7th International IFS and Contemporary Mathematics Conference

May, 25-29, 2021 Turkey

CONFERENCE PROCEEDING BOOK

EDITOR

ASSOC. PROF. DR. Gökhan ÇUVALCIOĞLU

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PREFACE

We are very pleased to introduce the abstracts of the 7th International IFS and Contemporary Mathematics Conference (IFSCOM2021).
As previous conferences, the theme was the link between the Mathematics by many valued logics and its applications.

In this context, there is a need to discuss the relationships and interactions between many valued logics and contemporary mathematics.

Finally,in the previous conference, it made suceessful activities to communicate with scientists working in similar fields and relations between the different disciplines.

This conference has papers in different areas; multi-valued logic, geometry, algebra, applied mathematics, theory of fuzzy sets, intuitionistic fuzzy set theory, mathematical physics, mathematics applications, etc.

Thank you to all paticipants scientists offering the most significant contribution to this conference.

Thank you to Scientific Committee Members, Referee Committee Members, Local Committee Members and MAJOR TEAM supporting this conference.

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$\mathbb{B}$-CONVEXITY AND $\mathbb{B}$-CONCAVITY PRESERVING PROPERTY OF TWO-DIMENSIONAL BERNSTEIN OPERATORS

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Abstract. In this study, we present some properties regarding $\mathbb{B}$-convex and $\mathbb{B}$-concave functions. Also, it has been determined whether the convexity properties of these functions are preserved by Bernstein operators of two variables. Consequently, we give some examples of which Bernstein polynomials of two variables do not preserve convexity properties of these functions. In addition, of these convexities, results are given regarding conditions it will be preserved.

1. Introduction

Let $f:[0,1]\to\mathbb{R}$ and $n\in\mathbb{N}$. The n. Bernstein polynomial related to f is defined by

$$B_n(f)(x)=\sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k}f\left(\frac{k}{n}\right).$$

For $f\in C[0,1]$, the sequence $\{B_n(f)\}_{n=1}^{\infty}$ uniformly converges to f [5]. Also, various convexities which functions have are provided by Bernstein polynomials. Some of them can be found in [4]. In addition, the Bernstein polynomials do not preserve $\mathbb{B}$-convexity but preserve the property $\mathbb{B}$-concavity of functions [6]. Similarly, in this work we study same argument regarding $\mathbb{B}$-convexity and $\mathbb{B}$-concavity for Bernstein poylnomials of two variables.

For $n,m\in\mathbb{N}$ and $f:[0,1]\times[0,1]\to\mathbb{R}$, two-dimensional generalization of the Bernstein polynomial of degree (n,m) related to f is defined by

$$B_{n,m}f(x,y)=\sum_{k=0}^{n}\sum_{i=0}^{m}p_{n,k}(x)p_{m,i}(y)f\left(\frac{k}{n},\frac{i}{m}\right)$$

where $p_{n,k}(t)=\binom{n}{k}t^k(1-t)^{n-k}$ [2].

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Key words and phrases. Bernstein operators, $\mathbb{B}$-convex function, $\mathbb{B}$-concave function, Shape preserving approximation.

2. Preliminaries

We use the notations $\mathbb{R}_+^n := \{(x_1,...,x_n) : x_i \geq 0, \quad i = 1,...,n\}$ and for $x = (x_1,...,x_n), y = (y_1,...,y_n) \in \mathbb{R}_+^n$,

$$x \vee y := (\max\{x_1,y_1\}, \max\{x_2,y_2\}, ..., \max\{x_n,y_n\}).$$

Definition 2.1. [3] A function $f : [0,1] \to \mathbb{R}$ is called a starshaped function if $f(\lambda x) \leq \lambda f(x)$ for all $x \in [0,1]$ and $\lambda \in [0,1]$.

Definition 2.2. [7] Let $U \subset \mathbb{R}_+^n$. U is $\mathbb{B}$-convex iff $\lambda x \vee y \in U$ for all $x, y \in U$ and $\lambda \in [0,1]$.

Definition 2.3. [1] A function, $f : U \subset \mathbb{R}_+^n \to \mathbb{R}_+$ is said to be a $\mathbb{B}$-convex function iff U is $\mathbb{B}$-convex and the inequality $f(\lambda x \vee y) \leq \lambda f(x) \vee f(y)$ holds for all $x, y \in U$ and $\lambda \in [0,1]$.
If U is $\mathbb{B}$-convex and the inequality $f(\lambda x \vee y) \geq \lambda f(x) \vee f(y)$ holds for all $x, y \in U$ and $\lambda \in [0,1]$ then f is called a $\mathbb{B}$-concave function.

Lemma 2.4. [6] *If $f : [0,1] \to \mathbb{R}_+$ is a starshaped function, then f is a $\mathbb{B}$-convex function.*

Theorem 2.5. [6] *Let $f : [0,1] \to \mathbb{R}_+$. Then f is a $\mathbb{B}$-concave function iff f is increasing and $f(\lambda x) \geq \lambda f(x)$ for all $x \in [0,1]$ and $\lambda \in [0,1]$.*

Definition 2.6. [4] Let $f : [0,1] \times [0,1] \to \mathbb{R}$. Then partial derivatives of $B_{n,m}f$ are given by

$$\frac{\partial B_{n,m}f(x,y)}{\partial x} = n \sum_{k=0}^{n-1} \sum_{i=0}^{m} p_{n-1,k}(x) p_{m,i}(y) \left[f\left(\frac{k+1}{n}, \frac{i}{m}\right) - f\left(\frac{k}{n}, \frac{i}{m}\right) \right],$$

$$\frac{\partial B_{n,m}f(x,y)}{\partial y} = m \sum_{k=0}^{n} \sum_{i=0}^{m-1} p_{n,k}(x) p_{m-1,i}(y) \left[f\left(\frac{k}{n}, \frac{i+1}{m}\right) - f\left(\frac{k}{n}, \frac{i}{m}\right) \right]$$

where $p_{n,k}(t) = \binom{n}{k} t^k (1-t)^{n-k}$ and $n, m \in \mathbb{N}$.

3. Main Results

In this section, we consider the set $U \subset \mathbb{R}_+^2$ as any $\mathbb{B}$-convex set and $V \subset \mathbb{R}_+^2$ as $\mathbb{B}$-convex set such that $(0,0) \in V$.

Theorem 3.1. *Let $f : [0,1] \to \mathbb{R}_+$ be a $\mathbb{B}$-concave function. Then f is continuous on $[0,1]$ iff f is right continuous at 0.*

Proof. If f is continuous on $[0,1]$ then it is right continuous at 0. Conversely, suppose that f is discontinuous at $a \in (0,1]$. Since f is increasing on $[0,1]$ (Teorem 2.5), then f has jump discontinuity at a. Therefore, following situations are valid:
If $a \in (0,1)$ then $f(a^+) \neq f(a^-)$ and $f(a^-) \leq f(a) \leq f(a^+)$.

