

As a new version of Reich's fixed point result [14], we give the following main theorem which implies that a fixed point is exist and unique on hyperbolic valued $b-$metric spaces.

Theorem 2.16. Let $\left(X, d_{\mathbb{D}}^{b}\right)$ be a chvbms with $s \succsim 1$, $E,F \subset X$ be $d_{\mathbb{D}}^{b}-$closed subsets, $E \neq \emptyset$ and $F \neq \emptyset$ and $T: E \cup F \to E \cup F$ be a Reich type $d_{\mathbb{D}}^{b}-$cyclic contraction. Then T has a unique fixed point in $E \cap F$.

Proof. Let $x \in E$. Then we write

$$d_{\mathbb{D}}^{b}\left(T^{2}x,Tx\right) \precsim \delta\left[d_{\mathbb{D}}^{b}\left(x,Tx\right)+d_{\mathbb{D}}^{b}\left(T^{2}x,Tx\right)+d\left(Tx,x\right)\right]$$

and so $d_{\mathbb{D}}^{b}\left(T^{2}x,Tx\right) \precsim \dfrac{2\delta}{1-\delta} d_{\mathbb{D}}^{b}\left(Tx,x\right)$ where $0 \prec \dfrac{2\delta}{1-\delta} \prec 1$. Continuing this process, we get $d_{\mathbb{D}}^{b}\left(T^{n+1}x,T^{n}x\right) \precsim \left(\dfrac{2\delta}{1-\delta}\right)^{n} d_{\mathbb{D}}^{b}\left(Tx,x\right)$ for any $n \in N$. Take $n \to \infty$, we get $d_{\mathbb{D}}^{b}\left(T^{n+1}x,T^{n}x\right) \to 0$. This implies that $\left(T^{n}x\right)$ is a $d_{\mathbb{D}}^{b}-$Cauchy sequence and there is an element $u \in E \cup F$ such that $T^{n}x \xrightarrow{d_{\mathbb{D}}^{b}} u.$ Since $x \in E,$ $\left(T^{2n}x\right) \subset E$ and $\left(T^{2n-1}x\right) \subset F$ in a way that both sequences converge to the same limit $u.$ Also we know the sets E and F are $d_{\mathbb{D}}^{b}-$closed, we obtain that $u \in E \cap F$ and $E \cap F \neq \emptyset$.

Now, we prove that u is a fixed point of T. For any $n \in N$ we have

$$\begin{aligned} d_{\mathbb{D}}^{b}\left(Tu,u\right) &= \lim_{n\to\infty} d_{\mathbb{D}}^{b}\left(Tu,T^{2n}x\right) \\ &\precsim \delta \lim_{n\to\infty}\left[d_{\mathbb{D}}^{b}\left(u,T^{2n-1}x\right)+d_{\mathbb{D}}^{b}\left(T^{2n}x,T^{2n-1}x\right)+d_{\mathbb{D}}^{b}\left(Tu,u\right)\right] \\ &\precsim \delta d_{\mathbb{D}}^{b}\left(Tu,u\right) \end{aligned}$$

and so $\left(1-\delta\right) d_{\mathbb{D}}^{b}\left(Tu,u\right)=0$. Since $0 \prec \delta \prec \dfrac{1}{3}$, we get $d_{\mathbb{D}}^{b}\left(Tu,u\right)=0$ and so $Tu = u.$

To show the uniqueness of fixed point, let $v \in E \cup F$ be another fixed point of T. Since T is $d_{\mathbb{D}}^{b}-$cyclic map we get $v \in E \cap F$. Then it follows from Definition 2.15 that

$$d_{\mathbb{D}}^{b}\left(u,v\right)=d_{\mathbb{D}}^{b}\left(Tu,Tv\right) \precsim \delta\left[d_{\mathbb{D}}^{b}\left(u,v\right)+d_{\mathbb{D}}^{b}\left(Tu,u\right)+d_{\mathbb{D}}^{b}\left(Tv,v\right)\right]$$

which is equivalent to $\left(1-\delta\right) d_{\mathbb{D}}^{b}\left(u,v\right)=0$ and hence $u=v$. Therefore, so u is a unique fixed point of T.

The next definition can be regarded as a generalization of Ćirić's contraction [15].

Definiton 2.17. Let $\left(X, d_{\mathbb{D}}^{b}\right)$ be a hvbms with $s \succsim 1$, $E,F \subset X$, $E \neq \emptyset$, $F \neq \emptyset$ and $T: E \cup F \to E \cup F$ be a $d_{\mathbb{D}}^{b}-$cyclic map. If there exists $0 \prec \eta \prec 1$ such that

$$d_{\mathbb{D}}^{b}\left(Tx,Ty\right) \precsim \eta U$$

where $U \in \left\{ d_{\mathrm{D}}^{b}\left(x,y\right), d_{\mathrm{D}}^{b}\left(Tx,x\right), d_{\mathrm{D}}^{b}\left(Ty,y\right) \right\}$ for all $x \in E$ and $y \in F$, then we say that T is a Ćirić type $d_{\mathrm{D}}^{b}-$cyclic contraction.

The following result is a new version of Ćirić's fixed point theorem [15] for classical metric spaces in hyperbolic valued $b-$metric spaces, as follows:

Theorem 2.18. Let $\left(X, d_{\mathrm{D}}^{b}\right)$ be a chvbms with $s \succsim 1$, $E,F \subset X$ be $d_{\mathrm{D}}^{b}-$closed subsets, $E \neq \varnothing$ and $F \neq \varnothing$ and $T : E \cup F \to E \cup F$ be a Ćirić type $d_{\mathrm{D}}^{b}-$cyclic contraction. Then T has a unique fixed point in $E \cap F$.

Proof. Let $x \in E$. Then we write $d_{\mathrm{D}}^{b}\left(Tx,Ty\right) \precsim \eta U$. If $U = d_{\mathrm{D}}^{b}\left(x,y\right)$, the desired result is seen from Theorem 2.4.