(i) Let $f(a^-) < f(a) \leq f(a^+)$. Then for $\lambda = (f(a) + f(a^-))/2f(a) < 1$, we have $\lambda a < a$ and $f(\lambda a) \leq f(a) < \lambda f(a) < f(a)$.
(ii) Let $f(a^-) = f(a)$. Then for $\lambda = (f(a^+) + f(a^-))/2f(a^+) < 1$ and $c \in (a,1)$ such that $\lambda c = a$, we have

$$f(\lambda c) = f(a) < \frac{(f(a^+) + f(a^-))}{2f(a^+)} f(a^+) \leq \frac{(f(a^+) + f(a^-))}{2f(a^+)} f(c) = \lambda f(c).$$

If $a = 1$, then $f(1^-) < f(1)$. For $\lambda = (f(1) + f(1^-))/2f(1) < 1$, we have $f(\lambda) < f(1)$ and $f(\lambda) < \lambda f(1)$. Since, in each case $\lambda \in [0,1]$ can be found such that $f(\lambda x) < \lambda f(x)$, this contradicts with $\mathbb{B}$-concavity of f. □

Using the term of strictly increasing instead of increasing, we correct the Lemma 2.3. in [6] as following:

Lemma 3.2. *Let $f : [0,1] \to \mathbb{R}_+$ be a strictly increasing function. Then f is a $\mathbb{B}$-convex function iff f is a starshaped function.*

Proof. If f is a $\mathbb{B}$-convex function then we have the inequality $f(\lambda x \vee y) \leq \lambda f(x) \vee f(y)$ for all $x, y \in [0,1]$ and $\lambda \in [0,1]$. Firstly, for $\lambda \neq 0$, $x \neq 0$, let us take $y = 0$. Then

$$f(\lambda x \vee 0) = f(\lambda x) \leq \lambda f(x) \vee f(0).$$

Also, considering $f(\lambda x) > f(0)$, we get the inequality $f(\lambda x) \leq \lambda f(x)$ for all $x, \lambda \in (0,1]$. Finally, let (λ_n) be a sequence in $(0,1]$ such that $\lim_{n\to\infty} \lambda_n = 0$. Since $f(\lambda_n x) \leq \lambda_n f(x)$ for all $n \in \mathbb{N}$ and $x \in (0,1]$ then we have

$$\lim_{n\to\infty} f(\lambda_n x) = f(0) \leq 0.$$

This inequality shows that $f(0) = 0$ because of $f(x) \geq 0$. Consequently $f(\lambda x) \leq \lambda f(x)$ holds for all $x \in [0,1]$ and $\lambda \in [0,1]$.
Conversely, if f is a starshaped function then f is a $\mathbb{B}$-convex function from Lemma 2.4. □

Lemma 3.3. *If $f : U \to \mathbb{R}_+$ is decreasing with respect to x and y, then f is a $\mathbb{B}$-convex function.*

Proof. Let (x_1, y_1), $(x_2, y_2) \in U$ and $\lambda \in [0,1]$. Then we have following inequalities:

(i) If $\lambda(x_1, y_1) \vee (x_2, y_2) = (\lambda x_1, \lambda y_1)$, then

$$f(\lambda x_1, \lambda y_1) \leq f(x_2, y_2) \leq \lambda f(x_1, y_1) \vee f(x_2, y_2)$$

(ii) If $\lambda(x_1, y_1) \vee (x_2, y_2) = (\lambda x_1, y_2)$, then

$$f(\lambda x_1, y_2) \leq f(x_2, y_2) \leq \lambda f(x_1, y_1) \vee f(x_2, y_2)$$

(iii) If $\lambda(x_1, y_1) \vee (x_2, y_2) = (x_2, \lambda y_1)$, then

$$f(x_2, \lambda y_1) \leq f(x_2, y_2) \leq \lambda f(x_1, y_1) \vee f(x_2, y_2)$$

(iv) If $\lambda(x_1, y_1) \vee (x_2, y_2) = (x_2, y_2)$, then

$$f(x_2, y_2) \leq \lambda f(x_1, y_1) \vee f(x_2, y_2).$$

Consequently, from above four cases, the inequality $f(\lambda(x_1, y_1) \vee (x_2, y_2)) \leq \lambda f(x_1, y_1) \vee f(x_2, y_2)$ is provided for all (x_1, y_1), $(x_2, y_2) \in U$ and $\lambda \in [0,1]$. □

Theorem 3.4. *If $f : [0,1] \times [0,1] \to \mathbb{R}_+$ is decreasing with respect to x and y, then $B_{n,m} f(x,y)$ is $\mathbb{B}$-convex for all $n, m \in \mathbb{N}$.*

Proof. From Definition 2.6 it is clear that if $f : [0,1] \times [0,1] \to \mathbb{R}_+$ is decreasing with respect to x and y, then for all $(x,y) \in [0,1] \times [0,1]$

$$\frac{\partial B_{n,m} f(x,y)}{\partial x} \leq 0$$

$$\frac{\partial B_{n,m} f(x,y)}{\partial y} \leq 0.$$

Since the Bernstein polynomials of functions which are decreasing with respect to x, y are also decreasing with respect to x, y, now the proof is clear from Lemma 3.3. □

Remark 3.5. The Bernstein operators $B_{n,m}$ do not preserve the property $\mathbb{B}$-convexity of functions.

Example
Let $f : [0,1] \times [0,1] \to \mathbb{R}_+$ be defined by $f(x) = (x-y)^2/(1+y)$. This function provides the inequality $f(\lambda(x_1,y_1) \vee (x_2,y_2)) \le \lambda f(x_1,y_1) \vee f(x_2,y_2)$ for all (x_1,y_1), $(x_2,y_2) \in U$ and $\lambda \in [0,1]$. But $B_{1,1}f$ do not provide the inequality. For example, if we take $(x_1,y_1) = (1,1)$, $(x_2,y_2) = (0,0)$ and $\lambda = 1/2$ then

$$B_{1,1}f(x,y) = \sum_{k=0}^{1}\sum_{i=0}^{1} x^k(1-x)^{1-k}y^i(1-y)^{1-i}f(k,i) = \frac{y+2x-3xy}{2},$$

$$B_{1,1}f(\lambda(x_1,y_1)) = B_{1,1}f\left(\frac{1}{2},\frac{1}{2}\right) = \frac{3}{8},$$
$$B_{1,1}f(x_2,y_2) = B_{1,1}f(0,0) = 0,$$
$$\lambda B_{1,1}f(x_1,y_1) = \frac{1}{2}B_{1,1}f(1,1) = 0.$$

From above equalities we get following inequality

$$B_{1,1}f(\lambda(x_1,y_1) \vee (x_2,y_2)) = B_{1,1}f\left(\frac{1}{2},\frac{1}{2}\right) = \frac{3}{8} > 0 = \lambda B_{1,1}f(x_1,y_1) \vee B_{1,1}(x_2,y_2).$$

Consequently, for $n = 1$, $m = 1$, Bernstein polynomial $B_{1,1}f$ related to the function f is not $\mathbb{B}$-convex.

Theorem 3.6. *Let $f : V \to \mathbb{R}_+$. Then f is a $\mathbb{B}$-concave function iff it has the following properties:*

(i) f is increasing with respect to x and y,
(ii) the inequality $f(\lambda x, \lambda y) \ge \lambda f(x,y)$ is provided for all $(x,y) \in V$ and $\lambda \in [0,1]$.