If $U = d_{\mathrm{D}}^{b}\left(Tx,x\right)$, then we obtain that $d_{\mathrm{D}}^{b}\left(T^{2}x,Tx\right) \precsim \eta d_{\mathrm{D}}^{b}\left(Tx,x\right)$ for $y = Tx \in F$. If we continue in the same manner, we get $d_{\mathrm{D}}^{b}\left(T^{n+1}x,T^{n}x\right) \precsim \eta^{n} d_{\mathrm{D}}^{b}\left(Tx,x\right)$. This implies that $\left(T^{n}x\right)$ is a $d_{\mathrm{D}}^{b}-$Cauchy sequence and there is an element $u \in E \cup F$ such that $T^{n}x \xrightarrow{d_{\mathrm{D}}^{b}} u.$ Since $x \in E$, $\left(T^{2n}x\right)$ is a sequence in E and $\left(T^{2n-1}x\right)$ is a sequence in F in a way that both sequences converge to u. Regarding $d_{\mathrm{D}}^{b}-$closedness of the sets E and F, we obtain that $u \in E \cap F$ and $E \cap F \neq \varnothing$.

Now, we show that $Tu = u$. We have

$$d_{\mathrm{D}}^{b}\left(Tu,u\right) = \lim_{n\to\infty} d_{\mathrm{D}}^{b}\left(Tu,T^{2n}x\right) \precsim \eta \lim_{n\to\infty} U = \eta d_{\mathrm{D}}^{b}\left(Tu,u\right)$$

for all $n \in N$ and so $\left(1-\eta\right) d_{\mathrm{D}}^{b}\left(Tu,u\right) = 0$. Since $0 \prec \eta \prec 1$, we get $d_{\mathrm{D}}^{b}\left(Tu,u\right) = 0$ which means that $Tu = u.$

To evidence the uniqueness of fixed point, let $v \in E \cup F$, $u \neq v$ and $Tv = v$. Since T is $d_{\mathrm{D}}^{b}-$cyclic map we get $v \in E \cap F$. Then it follows from Definiton 2.17 that

$$d_{\mathrm{D}}^{b}\left(u,v\right) = d_{\mathrm{D}}^{b}\left(Tu,Tv\right) \precsim \eta U = \eta d_{\mathrm{D}}^{b}\left(Tu,u\right) = 0,$$

and hence $u = v$. Therefore, so u is a unique fixed point of T.

If $U = d_{\mathrm{D}}^{b}\left(Ty,y\right)$, then we obtain that $d_{\mathrm{D}}^{b}\left(T^{2}x,Tx\right) \precsim \eta d_{\mathrm{D}}^{b}\left(T^{2}x,Tx\right)$ for $y = Tx \in F$. Since $0 \prec \eta \prec 1$, this is a contradiction.

On the other hand, we can state the following theorem as a new generalization of Zamfirescu's theorem [13] which are combination the contractive conditions of Banach [1], Kannan [11] and Chatterjea [12] in hyperbolic valued $b-$metric spaces.

Theorem 2.19. Let $\left(X, d_{\mathrm{D}}^{b}\right)$ be a chvbms with $s \succsim 1$, $E,F \subset X$ be $d_{\mathrm{D}}^{b}-$closed subsets, $E \neq \varnothing$ and $F \neq \varnothing$, $\alpha,\beta,\gamma \in \mathrm{D}^{+}$ such that $0 \prec \alpha \prec \frac{1}{s}$, $0 \prec \beta \prec \frac{1}{2}$ and $0 \prec \gamma \prec \frac{1}{2s}$ where $1-s\alpha$, $1-\beta$, $1-s\beta$ and $1-s\gamma$ are invertible, and $T : E \cup F \to E \cup F$. If T satisfies at least one of the conditions

$$d_{\mathrm{D}}^{b}\left(Tu,Tv\right) \precsim \alpha d_{\mathrm{D}}^{b}\left(u,v\right) \tag{3.1}$$

$$d_{\mathrm{D}}^{b}\left(Tu,Tv\right)\precsim \beta\left[d_{\mathrm{D}}^{b}\left(u,Tu\right)+d_{\mathrm{D}}^{b}\left(v,Tv\right)\right] \tag{3.2}$$

$$d_{\mathrm{D}}^{b}\left(Tu,Tv\right)\precsim \gamma\left[d_{\mathrm{D}}^{b}\left(u,Tv\right)+d_{\mathrm{D}}^{b}\left(v,Tu\right)\right] \tag{3.3}$$

for all $u\in E$ and $v\in F$, $u\neq v$, then T has a unique fixed point in $E\cap F$.

Remark 2.20. We notice that this theorem is analogous to Theorem 3.6 given in hyperbolic valued $b-$ metric spaces in [9] for cyclic contractions. But since the techniques are the same as others, we avoid proving the theorem. Also, Theorem 2.19 combines Theorem 2.4, Theorem 2.8 and Theorem 2.12.

3. CONCLUSION

In present work, we discuss the existence and uniqueness conditions for fixed points of self-mappings satisfying different types of cyclic contractive conditions on hyperbolic valued $b-$metric spaces by giving some numerical examples showing how our results can be used. We hope that our results will be attracted considerable interest from many authors for future works and applications to other related areas.

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ON FIXED CIRCLES IN HYPERBOLIC VALUED METRIC SPACES

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ABSTRACT

In this work, we give a concept of a fixed circle on a hyperbolic valued metric space. We also investigate some fixed circle theorems for self-mappings in different ways by supporting our newly obtained results with some numerical examples.

1. INTRODUCTION

Fixed point theory is a significant and popular topic in applications to various problems in nonlinear analysis, differential equations, approximation theory and control systems etc. Obtaining fixed points of different type contraction mappings has been the main goal of researchers. Many researchers have published various articles on fixed point theorems to extend the famous Banach contraction principle [1] with new and different contractive and expansive conditions in metric spaces or generalizations of metric spaces.

As a geometric approach to the fixed-point theory, fixed-circle problem in metric spaces raised by Özgür and Taş [2] has been developed very fast in recent times due to theoretical mathematical studies and some applications in various fields such as neural networks. New solutions of fixed-circle problem was investigated with various aspects and new contractive conditions on both metric spaces and some generalized metric spaces.

In 2021, Sager and Sağır [3] gave some existence and uniqueness theorems in hyperbolic valued metric spaces [presented by Kumar and Saini [4]] by defining hyperbolic contraction mapping. Other studies in this topic are [5], [6] and [7].

In the most existing literature, we observed that fixed circles have not been investigated in hyperbolic valued metric spaces. This motivates us to study fixed-circle problem in such spaces with geometric interpretation. So, in this work, we consider the fixed-circle problem on hyperbolic valued metric spaces and analyze its solutions for self-mappings on such spaces by defining some contractive conditions by means of some known techniques. Also, we discuss some nontrivial concrete examples to validate our hypotheses and to show the usability of our theoretical results.

Before starting our main theorems, we recall some known facts which will be needed in this discussion.

A bicomplex number is defined as $z = z_1 + jz_2$ where $j^2 = -1,\ ij = ji$, z_1 and z_2 are complex numbers, and i and j are independent imaginary units. The set of bicomplex numbers is denoted by $\mathbb{BC}$ and the set forms a Banach space with the operations $+, \cdot$ and the norm $|.|$

$$z + w = (z_1 + jz_2) + (w_1 + jw_2) = (z_1 + w_1) + j(z_2 + w_2),$$
$$\lambda.z = \lambda.(z_1 + jz_2) = \lambda z_1 + j\lambda z_2,$$
$$|.| : \mathbb{BC} \to R,\ z \to |z| = \sqrt{|z_1|^2 + |z_2|^2}$$

for all $z = z_1 + jz_2, w = w_1 + jw_2 \in \mathbb{BC}$ and for all $\lambda \in R.$ Also, $\mathbb{D} = \{a + kb\ :\ k = ij,\ a, b \in R\} \subset \mathbb{BC}$ is the set of hyperbolic numbers.

Three types of conjugates and moduli of $z = z_1 + jz_2 \in \mathbb{BC}$ are as follows:

$$z^{\dagger_1} = \overline{z_1} + j\overline{z_2},$$
$$z^{\dagger_2} = z_1 - jz_2,$$
$$z^{\dagger_3} = \overline{z_1} - j\overline{z_2},$$
$$|z|_i^2 = zz^{\dagger_2} = z_1^2 + z_2^2 \in C,$$
$$|z|_j^2 = zz^{\dagger_1} = \left(|z_1|^2 - |z_2|^2\right) + j\left(2\operatorname{Re}\left(z_1.\overline{z_2}\right)\right),$$
$$|z|_k^2 = zz^{\dagger_3} = \left(|z_1|^2 + |z_2|^2\right) + k\left(-\operatorname{Im}\left(z_1.\overline{z_2}\right)\right) \in \mathsf{D}.$$

The set $\left\{e_1 = \frac{1+ij}{2}, e_2 = \frac{1-ij}{2}\right\}$ is idempotent basis of the set of bicomplex numbers and so idempotent representation of $z = z_1 + jz_2$ is uniquely written as $z = e_1\beta_1 + e_2\beta_2$ where $\beta_1 = z_1 - iz_2,\ \ \beta_2 = z_1 + iz_2 \in C$ [8].

Let $\alpha = x + ky \in \mathsf{D}$. Then, $\alpha = e_1\alpha_1 + e_2\alpha_2 = e_1(x+y) + e_2(x-y)$ is the idempotent representation of α. Also, $\mathsf{D}^+ = \{\alpha = e_1\alpha_1 + e_2\alpha_2 \ :\ \alpha_1 \geq 0, \alpha_2 \geq 0\}$ is the set of positive hyperbolic numbers.

For $\alpha, \beta \in \mathsf{D}$, if $\beta - \alpha \in \mathsf{D}^+$ (or $\beta - \alpha \in \mathsf{D}^+ - \{0\}$), then we write $\alpha \precsim \beta$ (or $\alpha \underset{\neq}{\prec} \beta$). and also we have that

$$\alpha \precsim \beta \text{ if and only if } \alpha_1 \leq \beta_1 \text{ and } \alpha_2 \leq \beta_2,$$
$$\alpha \underset{\neq}{\prec} \beta \text{ if and only if } \alpha \neq \beta,\ \alpha_1 \leq \beta_1 \text{ and } \alpha_2 \leq \beta_2,$$
$$\alpha \prec \beta \text{ if and only if } \alpha_1 < \beta_1 \text{ and } \alpha_2 < \beta_2$$

for $\alpha = e_1\alpha_1 + e_2\alpha_2,\ \beta = \beta_1 e_1 + \beta_2 e_2 \in \mathsf{D}$ where $\alpha_1, \alpha_2, \beta_1, \beta_2 \in R$.

The followings hold for any $\alpha, \beta, \gamma, \delta \in \mathsf{D}$:

(i) If $\alpha \precsim \beta$ and $\gamma \precsim \delta$, then $\alpha + \gamma \precsim \beta + \delta$.

(ii) If $\alpha \precsim \beta$ and $0 \precsim \gamma$, then $\alpha\gamma \precsim \beta\gamma$.

(iii) If $\alpha, \beta \in \mathsf{D}^+$, then $\alpha \precsim \beta$ implies that $|\alpha| \leq |\beta|$.

(iv) If $\alpha \precsim \beta$ and $\beta \precsim \gamma$, then $\alpha \precsim \gamma$ [9].

Definition 1.1. [4] Let X be a nonempty set and let $d_{\mathsf{D}} \ :\ X \times X \to \mathsf{D}$ be a function such that the following properties hold:

(i) $0 \precsim d_{\mathsf{D}}(x,y)$, and $d_{\mathsf{D}}(x,y) = 0$ if and only if $x = y$.

(ii) $d_{\mathsf{D}}(x,y) = d_{\mathsf{D}}(y,x)$.

(iii) $d_{\mathsf{D}}(x,z) \precsim d_{\mathsf{D}}(x,y) + d_{\mathsf{D}}(y,z)$

for any $x, y, z \in X$. Then, d_{D} is said to be a $\mathsf{D}-$ valued or hyperbolic valued metric on X and the pair (X, d_{D}) is said to be a hyperbolic valued or $\mathsf{D}-$valued metric space.