Proof. Let f be a $\mathbb{B}$-concave function. So, the inequality $f(\lambda(x_1,y_1),(x_2,y_2)) \ge \lambda f(x_1,y_1) \vee f(x_2,y_2)$ holds for all $(x_1,y_1),(x_2,y_2) \in V$ and $\lambda \in [0,1]$. if we take $\lambda = 1$, then for all $h \ge 0$ providing $(x+h,y),(x,y) \in V$ and $k \ge 0$ providing $(x,y+k),(x,y) \in V$, we have

$$f((x+h,y) \vee (x,y)) = f(x+h,y) \ge f(x+h,y) \vee f(x,y) \ge f(x,y)$$
$$f((x,y+k) \vee (x,y)) = f(x,y+k) \ge f(x,y+k) \vee f(x,y) \ge f(x,y).$$

From these inequalities we get (i) . Also for all $(x,y) \in V$ and $\lambda \in [0,1]$, we have (ii), from the following inequality:

$$f(\lambda(x,y) \vee (0,0)) = f(\lambda x, \lambda y) \ge \lambda f(x,y) \vee f(0,0) \ge f(0,0)$$

For second part of proof, let f holds (i) and (ii). Let $(x_1,y_1),(x_2,y_2) \in V$ and $\lambda \in [0,1]$;
If $\lambda(x_1,y_1) \vee (x_2,y_2) = (\lambda x_1, \lambda y_1)$, then we get $f(\lambda x_1, \lambda y_1) \ge \lambda f(x_1,y_1)$ from (ii) and $f(\lambda x_1, \lambda y_1) \ge f(x_2,y_2)$ from (i). Therefore, the following inequality is true:

$$f(\lambda x_1, \lambda y_1) \ge \lambda f(x_1,y_1) \vee f(x_2,y_2).$$

If $\lambda(x_1, y_1) \vee (x_2, y_2) = (\lambda x_1, y_2)$, then we get $f(\lambda x_1, y_2) \geq f(x_2, y_2)$ from (i) and $f(\lambda x_1, y_2) \geq f(\lambda x_1, \lambda y_1) \geq \lambda f(x_1, y_1)$ from (i),(ii). Then, we have

$$f(\lambda x_1, y_2) \geq \lambda f(x_1, y_1) \vee f(x_2, y_2).$$

If $\lambda(x_1, y_1) \vee (x_2, y_2) = (x_2, \lambda y_1)$, then we get $f(x_2, \lambda y_1) \geq f(x_2, y_2)$ from (i) and $f(x_2, \lambda y_1) \geq f(\lambda x_1, \lambda y_1) \geq \lambda f(x_1, y_1)$ from (i),(ii). Then, we have

$$f(x_2, \lambda y_1) \geq \lambda f(x_1, y_1) \vee f(x_2, y_2).$$

If $\lambda(x_1, y_1) \vee (x_2, y_2) = (x_2, y_2)$, then we get $f(x_2, y_2) \geq f(\lambda x_1, \lambda y_1) \geq \lambda f(x_1, y_1)$ from (i),(ii). Then, we have

$$f(x_2, y_2) \geq \lambda f(x_1, y_1) \vee f(x_2, y_2).$$

Consequently, these inequalities show that f is $\mathbb{B}$-concave on V.

□

Proposition 1. The finite sum of nonnegative $\mathbb{B}$-concave functions defined on $V \subset \mathbb{R}_+$ is also $\mathbb{B}$-concave function.

Proof. Let $n > 1$ be an integer and $f_1, ...f_n$ are nonnegative $\mathbb{B}$-concave functions defined on V. Thus, for all $i = 1, ..., n$, f_i is increasing and satisfies the inequality $f_i(\lambda x, \lambda y) \geq \lambda f_i(x, y)$ where $x, y \in V$, $\lambda \in [0, 1]$. It is clear that the function $f : V \to \mathbb{R}_+$ defined by $f(x, y) = f_1(x, y) + ... + f_n(x, y)$ is also increasing and satisfies the inequality $f(\lambda x, \lambda y) \geq \lambda f(x, y)$ for all $x, y \in V$, $\lambda \in [0, 1]$. □

Remark 3.7. The Bernstein operators $B_{n,m}$ do not preserve the property $\mathbb{B}$-concavity of functions.

Example
Let $f : [0, 1] \times [0, 1] \to \mathbb{R}_+$ be defined by

$$f(x, y) = \begin{cases} \frac{xy}{x+y}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0). \end{cases}$$

From Theorem 3.6, this function is $\mathbb{B}$-concave. For $n = 1$ and $m = 1$, we have

$$B_{1,1}f(x, y) = \sum_{k=0}^{1} \sum_{i=0}^{1} x^k (1 - x)^{1-k} y^i (1 - y)^{1-i} f(k, i) = \frac{xy}{2}.$$

In this case for all $x, y, \lambda \in (0, 1)$, the inequality

$$B_{1,1}f(\lambda x, \lambda y) = \frac{\lambda^2 xy}{2} < \frac{\lambda xy}{2} = \lambda B_{1,1}f(x, y)$$

holds. Therefore, the condition (ii) in Theorem 3.6 fails for $B_{1,1}f$. Consequently, $B_{1,1}f$ do not provide the property of $\mathbb{B}$-concavity of function f.

We are to use the following lemmas to give $\mathbb{B}$-concave function examples preserved by Bernstein polynomials of two variables.

Lemma 3.8. *Let $g : [0, 1] \to \mathbb{R}_+$ be any nonnegative function and $f : [0, 1] \times [0, 1] \to \mathbb{R}_+$ be defined by $f(x, y) = g(x)$. If $f_x \geq 0$ and $x f_x(x, y) - f(x, y) \leq 0$ for all $(x, y) \in [0, 1] \times [0, 1]$ then f is a $\mathbb{B}$-concave function.*

Proof. Because of $f_x(x,y) \geq 0$ and $f_y(x,y) = 0$, f is increasing with respect to x and y. Also, from the following inequality

$$\left(\frac{f(x,y)}{x}\right)_x = xf_x(x,y) - f(x,y) \leq 0,$$

$f(x,y)/x$ is decreasing. For this reason, we have

$$\frac{f(\lambda x, \lambda y)}{\lambda x} = \frac{f(\lambda x, y)}{\lambda x} \geq \frac{f(x,y)}{x}.$$

This shows that $f(\lambda x, \lambda y) \geq \lambda f(x,y)$. Therefore, f is $\mathbb{B}$-concave from Teorem 3.6. □

Lemma 3.9. *Let $h : [0,1] \to \mathbb{R}_+$ be any nonnegative function and $f : [0,1]\times[0,1] \to \mathbb{R}_+$ be defined by $f(x,y) = h(y)$. If $f_y \geq 0$ and $yf_y(x,y) - f(x,y) \leq 0$ for all $(x,y) \in [0,1] \times [0,1]$, then f is a $\mathbb{B}$-concave function..*

Proof. The proof is similar to the proof of Lemma 3.8. □

Theorem 3.10. *Let $f : [0,1] \times [0,1] \to \mathbb{R}_+$ be given by $f(x,y) = g(x)$. If f is a $\mathbb{B}$-concave function then the Bernstein polynomial $B_{n,m}f$ is also $\mathbb{B}$-concave for all $n, m \in \mathbb{N}$.*