2. MAIN RESULTS

Definition 2.1. Let (X, d_{D}) be a hyperbolic valued metric space, $x_0 \in X$ and $r \in \mathsf{D}_+$. Then, the circle with the centered x_0 and the radius r is defined by $C^{\mathbb{D}}_{x_0,r}=\left\{x\in X\ :\ d_{\mathbb{D}}\left(x,x_0\right)=r\right\}$.

Definition 2.2. Let $\left(X,d_{\mathbb{D}}\right)$ be a hyperbolic valued metric space, $C^{\mathbb{D}}_{x_0,r}$ be a circle on X and $T\ :\ X\rightarrow X$ be a self-mapping. The circle $C^{\mathbb{D}}_{x_0,r}$ is called as the fixed circle of T, if $Tx=x$ for all $x\in C^{\mathbb{D}}_{x_0,r}$.

2.1. The existence of fixed circles

In this part, we investigate the existence conditions of fixed-circles for self-mappings defining some contractive conditions in hyperbolic valued metric spaces.

Theorem 2.1.1. Let $C^{\mathbb{D}}_{x_0,r}$ be any circle on a hyperbolic valued metric space $\left(X,d_{\mathbb{D}}\right)$. Define the mapping $\phi\ :\ X\rightarrow \mathbb{D}^{+}$ as

$$\phi\left(x\right)=d_{\mathbb{D}}\left(x,x_0\right), \tag{1}$$

for all $x\in X$. If $T:X\rightarrow X$ satisfies the conditions

$$d_{\mathbb{D}}\left(x,Tx\right)\precsim\phi\left(x\right)+\phi\left(Tx\right)-2r \tag{2}$$

and

$$d_{\mathbb{D}}\left(Tx,x_0\right)\precsim r, \tag{3}$$

for all $x\in C^{\mathbb{D}}_{x_0,r}$, then $C^{\mathbb{D}}_{x_0,r}$ is a fixed circle of T.

Proof. Let x be arbitrary point in $C^{\mathbb{D}}_{x_0,r}$. Then, regarding (2), (1), the fact that $x\in C^{\mathbb{D}}_{x_0,r}$, (3) and the definition of the relation $\precsim$, we obtain

$$\begin{aligned} d_{\mathbb{D}}\left(x,Tx\right)&\precsim\phi\left(x\right)+\phi\left(Tx\right)-2r\\ &=d_{\mathbb{D}}\left(x,x_0\right)+d_{\mathbb{D}}\left(Tx,x_0\right)-2r\\ &=d_{\mathbb{D}}\left(Tx,x_0\right)-r\\ &\precsim r-r=0 \end{aligned}$$

and hence $d_{\mathbb{D}}\left(x,Tx\right)=0$ and $Tx=x$. Consequently, we observe that $C^{\mathbb{D}}_{x_0,r}$ is a fixed circle of T.

Remark 2.1.2. The inequality (2) says that Tx is not in the interior of $C^{\mathbb{D}}_{x_0,r}$ for any $x\in C^{\mathbb{D}}_{x_0,r}$. On the other hand, the inequality (3) implies that Tx is not in the exterior of $C^{\mathbb{D}}_{x_0,r}$ for any $x\in C^{\mathbb{D}}_{x_0,r}$. It follows that $T\left(C^{\mathbb{D}}_{x_0,r}\right)\subset C^{\mathbb{D}}_{x_0,r}$ regarding the conditions (2) and (3).

Example 2.1.3. Let $\left(X,d_{\mathbb{D}}\right)$ be a hyperbolic valued metric space and $x_0,\alpha\in X$ such that $d_{\mathbb{D}}\left(x_0,\alpha\right)\prec r$. Consider the circle $C^{\mathbb{D}}_{x_0,r}$. Let us define $T:X\rightarrow X$ as

$$Tx=\begin{cases} x &, \quad x\in C^{\mathbb{D}}_{x_0,r}\\ \alpha &, \quad x\notin C^{\mathbb{D}}_{x_0,r}\end{cases}.$$

Then, a simple calculation yields that T satisfies the inequalities (2) and (3). This implies that $C^{\mathbb{D}}_{x_0,r}$ is a fixed circle of T.

Theorem 2.1.4. Let $(X, d_{\mathbb{D}})$ be a hyperbolic valued metric space, the mapping ϕ be as in (1) and $C_{x_0,r}^{\mathbb{D}}$ be any circle on X. If T is a self-mapping defined on X satisfying the conditions

$$d_{\mathbb{D}}(x, Tx) \precsim \phi(x) - \phi(Tx) \tag{4}$$

and

$$d_{\mathbb{D}}(Tx, x_0) \succsim r \tag{5}$$

for all $x \in C_{x_0,r}^{\mathbb{D}}$, then $C_{x_0,r}^{\mathbb{D}}$ is a fixed circle of T.

Proof. Let x be any element in $C_{x_0,r}^{\mathbb{D}}$. Taking (4), (1), the fact that $x \in C_{x_0,r}^{\mathbb{D}}$, (5) and the definition of the relation $\precsim$ into account we write

$$d_{\mathbb{D}}(x, Tx) \precsim \phi(x) - \phi(Tx) = d_{\mathbb{D}}(x, x_0) - d_{\mathbb{D}}(Tx, x_0) = r - d_{\mathbb{D}}(Tx, x_0) \succsim r - r = 0$$

and so $d_{\mathbb{D}}(x, Tx) = 0$. This yields the equality $Tx = x$. Therefore, we get that $C_{x_0,r}^{\mathbb{D}}$ is a fixed circle of T.

Remark 2.1.5. The condition (4) indicates that Tx is not in the exterior of $C_{x_0,r}^{\mathbb{D}}$ for any $x \in C_{x_0,r}^{\mathbb{D}}$. In the same way, the inequality (5) shows that Tx is not in the interior of $C_{x_0,r}^{\mathbb{D}}$ for any $x \in C_{x_0,r}^{\mathbb{D}}$. As a result, these results say that $T\left(C_{x_0,r}^{\mathbb{D}}\right) \subset C_{x_0,r}^{\mathbb{D}}$ due to the inequalities (4) and (5).

Example 2.1.6. Let $(X, d_{\mathbb{D}})$ be a hyperbolic valued metric space and $x_0, \alpha \in X$ such that $d_{\mathbb{D}}(x_0, \alpha) \succ r$. Consider the circle $C_{x_0,r}^{\mathbb{D}}$. Let us define $T : X \to X$ as

$$Tx = \begin{cases} x & , \quad x \in C_{x_0,r}^{\mathbb{D}} \\ \alpha & , \quad x \notin C_{x_0,r}^{\mathbb{D}} \end{cases}.$$

Then, with a direct computation it can be seen that T satisfies (4) and (5). That is to say that $C_{x_0,r}^{\mathbb{D}}$ is a fixed circle of T.

Theorem 2.1.7. Let $(X, d_{\mathbb{D}})$ be a hyperbolic valued metric space, the mapping ϕ be as in (1) and $C_{x_0,r}^{\mathbb{D}}$ be any circle on X. If T is a self-mapping defined on X satisfying the conditions

$$d_{\mathbb{D}}(x, Tx) \precsim \phi(x) - \phi(Tx) \tag{6}$$

and

$$\lambda d_{\mathbb{D}}(x, Tx) + d_{\mathbb{D}}(Tx, x_0) \succsim r \tag{7}$$

for all $x \in C_{x_0,r}^{\mathbb{D}}$ and some $\lambda \in \mathbb{D}^+$ with $\lambda \prec 1$, then the circle $C_{x_0,r}^{\mathbb{D}}$ is a fixed circle of T.

Proof. Assume that $x \in C_{x_0,r}^{\mathbb{D}}$ and $x \neq Tx$. Then, we get

$$\begin{aligned} d_{\mathbb{D}}(x, Tx) \precsim \phi(x) - \phi(Tx) &= d_{\mathbb{D}}(x, x_0) - d_{\mathbb{D}}(Tx, x_0) \\ &= r - d_{\mathbb{D}}(Tx, x_0) \\ &\precsim \lambda d_{\mathbb{D}}(x, Tx) + d_{\mathbb{D}}(Tx, x_0) - d_{\mathbb{D}}(Tx, x_0) \\ &= \lambda d_{\mathbb{D}}(x, Tx) \end{aligned}$$

using the (6), (1) and (5), and so $d_{\mathbb{D}}(x,Tx)=0$. Thus, it should be $x=Tx$ for all $x\in C_{x_0,r}^{\mathbb{D}}$ and so this yields that $C_{x_0,r}^{\mathbb{D}}$ is a fixed circle of T.

Remark 2.1.8. The condition (6) shows that Tx is not in the exterior of the circle $C_{x_0,r}^{\mathbb{D}}$ for each $x\in C_{x_0,r}^{\mathbb{D}}$. In the same manner, the condition (7) guarantees that Tx is not in the interior of the circle $C_{x_0,r}^{\mathbb{D}}$ for each $x\in C_{x_0,r}^{\mathbb{D}}$. Then, it is apparent that $T\left(C_{x_0,r}^{\mathbb{D}}\right)\subset C_{x_0,r}^{\mathbb{D}}$ taking the conditions (6) and (7) into account.

Example 2.1.9. Define a function $d_{\mathbb{D}}:[0,\infty)\times[0,\infty)\to\mathbb{D}^{+}$ as

$$d_{\mathbb{D}}(x,y)=|x-y|e_1+2|x-y|e_2.$$

One can simply see that $\left([0,\infty),d_{\mathbb{D}}\right)$ is a hyperbolic valued metric space. Consider the

$$\begin{aligned}C_{0,e_1+2e_2}^{\mathbb{D}}&=\left\{x\in[0,\infty):d_{\mathbb{D}}(x,0)=e_1+2e_2\right\}\\&=\left\{x\in[0,\infty):|x|e_1+2|x|e_2=e_1+2e_2\right\}\\&=\left\{x\in[0,\infty):|x|=1\right\}\\&=\{1\}.\end{aligned}$$

Take a self-mapping T on $[0,\infty)$ as

$$Tx=\begin{cases}1 & , \quad x\in C_{0,e_1+2e_2}^{\mathbb{D}}\\ e_1+2e_2 & , \quad x\notin C_{0,e_1+2e_2}^{\mathbb{D}}\end{cases}.$$

Then, we get that T satisfies the conditions (6) and (7) with $\lambda=\frac{3}{10}e_1+\frac{1}{7}e_2\prec 1$, and we observe that the circle $C_{0,e_1+2e_2}^{\mathbb{D}}$ is a fixed circle of T.

Theorem 2.1.10. Let $(X,d_{\mathbb{D}})$ be a hyperbolic valued metric space, the mapping ϕ be as in (1) and $C_{x_0,r}^{\mathbb{D}}$ be any circle on X. T is a self-mapping defined on X satisfying the conditions

$$d_{\mathbb{D}}(x,Tx)\precsim\phi(x)-r \tag{8}$$

or

$$d_{\mathbb{D}}(x,Tx)\precsim\phi(Tx)-r, \tag{9}$$

and

$$d_{\mathbb{D}}(Tx,x_0)\precsim r+\lambda d_{\mathbb{D}}(x,Tx), \tag{10}$$

for all $x\in C_{x_0,r}^{\mathbb{D}}$ and some $\lambda\in\mathbb{D}^{+}$ with $\lambda\prec 1$, then the circle $C_{x_0,r}^{\mathbb{D}}$ is a fixed circle of T.

Proof. Assume that $x\in C_{x_0,r}^{\mathbb{D}}$. Then, if the condition (8) holds, we obtain

$$d_{\mathbb{D}}(x,Tx)\precsim\phi(x)-r=d_{\mathbb{D}}(x,x_0)-r=r-r=0$$

and so $d_{\mathbb{D}}(x,Tx)=0$. This implies that $Tx=x$.

Also, if the condition (9) holds, we establish by (10)

$$\begin{aligned} d_{\mathbb{D}}(x,Tx) &\precsim \phi(Tx)-r = d_{\mathbb{D}}(Tx,x_0)-r \\ &\precsim r+\lambda d_{\mathbb{D}}(x,Tx)-r \\ &= \lambda d_{\mathbb{D}}(x,Tx) \end{aligned}$$

and so $d_{\mathbb{D}}(x,Tx)=0$. Thus, we derive that $x=Tx$ for all $x \in C^{\mathbb{D}}_{x_0,r}$ and more precisely, $C^{\mathbb{D}}_{x_0,r}$ is a fixed circle of T.