Proof. Let $n, m \in \mathbb{N}$ and $T_{n,m} := (B_{n,m}f(x,y))_x - (B_{n,m}f(x,y)/x)$. Due to $\mathbb{B}$-concavity of f, f is increasing and satisfies the inequality $f(\lambda x, \lambda y) = f(\lambda x, y) \geq \lambda f(x,y)$ for all $(x,y) \in [0,1] \times [0,1]$ and $\lambda \in [0,1]$. Therefore,

$$\frac{\partial B_{n,m}f(x,y)}{\partial x} = n\sum_{k=0}^{n-1}\sum_{i=0}^{m} p_{n-1,k}(x)p_{m,i}(y)\left[f\left(\frac{k+1}{n}, \frac{i}{m}\right) - f\left(\frac{k}{n}, \frac{i}{m}\right)\right] \geq 0$$

and considering the following equality:

$$\frac{1}{x}\sum_{k=1}^{n}\sum_{i=0}^{m} p_{n,k}(x)p_{m,i}(y)f\left(\frac{k}{n}, \frac{i}{m}\right) = \frac{n}{k+1}\sum_{k=0}^{n-1}\sum_{i=0}^{m} p_{n-1,k}(x)p_{m,i}(y)f\left(\frac{k+1}{n}, \frac{i}{m}\right),$$

then we get

$$\begin{aligned} T_{n,m} &= n\sum_{k=0}^{n-1}\sum_{i=0}^{m} p_{n-1,k}(x)p_{m,i}(y)\left[f\left(\frac{k+1}{n}, \frac{i}{m}\right) - f\left(\frac{k}{n}, \frac{i}{m}\right)\right] \\ &- x^{-1}\sum_{k=0}^{n}\sum_{i=0}^{m} p_{n,k}(x)p_{m,i}(y)f\left(\frac{k}{n}, \frac{i}{m}\right) \\ &\leq n\sum_{k=0}^{n-1}\sum_{i=0}^{m} p_{n-1,k}(x)p_{m,i}(y)\left[f\left(\frac{k+1}{n}, \frac{i}{m}\right) - f\left(\frac{k}{n}, \frac{i}{m}\right)\right] \\ &- x^{-1}\sum_{k=1}^{n}\sum_{i=0}^{m} p_{n,k}(x)p_{m,i}(y)f\left(\frac{k}{n}, \frac{i}{m}\right) \\ &= n\sum_{k=0}^{n-1}\sum_{i=0}^{m} p_{n-1,k}(x)p_{m,i}(y)\left[\frac{k}{k+1}f\left(\frac{k+1}{n}, \frac{i}{m}\right) - f\left(\frac{k}{n}, \frac{i}{m}\right)\right] \end{aligned}$$

From the last equality, we have

$$\frac{k}{k+1}f\left(\frac{k+1}{n},\frac{i}{m}\right)-f\left(\frac{k}{n},\frac{i}{m}\right)\leq f\left(\frac{k}{k+1}\frac{k+1}{n},\frac{i}{m}\right)-f\left(\frac{k}{n},\frac{i}{m}\right)=0$$

and considering $p_{n,k}(t)\geq 0$, we get the following inequality:

$$\left(B_{n,m}f(x,y)\right)_x-\frac{B_{n,m}f(x,y)}{x}\leq 0.$$

Thus, the conditions in Lemma 3.8 are provided. □

Theorem 3.11. *Let $f:[0,1]\times[0,1]\to\mathbb{R}_+$ given by $f(x,y)=h(y)$. If f is a $\mathbb{B}$-concave function then the Bernstein polynomial $B_{n,m}f$ is also $\mathbb{B}$-concave for all $n,m\in\mathbb{N}$.*

Proof. The proof is similar to the proof of Theorem 3.10. □

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GLOBAL NONEXISTENCE OF SOLUTIONS FOR A SYSTEM OF VISCOELASTIC PLATE EQUATIONS

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Abstract. We study a system of viscoelastic plate equations with degenerate damping and source terms under Dirichlet boundary condition. We obtain the blow up of solutions.

1. Introduction

In this work, we consider the following system of viscoelastic plate equations with degenerate damping terms:

$$\begin{cases} u_{tt}+\Delta^2 u-\int_0^t \lambda_1(t-s)\Delta^2 u(s)ds+\left(|u|^p+|v|^q\right)|u_t|^{d-1}u_t \\ \qquad =f_1(u,v), \ \ (x,t)\in\Omega\times(0,T), \\ v_{tt}+\Delta^2 v-\int_0^t \lambda_2(t-s)\Delta^2 v(s)ds+\left(|v|^l+|u|^{\upsilon}\right)|v_t|^{r-1}v_t \\ \qquad =f_2(u,v), \ \ (x,t)\in\Omega\times(0,T), \\ u(x,t)=v(x,t)=0, \qquad (x,t)\in\partial\Omega\times(0,T), \\ u(x,0)=u_0(x), \ u_t(x,0)=u_1(x), \qquad x\in\Omega, \\ v(x,0)=v_0(x), \ v_t(x,0)=v_1(x), \qquad x\in\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain with a sufficiently smooth boundary in R^n $(n\geq 1)$, $d,r\geq 1$, $p,q,l,\upsilon\geq 0$; $\lambda_i(.):R^+\to R^+$ $(i=1,2)$ are positive relaxation functions.

For the last several decades, the mathematical analysis of plate equations has attracted a lot of attention. For example, Lagnese [1], Rivera et al. [2] and Alabau-Boussouira et al. [3] investigated plate equation. Messaoudi [4] considered the following problem

$$u_{tt}+\Delta^2 u+|u_t|^{d-1}u_t=|u|^{p-2}u.$$

He obtained an existence result and studied global solution in case $d\geq p$. Then, blow-up of solutions with nonpositive initial energy and $d<p$ was obtained.

The evolution equations with degenerate damping are of much interest in material science and physics. Now, we state some present results in the literature:

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Rammaha and Sakuntasathien [5] firstly study system with degenerate damping terms and they considered the following system

$$\begin{cases} u_{tt} - \triangle u + \left(|u|^p + |v|^q\right)|u_t|^{d-1} u_t = f_1(u,v), \\ v_{tt} - \triangle v + \left(|v|^l + |u|^v\right)|v_t|^{r-1} v_t = f_2(u,v). \end{cases} \tag{1.2}$$

They considered the global well posedness of the solution under some restriction on the parameters. In [6, 7], authors studied the same problem treated in [5], and they investigated the growth and blow up properties. In addition, some authors studied the system with degenerate damping terms [8, 9, 10, 11, 12].

The outline of the paper is as follows: In Section 2, as preliminaries, we give necessary assumptions, lemma that will be used later and Local existence theorem. The blow up of solution is presented in last section.

2. Preliminaries

In this section, we will present some assumptions, notations, and lemma that will be used later for our main results. Throughout this paper, we denote the standart $L^2(\Omega)$ norm by $\|.\| = \|.\|_{L^2(\Omega)}$ and $L^p(\Omega)$ norm $\|.\|_p = \|.\|_{L^p(\Omega)}$.

To state and prove our result, we need some assumptions:

(**A1**) Regarding $\lambda_i : R^+ \to R^+$, $(i=1,2)$ are C^1- nonincreasing functions satisfying

$$\lambda_i(\alpha) > 0, \quad \lambda_i'(\alpha) \leq 0, \quad 1 - \int_0^\infty \lambda_i(\alpha) d\alpha = \mu_i > 0, \ \alpha \geq 0.$$

(**A2**) For the nonlinearity, we assume that

$$\begin{cases} 1 \leq d, r & \text{if } n = 1,2, \\ 1 \leq d, r \leq \frac{n+2}{n-2} & \text{if } n \geq 3. \end{cases}$$

We take $f_i\ (u,v)\ (i=1,2)$ function such that

$$\begin{aligned} f_1(u,v) &= a|u+v|^{2(\alpha+1)}(u+v) + b|u|^\alpha u |v|^{\alpha+2}, \\ f_2(u,v) &= a|u+v|^{2(\alpha+1)}(u+v) + b|v|^\alpha v |u|^{\alpha+2}, \end{aligned}$$

where $a > 0$, $b > 0$ and

$$\begin{cases} -1 < \alpha & \text{if } n = 1,2, \\ -1 < \alpha \leq \frac{3-n}{n-2} & \text{if } n \geq 3. \end{cases} \tag{2.1}$$