Remark 2.1.11. The inequalities (8) and (9) guarantees that Tx is not in the interior of $C^{\mathbb{D}}_{x_0,r}$ for any $x \in C^{\mathbb{D}}_{x_0,r}$. In the same way, taking the inequality (10) into account, Tx is not in the exterior of $C^{\mathbb{D}}_{x_0,r}$ for any $x \in C^{\mathbb{D}}_{x_0,r}$. We obtain that $T\left(C^{\mathbb{D}}_{x_0,r}\right) \subset C^{\mathbb{D}}_{x_0,r}$ under the conditions (8) or (9) and (10).

Example 2.1.12. Define a function $d_{\mathbb{D}}: R \times R \to \mathbb{D}^+$ as

$$d_{\mathbb{D}}(x,y)=|x-y|e_1+2|x-y|e_2.$$

We can easily show that $(R,d_{\mathbb{D}})$ is a hyperbolic valued metric space. Consider the

$$\begin{aligned} C^{\mathbb{D}}_{0,e_1+2e_2} &= \{x \in R : d_{\mathbb{D}}(x,0)=e_1+2e_2\} \\ &= \{x \in R : |x|e_1+2|x|e_2=e_1+2e_2\} \\ &= \{x \in R : |x|=1\} \\ &= \{-1,+1\}. \end{aligned}$$

Take a self-mapping T on R as

$$Tx=\begin{cases} 1, & x=1 \\ -1, & x=-1 \\ e_1+2e_2, & x \notin C^{\mathbb{D}}_{0,e_1+2e_2} \end{cases}.$$

Then, we get that T satisfies the conditions (8), (9) and (10) with $\lambda=\frac{1}{3}e_1+\frac{1}{2}e_2 \prec 1$, and we observe that the circle $C^{\mathbb{D}}_{0,e_1+2e_2}$ is a fixed circle of T.

Let I_X be the identity map defined as $I_X(x)=x$ for all $x \in X$. We notice that the identity map satisfies the conditions in Theorem 2.1.1, Theorem 2.1.4, Theorem 2.1.7 and Theorem 2.1.10 for any circle. Now we find a new condition which excludes I_X in these theorems as follows:

Theorem 2.1.13. Let $(X,d_{\mathbb{D}})$ be a hyperbolic valued metric space, the mapping ϕ be as in (1) and $C^{\mathbb{D}}_{x_0,r}$ be any circle on X. Then, T is a self-mapping defined on X satisfying the condition

$$\lambda d_{\mathbb{D}}(x,Tx) \precsim \phi(x)-\phi(Tx) \tag{11}$$

for all $x \in X$ where $\lambda \in \mathbb{D}^+$ is an invertible element and $\lambda^{-1} \prec 1$ if and only if T fixes the circle $C^{\mathbb{D}}_{x_0,r}$ and $T=I_X$.

Proof. Assume that $T: X \to X$ satisfies the condition (11). Let $x \in X$. We imply that $x=Tx$. Suppose, contrarily that $x \neq Tx$. Then, utilizing the (11), (1) and (iii) given in Definition 1.1, we get

$$\begin{aligned}\lambda d_{\mathrm{D}}(x,Tx) &\precsim \phi(x)-\phi(Tx)\\ &= d_{\mathrm{D}}(x,x_0)-d_{\mathrm{D}}(Tx,x_0)\\ &\precsim d_{\mathrm{D}}(x,Tx)+d_{\mathrm{D}}(Tx,x_0)-d_{\mathrm{D}}(Tx,x_0)\\ &= d_{\mathrm{D}}(x,Tx)\end{aligned}$$

and so

$$d_{\mathrm{D}}(x,Tx)\precsim \lambda^{-1}d_{\mathrm{D}}(x,Tx).$$

But it is not possible. So, our assertion is held, that is, $x=Tx$ for all $x\in X$ and $T=I_X$.

On the contrary, suppose that T fixes $C^{\mathrm{D}}_{x_0,r}$ and $T=I_X$. Since $Tx=x$ for all $x\in X$, the inequality (11) holds for any invertible element $\lambda\in\mathrm{D}^+$ with $\lambda^{-1}\prec 1$. The proof is completed.

Remark 2.1.14. Theorem 2.1.13 indicates that if a self-mapping fixes a circle by satisfying the conditions (2) and (3) (or the conditions (4) and (5), or the conditions (6) and (7), or the conditions [(8) or (9)] and (10), but does not satisfy the condition (11), then the self-mapping cannot be identity map.

2.2. The uniqueness of fixed circles

In this part, we discuss the uniqueness of fixed circles in the existence theorems in Subsection 2.1. Also, the following example emphasizes that fixed circles of a self-mapping may not be unique.

Example 2.2.1. Define a function $d_{\mathrm{D}}:[0,1]\times[0,1]\to\mathrm{D}^+$ as

$$d_{\mathrm{D}}(x,y)=|x-y|e_1+2|x-y|e_2.$$

Then, it can be simply obtained that $([0,1],d_{\mathrm{D}})$ is a hyperbolic valued metric space. Consider the circles $C^{\mathrm{D}}_{0,\frac{1}{3}e_1+\frac{2}{3}e_2}$ and $C^{\mathrm{D}}_{1,\frac{1}{3}e_1+\frac{2}{3}e_2}$ as follows:

$$\begin{aligned}C^{\mathrm{D}}_{0,\frac{1}{3}e_1+\frac{2}{3}e_2} &= \left\{x\in[0,1]:d_{\mathrm{D}}(x,0)=\frac{1}{3}e_1+\frac{2}{3}e_2\right\}\\ &= \left\{x\in[0,1]:|x|e_1+2|x|e_2=\frac{1}{3}e_1+\frac{2}{3}e_2\right\}\\ &= \left\{x\in[0,1]:|x|=\frac{1}{3}\right\}\\ &= \left\{\frac{1}{3}\right\},\end{aligned}$$

$$\begin{aligned}C^{\mathrm{D}}_{1,\frac{1}{3}e_1+\frac{2}{3}e_2} &= \left\{x\in[0,1]:d_{\mathrm{D}}(x,1)=\frac{1}{3}e_1+\frac{2}{3}e_2\right\}\\ &= \left\{x\in[0,1]:|x-1|e_1+2|x-1|e_2=\frac{1}{3}e_1+\frac{2}{3}e_2\right\}\\ &= \left\{x\in[0,1]:|x-1|=\frac{1}{3}\right\}\\ &= \left\{\frac{2}{3}\right\}.\end{aligned}$$

Take a self-mapping T on $[0,1]$ as

$$Tx = \begin{cases} x^2 + \frac{2}{9} & , \quad x \in \left\{ \frac{1}{3}, \frac{2}{3} \right\} \\ 0 & , \quad x \notin \left\{ \frac{1}{3}, \frac{2}{3} \right\} \end{cases}.$$

Then, we get that T fixes the circles $C^{\mathrm{D}}_{0,\frac{1}{3}e_1+\frac{2}{3}e_2}$ and $C^{\mathrm{D}}_{1,\frac{1}{3}e_1+\frac{2}{3}e_2}$. This shows that the fixed circles of T is not unique.

Firstly, we focus on the uniqueness of fixed circles in Theorem 2.1.1 using Theorem 3.1 in [3] which are modified versions of Banach's fixed-point theorem [1] in the following theorem.

Theorem 2.2.2. Let $C^{\mathrm{D}}_{x_0,r}$ be any circle on a hyperbolic valued metric space (X, d_{D}). Suppose that $T : X \to X$ satisfies the conditions (2) and (3) given in Theorem 2.1.1. If

$$d_{\mathrm{D}}(Tx, Ty) \precsim \lambda d_{\mathrm{D}}(x, y), \tag{12}$$

for all $x \in C^{\mathrm{D}}_{x_0,r}$, $y \in X - C^{\mathrm{D}}_{x_0,r}$ and some $\lambda \in \mathrm{D}^+$ with $\lambda \prec 1,$ then $C^{\mathrm{D}}_{x_0,r}$ is unique fixed circle of $T.$

Proof. Suppose that $C^{\mathrm{D}}_{x_1,\delta},$ is another fixed circle of $T.$ Let x and y be any points in the circles $C^{\mathrm{D}}_{x_0,r}$ and $C^{\mathrm{D}}_{x_1,\delta},$ respectively. Then, considering the condition (12) we get

$$d_{\mathrm{D}}(x, y) = d_{\mathrm{D}}(Tx, Ty) \precsim \lambda d_{\mathrm{D}}(x, y).$$

So, $d_{\mathrm{D}}(x, y) = 0$ and $x = y$. Hence, the self-mapping T fixes only $C^{\mathrm{D}}_{x_0,r}.$

Now, we assign the uniqueness condition for the fixed circles in Theorem 2.1.4 which is an extension of Kannan's fixed-point condition [10].

Theorem 2.2.3. Let (X, d_{D}) be hyperbolic valued metric space, $C^{\mathrm{D}}_{x_0,r}$ be any circle on X and the T be a self-mapping providing the conditions (4) and (5) given in Theorem 2.1.4. If T satisfies the contraction condition

$$d_{\mathrm{D}}(Tx, Ty) \precsim \lambda\left[d_{\mathrm{D}}(Tx, x) + d_{\mathrm{D}}(Ty, y)\right] \tag{13}$$

for all $x \in C^{\mathrm{D}}_{x_0,r}$, $y \in X - C^{\mathrm{D}}_{x_0,r}$ and some $\lambda \in \mathrm{D}^+$ with $\lambda \prec \frac{1}{2},$ then the circle $C^{\mathrm{D}}_{x_0,r}$ is unique fixed circle of $T.$

Proof. Assume that $C^{\mathrm{D}}_{x_1,\delta}$ is another fixed circle of $T.$ Let x and y be arbitrary points in $C^{\mathrm{D}}_{x_0,r}$ and $C^{\mathrm{D}}_{x_1,\delta},$ respectively. Thus, we get by (13)

$$d_{\mathrm{D}}(x, y) = d_{\mathrm{D}}(Tx, Ty) \precsim \lambda\left[d_{\mathrm{D}}(Tx, x) + d_{\mathrm{D}}(Ty, y)\right] = 0$$

which implies $x = y.$ This guarantees that T fixes only $C^{\mathrm{D}}_{x_0,r}.$

In what follows, we find the uniqueness condition for the fixed circles in Theorem 2.1.7 which is a new version of Chatterjea's contractive condition [11].

Theorem 2.2.4. Let $C_{x_0,r}^{\mathbb{D}}$ be any circle on a hyperbolic valued metric space $(X, d_{\mathbb{D}})$. Assume that $T: X \to X$ satisfies the conditions (6) and (7) given in Theorem 2.1.7. If

$$d_{\mathbb{D}}(Tx,Ty) \precsim \lambda\left[d_{\mathbb{D}}(Tx,y)+d_{\mathbb{D}}(Ty,x)\right] \tag{14}$$

for all $x \in C_{x_0,r}^{\mathbb{D}}$, $y \in X - C_{x_0,r}^{\mathbb{D}}$ and some $\lambda \in \mathbb{D}^+$ with $\lambda \prec \frac{1}{2}$, then the circle $C_{x_0,r}^{\mathbb{D}}$ is unique fixed circle of T.

Proof. Assume that $C_{x_1,\delta}^{\mathbb{D}}$ is another fixed circle of T. Let $x \in C_{x_0,r}^{\mathbb{D}}$ and $y \in C_{x_1,\delta}^{\mathbb{D}}$. So we have by (14)

$$d_{\mathbb{D}}(x,y) = d_{\mathbb{D}}(Tx,Ty) \precsim \lambda\left[d_{\mathbb{D}}(Tx,y)+d_{\mathbb{D}}(Ty,x)\right] = 2\lambda d_{\mathbb{D}}(x,y).$$

This causes $d_{\mathbb{D}}(x,y)=0$ which means that $x=y$. Observe that the self-mapping T fixes only circle $C_{x_0,r}^{\mathbb{D}}$.

Now, we give our last three uniqueness theorems for the fixed circles in Theorem 2.1.1, Theorem 2.1.10 and Theorem 2.1.4 by modifying Ćirić's fixed point theorem [12] and Reich's fixed point theorem [13].