It is easy to show that

$$u f_1(u,v) + v f_2(u,v) = 2(\alpha+2) F(u,v), \ \forall (u,v) \in R^2, \tag{2.2}$$

where

$$F(u,v) = \frac{1}{2(\alpha+2)} \left[a|u+v|^{2(\alpha+2)} + 2b|uv|^{\alpha+2} \right]. \tag{2.3}$$

In addition, we will use the following notation:

$$(\lambda_i \diamond \Delta\vartheta)(t) = \int_0^t \lambda_i(t-s) \|\Delta\vartheta(s) - \Delta\vartheta(t)\|^2 ds.$$

Lemma 2.1. [13]. *Assume that*

$$s \leq 2\frac{n-1}{n-2},\ n \geq 3$$

holds. Then, there exists a positive constant $C > 1$ depending on Ω only such that

$$\|u\|_s^\alpha \leq C\left(\|\nabla u\|^2 + \|u\|_s^s\right)$$

for any $u \in H_0^1(\Omega)$, $2 \leq \alpha \leq s$.

We define the energy function as follows

$$\begin{aligned} E(t) &= \frac{1}{2}\left(\|u_t\|_2^2 + \|v_t\|_2^2\right) + \frac{1}{2}\left[(\lambda_1 \diamond \Delta u)(t) + (\lambda_2 \diamond \Delta v)(t)\right] \\ &+ \frac{1}{2}\left[(1 - \int_0^t \lambda_1(s)ds)\|\Delta u(t)\|^2 + (1 - \int_0^t \lambda_2(s)ds)\|\Delta v(t)\|^2\right] - \int_\Omega F(u,v)\,dx. \end{aligned} \tag{2.4}$$

By computation, we get

$$\begin{aligned} \frac{d}{dt}E(t) &\leq \frac{1}{2}\left[(\lambda_1' \diamond \Delta u)(t) + (\lambda_2' \diamond \Delta v)(t)\right] - \frac{1}{2}\left(\lambda_1(t)\|\Delta u\|^2 + \lambda_2(t)\|\Delta v\|^2\right) \\ &\quad - \int_\Omega \left(|u|^p + |v|^q\right)|u_t|^{d+1}dx - \int_\Omega \left(|v|^l + |u|^\upsilon\right)|v_t|^{r+1}dx \\ &\leq 0. \end{aligned} \tag{2.5}$$

Now, we complete this section by giving a local existence results of (1.1), which can be established by combinining arguments of [4, 5, 14].

Theorem 2.2. *(Local existence). Assume assumptions $(A1)$, $(A2)$, $(A3)$ and (2.1)-(2.3) hold if $u_0, v_0 \in H_0^2(\Omega)$ and $u_1, v_1 \in L^2(\Omega)$. Then, for some $T > 0$ problem (1.1) has a unique local solution (u,v) which satisfies*

$$u, v \in C\left([0,T); H_0^2(\Omega)\right),$$

$$u_t \in C\,[0,T); L^2(\Omega) \cap L^{d+1}(\Omega \times [0,T))),$$

$$v_t \in C\left([0,T); L^2(\Omega) \cap L^{r+1}(\Omega \times [0,T))\right).$$

3. Blow up

In this part, we shall state and obtain the blow up result.

Theorem 3.1. *Suppose that the initial energy $E(0) < 0$ and*

$$2(\alpha+2) > \max\left\{p+d+1, q+d+1, l+r+1, \upsilon+r+1, \frac{\mu_1}{1-\mu_1}, \frac{\mu_2}{1-\mu_2}\right\}.$$

Then, the solution of problem (1.1) blows up in finite time T^*, and

$$T^* \leq \frac{1-\rho}{\xi\rho G^{\frac{\rho}{1-\rho}}(0)}$$

where $G(t)$ and ρ are given in (3.2) and (3.3), respectively.

Proof. We assume that the solution exists for all time and we arrive to a contradiction. We get

$$\|u_t\|^2+\|v_t\|^2+H(t)+\|\Delta u\|^2+\|\Delta v\|^2+\|u\|_{2(\alpha+2)}^{2(\alpha+2)}+\|v\|_{2(\alpha+2)}^{2(\alpha+2)}\leq C,\ \forall t\geq 0. \tag{3.1}$$

We define

$$H(t)=-E(t),$$

then since $E(0)<0$, and (2.5) gives $H(t)\geq H(0)>0$. Set

$$G(t)=H^{1-\rho}(t)+\varepsilon\left(\int_\Omega u_t u dx+\int_\Omega v_t v dx\right) \tag{3.2}$$

where $\varepsilon>0$ small to chosen later and

$$\begin{aligned} 0 \quad < \quad & \rho\leq\min\left\{\frac{\alpha+1}{2(\alpha+2)},\frac{2\alpha+3-(p+d)}{2d\,(\alpha+2)},\right.\\ & \left.\frac{2\alpha+3-(q+d)}{2d\,(\alpha+2)},\frac{2\alpha+3-(v+r)}{2r\,(\alpha+2)},\frac{2\alpha+3-(l+r)}{2r\,(\alpha+2)}\right\}. \end{aligned} \tag{3.3}$$

Differentiating (3.2) and using Eq.(1.1), we haveλ

$$\begin{aligned} G'(t) &= (1-\rho)H^{-\rho}(t)H'(t)+\varepsilon\left(\int_\Omega |u_t|^2dx+\int_\Omega |v_t|^2dx\right)\\ &\quad+\varepsilon\left(\int_\Omega u_{tt}udx+\int_\Omega v_{tt}vdx\right)\\ &= (1-\rho)H^{-\rho}(t)H'(t)+\varepsilon\left(\|u_t\|^2+\|v_t\|^2\right)-\varepsilon\left(\|\Delta u\|^2+\|\Delta v\|^2\right)\\ &\quad+2\varepsilon(\alpha+2)\int_\Omega F(u,v)dx\\ &\quad+\varepsilon\int_\Omega\int_0^t\lambda_1(t-s)\Delta u(s)\Delta u(t)dsdx+\varepsilon\int_\Omega\int_0^t\lambda_2(t-s)\Delta v(s)\Delta v(t)dsdx\\ &\quad-\varepsilon\left(\int_\Omega u\left(|u|^p+|v|^q\right)u_t|u_t|^{d-1}dx+\int_\Omega v\left(|v|^l+|u|^v\right)v_t|v_t|^{r-1}dx\right). \end{aligned} \tag{3.4}$$

Now, the ninth term in the right hand side of (3.4) can be estimated, as follows (see [15]):

$$\begin{aligned} &\int_\Omega\Delta u(t)\int_0^t\lambda_1(t-s)\Delta u(s)dsdx\\ \leq\ &\frac{1}{2}\|\Delta u\|^2+\frac{1}{2}\int_\Omega\left(\int_0^t\lambda_1(t-s)\left(|\Delta u(s)-\Delta u(t)|+|\Delta u(t)|\right)ds\right)^2dx. \end{aligned} \tag{3.5}$$

By using Young's inequality and since

$$\int_0^t\lambda_1(s)ds\leq\int_0^\infty\lambda_1(s)ds\leq 1-\mu_1$$

we have, for any $\eta_1 > 0$,

$$
\begin{aligned}
\int_\Omega \Delta u(t)\int_0^t \lambda_1(t-s)\Delta u(s)dsdx &\leq \frac{1}{2}\left\|\Delta u\right\|^2 + \frac{1}{2}(1+\eta_1)\int_\Omega\left(\int_0^t \lambda_1(t-s)\Delta u(s)ds\right)^2 dx \\
&\quad + \frac{1}{2}\left(1+\frac{1}{\eta_1}\right)\int_\Omega\left(\int_0^t \lambda_1(t-s)\left|\Delta u(s)-\Delta u(t)\right|ds\right)^2 dx \\
&\leq \frac{1+(1+\eta_1)(1-\mu_1)^2}{2}\left\|\Delta u\right\|^2 \\
&\quad + \frac{(1+\frac{1}{\eta_1})(1-\mu_1)}{2}(\lambda_1 \diamond \Delta u)(t). \qquad (3.6)
\end{aligned}
$$

Similar calculations also yield, for any $\eta_2 > 0$,

$$
\begin{aligned}
\int_\Omega \Delta v(t)\int_0^t \lambda_2(t-s)\Delta v(s)dsdx &\leq \frac{1+(1+\eta_2)(1-\mu_2)^2}{2}\left\|\Delta v\right\|^2 \\
&\quad + \frac{(1+\frac{1}{\eta_2})(1-\mu_2)}{2}(\lambda_2 \diamond \Delta v)(t). \qquad (3.7)
\end{aligned}
$$

A substitution of (3.5)-(3.7) into (3.4) leads to

$$
\begin{aligned}
G'(t) &\leq (1-\rho)\,H^{-\rho}(t)\,H'(t) + \varepsilon\left(\left\|u_t\right\|^2 + \left\|v_t\right\|^2\right) + 2\varepsilon(\alpha+2)\int_\Omega F(u,v)dx \\
&\quad + \varepsilon\left(\frac{(1+\eta_1)(1-\mu_1)^2-1}{2}\right)\left\|\Delta u\right\|^2 + \varepsilon\frac{(1+\frac{1}{\eta_1})(1-\mu_1)}{2}(\lambda_1 \diamond \Delta u)(t) \\
&\quad + \varepsilon\left(\frac{(1+\eta_2)(1-\mu_2)^2-1}{2}\right)\left\|\Delta v\right\|^2 + \varepsilon\frac{(1+\frac{1}{\eta_2})(1-\mu_2)}{2}(\lambda_2 \diamond \Delta u)(t) \\
&\quad - \varepsilon\left(\int_\Omega u\left(|u|^p + |v|^q\right)u_t\left|u_t\right|^{d-1}dx + \int_\Omega v\left(|v|^l + |u|^\upsilon\right)v_t\left|v_t\right|^{r-1}dx\right) \qquad (3.8)
\end{aligned}
$$

From the definition of $H(t)$, as follows

$$
\begin{aligned}
&\int_\Omega F(u,v)dx \\
&= H(t) + \frac{1}{2}\left(\left\|u_t\right\|^2 + \left\|v_t\right\|^2\right) + \frac{1}{2}\left[(\lambda_1 \diamond \Delta u)(t) + (\lambda_2 \diamond \Delta v)(t)\right] \\
&\quad + \frac{1}{2}\left[(1-\int_0^t \lambda_1(s)ds)\left\|\Delta u(t)\right\|^2 + (1-\int_0^t \lambda_2(s)ds)\left\|\Delta v(t)\right\|^2\right]. \qquad (3.9)
\end{aligned}
$$

Inserting (3.9) into (3.8), we have

$$
\begin{aligned}
G'(t) &= (1-\rho) H^{-\rho}(t) H'(t) + \varepsilon(\alpha+3)\left(\|u_t\|^2 + \|v_t\|^2\right) + 2\varepsilon(\alpha+2)H(t) \\
&+\varepsilon\left[(\alpha+2)(1-\mu_1) + \frac{(1+\eta_1)(1-\mu_1)^2-1}{2}\right]\|\Delta u\|^2 \\
&+\varepsilon\left[(\alpha+2)(1-\mu_2) + \frac{(1+\eta_2)(1-\mu_2)^2-1}{2}\right]\|\Delta v\|^2 \\
&+\varepsilon\left[(\alpha+2) + \frac{(1+\frac{1}{\eta_1})(1-\mu_1)}{2}\right](\lambda_1 \diamond \Delta u)(t) \\
&+\varepsilon\left[(\alpha+2) + \frac{(1+\frac{1}{\eta_2})(1-\mu_2)}{2}\right](\lambda_2 \diamond \Delta u)(t) \\
&-\varepsilon\left(\int_\Omega u\left(|u|^p + |v|^q\right) u_t |u_t|^{d-1}\,dx + \int_\Omega v\left(|v|^l + |u|^v\right) v_t |v_t|^{r-1}\,dx\right).
\end{aligned} \tag{3.10}
$$

In order to estimate the last two terms in the right hand side of (3.10), we use the next Young inequality

$$
KL \le \frac{\delta^x K^x}{x} + \frac{\delta^{-y} K^y}{y}, \tag{3.11}
$$

in which $K, L \ge 0,\ \delta > 0,\ x, y \in R^+$ like that $\frac{1}{x} + \frac{1}{y} = 1$. Thus, we get

$$
\int_\Omega u u_t |u_t|^{d-1}\,dx \le \frac{\delta_1^{d+1}}{d+1}\|u\|_{d+1}^{d+1} + \frac{d\delta_1^{-\frac{d+1}{d}}}{d+1}\|u_t\|_{d+1}^{d+1},
$$

and then

$$
\begin{aligned}
\int_\Omega u\left(|u|^p + |v|^q\right) u_t |u_t|^{d-1}\,dx &\le \frac{\delta_1^{d+1}}{d+1}\int_\Omega \left(|u|^p + |v|^q\right)|u|^{d+1}\,dx \\
&+\frac{d\delta_1^{-\frac{d+1}{d}}}{d+1}\int_\Omega \left(|u|^p + |v|^q\right)|u_t|^{d+1}\,dx.
\end{aligned}
$$

Similarly, we obtain

$$
\int_\Omega v v_t |v_t|^{r-1}\,dx \le \frac{\delta_2^{r+1}}{r+1}\|v\|_{r+1}^{r+1} + \frac{r\delta_2^{-\frac{r+1}{r}}}{r+1}\|v_t\|_{r+1}^{r+1},
$$

and

$$
\begin{aligned}
\int_\Omega v\left(|v|^l + |u|^v\right) v_t |v_t|^{r-1}\,dx &\le \frac{\delta_2^{r+1}}{r+1}\int_\Omega \left(|v|^l + |u|^v\right)|v|^{r+1}\,dx \\
&+\frac{r\delta_2^{-\frac{r+1}{r}}}{r+1}\int_\Omega \left(|v|^l + |u|^v\right)|v_t|^{r+1}\,dx,
\end{aligned}
$$

where $\delta_1,\ \delta_2$ are constants depending on the time t and specified later. Then (3.10), becomes