Theorem 2.2.5. Let $C_{x_0,r}^{\mathbb{D}}$ be any circle on a hyperbolic valued metric space $(X, d_{\mathbb{D}})$. Assume that $T: X \to X$ satisfies the conditions (2) and (3) given in Theorem 2.1.1. If there exists

$$U \in \left\{d_{\mathbb{D}}(x,y), d_{\mathbb{D}}(Tx,x), d_{\mathbb{D}}(Ty,y), d_{\mathbb{D}}(Ty,x), d_{\mathbb{D}}(Tx,y)\right\}$$

such that

$$d_{\mathbb{D}}(Tx,Ty) \precsim \lambda U, \tag{15}$$

for all $x \in C_{x_0,r}^{\mathbb{D}}$, $y \in X - C_{x_0,r}^{\mathbb{D}}$ and some $\lambda \in \mathbb{D}^+$ with $\lambda \prec 1$, then the circle $C_{x_0,r}^{\mathbb{D}}$ is unique fixed circle of T.

Proof. Assume that $C_{x_1,\delta}^{\mathbb{D}}$ is another fixed circle of T. Let x and y be arbitrary elements in $C_{x_0,r}^{\mathbb{D}}$ and $C_{x_1,\delta}^{\mathbb{D}}$, respectively. Thus, we get

$$d_{\mathbb{D}}(x,y) = d_{\mathbb{D}}(Tx,Ty) \precsim \lambda U$$

from the condition (15). This implies that $d_{\mathbb{D}}(x,y) \precsim \lambda d_{\mathbb{D}}(x,y)$ or $d_{\mathbb{D}}(x,y) \precsim 0$. So $d_{\mathbb{D}}(x,y)=0$ and $x=y$. This means that the self-mapping T fixes only one circle.

Theorem 2.2.6. Let $C_{x_0,r}^{\mathbb{D}}$ be any circle on a hyperbolic valued metric space $(X, d_{\mathbb{D}})$. Assume that $T: X \to X$ satisfies the conditions (8) or (9) and (10) given in Theorem 2.1.10. If there exists

$$V \in \left\{d_{\mathbb{D}}(Tx,x), d_{\mathbb{D}}(Ty,y), d_{\mathbb{D}}(Ty,x), d_{\mathbb{D}}(Tx,y)\right\}$$

such that

$$d_{\mathbb{D}}(Tx,Ty) \precsim \lambda d_{\mathbb{D}}(x,y) + \mu V, \tag{16}$$

for all $x \in C_{x_0,r}^{\mathbb{D}}$, $y \in X - C_{x_0,r}^{\mathbb{D}}$ some $\lambda, \mu \in \mathbb{D}^+$ with $\lambda \prec \frac{1}{2}$ and $\mu \prec \frac{1}{2}$ then the circle $C_{x_0,r}^{\mathbb{D}}$ is unique fixed circle of T.

Proof. Let $x \in C_{x_0,r}^{\mathbb{D}}$ and $y \in C_{x_1,\delta}^{\mathbb{D}}$ assuming that $C_{x_1,\delta}^{\mathbb{D}}$ is another fixed circle of T. Then, we get by (16)

$$d_{\mathsf{D}}(x,y) = d_{\mathsf{D}}(Tx,Ty) \precsim \lambda d_{\mathsf{D}}(x,y) + \mu V.$$

This implies that $d_{\mathsf{D}}(x,y) \precsim \lambda d_{\mathsf{D}}(x,y)$ or $d_{\mathsf{D}}(x,y) \precsim (\lambda+\mu) d_{\mathsf{D}}(x,y)$. Since $\lambda \prec \frac{1}{2}$ and $\lambda + \mu \prec 1$, we have $d_{\mathsf{D}}(x,y) = 0$. As a result it apparent that $x = y$, so T fixes only circle $C^{\mathsf{D}}_{x_0,r}$.

Theorem 2.2.7. Let (X, d_{D}) be a hyperbolic valued metric space, $C^{\mathsf{D}}_{x_0,r}$ be any circle on X and T be a self-mapping satisfying the conditions (4) and (5) given in Theorem 2.1.4. If T satisfies the contraction condition such that

$$d_{\mathsf{D}}(Tx,Ty) \precsim \lambda d_{\mathsf{D}}(x,y) + \mu d_{\mathsf{D}}(Tx,x) + \nu d_{\mathsf{D}}(Ty,y), \tag{17}$$

for all $x \in C^{\mathsf{D}}_{x_0,r}$, $y \in X - C^{\mathsf{D}}_{x_0,r}$ and some $\lambda, \mu, \nu \in \mathsf{D}^+$ with $\lambda \prec \frac{1}{3}$, $\mu \prec \frac{1}{3}$ and $\nu \prec \frac{1}{3}$ then the circle $C^{\mathsf{D}}_{x_0,r}$ is unique fixed circle of T.

Proof. Suppose that $C^{\mathsf{D}}_{x_1,\delta}$ is another fixed circle of T. Let x and y be any points in $C^{\mathsf{D}}_{x_0,r}$ and $C^{\mathsf{D}}_{x_1,\delta}$, respectively. Therefore, we get by (17)

$$d_{\mathsf{D}}(x,y) = d_{\mathsf{D}}(Tx,Ty) \precsim \lambda d_{\mathsf{D}}(x,y) + \mu d_{\mathsf{D}}(Tx,x) + \nu d_{\mathsf{D}}(Ty,y) \prec \frac{1}{3} d_{\mathsf{D}}(x,y),$$

which implies $d_{\mathsf{D}}(x,y) = 0$ and $x = y$. This gives that T fixes only circle $C^{\mathsf{D}}_{x_0,r}$.

Remark 2.2.8. The uniqueness theorems obtained in this subsection for hyperbolic valued metric spaces are analogues of Theorem 3.1, Theorem 3.2 and Theorem 3.3 in [2] for metric spaces. The uniqueness theorems given in subsection 2.2 can be also rewritten using the contraction conditions in other uniqueness theorems given in the same subsection. But since the techniques are the same as another, we avoid listing all possible corollaries.

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