$$\begin{aligned} G'(t) \geq\ & (1-\rho) H^{-\rho}(t) H'(t) + 2\varepsilon(\alpha+2) H(t) \\ & +\varepsilon(\alpha+3)\left(\|u_t\|^2+\|v_t\|^2\right) \\ & +\varepsilon\left[(\alpha+2)(1-\mu_1)+\frac{(1+\eta_1)(1-\mu_1)^2-1}{2}\right]\|\Delta u\|^2 \\ & +\varepsilon\left[(\alpha+2)(1-\mu_2)+\frac{(1+\eta_2)(1-\mu_2)^2-1}{2}\right]\|\Delta v\|^2 \\ & +\varepsilon\left[(\alpha+2)+\frac{(1+\frac{1}{\eta_1})(1-\mu_1)}{2}\right](\lambda_1\diamond\Delta u)(t) \\ & +\varepsilon\left[(\alpha+2)+\frac{(1+\frac{1}{\eta_2})(1-\mu_2)}{2}\right](\lambda_2\diamond\Delta v)(t) \\ & -\varepsilon\frac{\delta_1^{d+1}}{d+1}\int_\Omega\left(|u|^p+|v|^q\right)|u|^{d+1}dx-\varepsilon\frac{d\delta_1^{-\frac{d+1}{d}}}{d+1}\int_\Omega\left(|u|^p+|v|^q\right)|u_t|^{d+1}dx \\ & -\varepsilon\frac{\delta_2^{r+1}}{r+1}\int_\Omega\left(|v|^l+|u|^\upsilon\right)|v|^{r+1}dx-\varepsilon\frac{r\delta_2^{-\frac{r+1}{s}}}{r+1}\int_\Omega\left(|v|^l+|u|^\upsilon\right)|v_t|^{r+1}dx. \end{aligned} \tag{3.12}$$

Therefore by taking δ_1 and δ_2 so that $\delta_1^{-\frac{d+1}{d}}=k_1H^{-\sigma}(t),\ \delta_2^{-\frac{r+1}{r}}=k_2H^{-\sigma}(t)$ where $k_1,\ k_2>0$ are specified later, we get

$$\begin{aligned} G'(t) \geq\ & ((1-\rho)-K\varepsilon) H^{-\rho}(t) H'(t) + 2\varepsilon(\alpha+2) H(t) \\ & +\varepsilon(\alpha+3)\left(\|u_t\|^2+\|v_t\|^2\right) \\ & +\varepsilon\left[(\alpha+2)(1-\mu_1)+\frac{(1+\eta_1)(1-\mu_1)^2-1}{2}\right]\|\Delta u\|^2 \\ & +\varepsilon\left[(\alpha+2)(1-\mu_2)+\frac{(1+\eta_2)(1-\mu_2)^2-1}{2}\right]\|\Delta v\|^2 \\ & +\varepsilon\left[(\alpha+2)+\frac{(1+\frac{1}{\eta_1})(1-\mu_1)}{2}\right](\lambda_1\diamond\Delta u)(t) \\ & +\varepsilon\left[(\alpha+2)+\frac{(1+\frac{1}{\eta_2})(1-\mu_2)}{2}\right](\lambda_2\diamond\Delta v)(t) \\ & -\varepsilon\frac{k_1^{-d}H^{\rho d}(t)}{m+1}\int_\Omega\left(|u|^p+|v|^q\right)|u|^{d+1}dx \\ & -\varepsilon\frac{k_2^{-r}H^{\rho r}(t)}{r+1}\int_\Omega\left(|v|^l+|u|^\upsilon\right)|v|^{r+1}dx \end{aligned} \tag{3.13}$$

where $K=\frac{k_1 d}{d+1}+\frac{k_2 r}{r+1}$.

Applying the Young inequality, we obtain

$$\begin{aligned}
\int_\Omega \left(|u|^p+|v|^q\right)|u|^{d+1}dx &\leq \int_\Omega |u|^{p+d+1}dx+\int_\Omega |v|^q|u|^{d+1}dx\\
&\leq \int_\Omega |u|^{p+d+1}dx+\frac{l}{q+d+1}\delta_1^{\frac{q+d+1}{q}}\int_\Omega |v|^{q+d+1}dx\\
&\quad+\frac{d+1}{q+d+1}\delta_1^{-\frac{q+d+1}{d+1}}\int_\Omega |u|^{q+d+1}dx\\
&= \|u\|_{p+d+1}^{p+d+1}+\frac{q}{q+d+1}\delta_1^{\frac{q+d+1}{q}}\|v\|_{q+d+1}^{q+d+1}\\
&\quad+\frac{d+1}{q+d+1}\delta_1^{-\frac{q+d+1}{d+1}}\|u\|_{q+d+1}^{q+d+1}. \qquad (3.14)
\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}
\int_\Omega \left(|v|^l+|u|^\upsilon\right)|v|^{r+1}dx &\leq \|v\|_{l+r+1}^{l+r+1}+\frac{\upsilon}{\upsilon+r+1}\delta_2^{\frac{\upsilon+r+1}{\upsilon}}\|u\|_{\upsilon+r+1}^{\upsilon+r+1}\\
&\quad+\frac{r+1}{\upsilon+r+1}\delta_2^{-\frac{\upsilon+r+1}{r+1}}\|v\|_{\upsilon+r+1}^{\upsilon+r+1}. \qquad (3.15)
\end{aligned}$$

Inserting (3.14) and (3.15) into (3.13), we have

$$\begin{aligned}
G'(t) \geq\ & \left((1-\rho)-K\varepsilon\right)H^{-\rho}(t)H'(t)+2\varepsilon(\alpha+2)H(t)\\
&+\varepsilon(\alpha+3)\left(\|u_t\|^2+\|v_t\|^2\right)\\
&+\varepsilon\left[(\alpha+2)(1-\mu_1)+\frac{(1+\eta_1)(1-\mu_1)^2-1}{2}\right]\|\Delta u\|^2\\
&+\varepsilon\left[(\alpha+2)(1-\mu_2)+\frac{(1+\eta_2)(1-\mu_2)^2-1}{2}\right]\|\Delta v\|^2\\
&+\varepsilon\left[(\alpha+2)+\frac{(1+\frac{1}{\eta_1})(1-\mu_1)}{2}\right](\lambda_1\diamond\Delta u)(t)\\
&+\varepsilon\left[(\alpha+2)+\frac{(1+\frac{1}{\eta_2})(1-\mu_2)}{2}\right](\lambda_2\diamond\Delta v)(t)\\
&-\varepsilon\frac{k_1^{-d}H^{\rho d}(t)}{d+1}\left(\|u\|_{p+d+1}^{p+d+1}+\frac{q}{q+d+1}\delta_1^{\frac{q+d+1}{q}}\|v\|_{q+d+1}^{q+d+1}\right)\\
&-\varepsilon\frac{k_1^{-d}H^{\rho d}(t)}{d+1}\frac{d+1}{q+d+1}\delta_1^{-\frac{q+d+1}{d+1}}\|u\|_{q+d+1}^{q+d+1}\\
&-\varepsilon\frac{k_2^{-r}H^{\rho r}(t)}{r+1}\left(\|v\|_{l+r+1}^{l+r+1}+\frac{\upsilon}{\upsilon+r+1}\delta_2^{\frac{\upsilon+r+1}{\upsilon}}\|u\|_{\upsilon+r+1}^{\upsilon+r+1}\right)\\
&-\varepsilon\frac{k_2^{-r}H^{\rho r}(t)}{r+1}\frac{r+1}{\upsilon+r+1}\delta_2^{-\frac{\upsilon+r+1}{r+1}}\|v\|_{\upsilon+r+1}^{\upsilon+r+1}. \qquad (3.16)
\end{aligned}$$

Since $2(\alpha+2)>\max\{p+d+1,\ q+d+1,\ l+r+1,\ \upsilon+r+1\}$, we have

$$H^{\rho d}(t)\|u\|_{p+d+1}^{p+d+1}\leq C\left(\|u\|_{2(\alpha+2)}^{2\rho d(\alpha+2)+p+d+1}+\|v\|_{2(\alpha+2)}^{2\rho d(\alpha+2)}\|u\|_{p+d+1}^{p+d+1}\right), \qquad (3.17)$$

$$H^{\rho r}(t)\|v\|_{l+r+1}^{l+r+1}\leq C\left(\|v\|_{2(\alpha+2)}^{2\rho r(\alpha+2)+l+r+1}+\|u\|_{2(\alpha+2)}^{2\rho r(\alpha+2)}\|v\|_{l+r+1}^{l+r+1}\right), \qquad (3.18)$$

and

$$\begin{aligned} & \frac{q}{q+d+1}\delta_1^{\frac{q+d+1}{q}} H^{\rho d}(t) \|v\|_{q+d+1}^{q+d+1} \\ \leq \quad & C\frac{q}{q+d+1}\delta_1^{\frac{q+d+1}{q}} \left(\|v\|_{2(\alpha+2)}^{2\rho d(\alpha+2)+q+d+1} + \|u\|_{2(\alpha+2)}^{2\rho d(\alpha+2)} \|v\|_{q+d+1}^{q+d+1}\right) \end{aligned} \tag{3.19}$$

$$\begin{aligned} & \frac{\upsilon}{\upsilon+r+1}\delta_2^{\frac{\upsilon+r+1}{\upsilon}} H^{\rho r}(t) \|u\|_{\upsilon+r+1}^{\upsilon+r+1} \\ \leq \quad & C\frac{\upsilon}{\upsilon+r+1}\delta_2^{\frac{\upsilon+r+1}{\upsilon}} \left(\|u\|_{2(\alpha+2)}^{2\rho r(\alpha+2)+\upsilon+r+1} + \|v\|_{2(\alpha+2)}^{2\rho r(\alpha+2)} \|u\|_{\upsilon+r+1}^{\upsilon+r+1}\right). \end{aligned} \tag{3.20}$$

By (3.3) and using the following algebraic inequality

$$x^{\varrho} \leq x+1 \leq \left(1+\frac{1}{\beta}\right)(x+\beta), \quad \forall x \geq 0,\ 0<\varrho\leq 1,\ \beta\geq 0, \tag{3.21}$$

for all $t \geq 0$,

$$\begin{aligned} \|u\|_{2(\alpha+2)}^{2\rho d(\alpha+2)+p+d+1} & \leq m\left(\|u\|_{2(\alpha+2)}^{2(\alpha+2)} + H(0)\right) \\ & \leq m\left(\|u\|_{2(\alpha+2)}^{2(\alpha+2)} + H(t)\right), \end{aligned} \tag{3.22}$$

$$\|v\|_{2(\alpha+2)}^{2\rho r(\alpha+2)+l+r+1} \leq m\left(\|v\|_{2(\alpha+2)}^{2(\alpha+2)} + H(t)\right) \tag{3.23}$$

where $m = 1+\frac{1}{H(0)}$. Likewise

$$\|u\|_{2(\alpha+2)}^{2\rho r(\alpha+2)+\upsilon+r+1} \leq m\left(\|u\|_{2(\alpha+2)}^{2(\alpha+2)} + H(t)\right), \tag{3.24}$$

$$\|v\|_{2(\alpha+2)}^{2\rho d(\alpha+2)+q+d+1} \leq m\left(\|v\|_{2(\alpha+2)}^{2(\alpha+2)} + H(t)\right). \tag{3.25}$$

Next, using Young inequality, (3.3) and (3.21) and the inequality $(x+y)^{\alpha} \leq C(x^{\alpha}+y^{\alpha}),\ x,y>0$, we obtain

$$\begin{aligned} \|v\|_{2(\alpha+2)}^{2\rho d(\alpha+2)} \|u\|_{p+d+1}^{p+d+1} & \leq |\Omega|^{\frac{2(\alpha+2)-(p+d+1)}{2(\alpha+2)}} \left(\|v\|_{2(\alpha+2)}^{2\rho d(\alpha+2)} \|u\|_{2(\alpha+2)}^{p+d+1}\right) \\ & = |\Omega|^{\frac{2(\alpha+2)-(p+d+1)}{2(\alpha+2)}} \left(\|v\|_{2(\alpha+2)}^{\rho d} \|u\|_{p+d+1}^{p+d+1}\right) \\ & \leq |\Omega|^{\frac{2(\alpha+2)-(p+d+1)}{2(\alpha+2)}} \left(c' \|v\|_{2(\alpha+2)}^{\frac{2\rho d(\alpha+2)+p+d+1}{2(\alpha+2)}} + c'' \|u\|_{2(\alpha+2)}^{\frac{2\rho d(\alpha+2)+p+d+1}{2(\alpha+2)}}\right) \\ & \leq C\left(\|u\|_{2(\alpha+2)}^{2(\alpha+2)} + \|v\|_{2(\alpha+2)}^{2(\alpha+2)}\right). \end{aligned} \tag{3.26}$$

Similarly, we obtain

$$\|u\|_{2(\alpha+2)}^{2\rho r(\alpha+2)} \|v\|_{l+r+1}^{l+r+1} \leq C\left(\|u\|_{2(\alpha+2)}^{2(\alpha+2)} + \|v\|_{2(\alpha+2)}^{2(\alpha+2)}\right), \tag{3.27}$$

$$\|u\|_{2(\alpha+2)}^{2\rho d(\alpha+2)} \|v\|_{q+d+1}^{q+d+1} \leq C\left(\|u\|_{2(\alpha+2)}^{2(\alpha+2)} + \|v\|_{2(\alpha+2)}^{2(\alpha+2)}\right), \tag{3.28}$$

and

$$\|v\|_{2(\alpha+2)}^{2\rho r(\alpha+2)} \|u\|_{\upsilon+r+1}^{\upsilon+r+1} \leq C\left(\|u\|_{2(\alpha+2)}^{2(\alpha+2)} + \|v\|_{2(\alpha+2)}^{2(\alpha+2)}\right). \tag{3.29}$$

Combining (3.17)-(3.20) and (3.22)-(3.29) into (3.16), we get

$$\begin{aligned}
G'(t) &\geq ((1-\rho)-K\varepsilon) H^{-\rho}(t) H'(t) + 2\varepsilon(\alpha+2) H(t) \\
&+\varepsilon(\alpha+3)\left(\|u_t\|^2+\|v_t\|^2\right) \\
&+\varepsilon C_1(\lambda_1 \diamond \Delta u)(t)+\varepsilon C_2(\lambda_2 \diamond \Delta v)(t) \\
&+\varepsilon C_3\|\Delta u\|^2+\varepsilon C_4\|\Delta v\|^2 \\
&+\varepsilon\left[2(\alpha+2)-Ck_1^{-d}\left(1+\frac{q}{q+d+1}\delta_1^{\frac{q+d+1}{q}}+\frac{d+1}{q+d+1}\delta_1^{-\frac{q+d+1}{d+1}}\right)\right. \\
&\left.-Ck_2^{-r}\left(1+\frac{\upsilon}{\upsilon+r+1}\delta_2^{\frac{\upsilon+r+1}{\upsilon}}+\frac{r+1}{\upsilon+r+1}\delta_2^{-\frac{\upsilon+r+1}{r+1}}\right)\right] H(t) \\
&+\varepsilon\left[Ck_1^{-d}\left(1+\frac{q}{q+d+1}\delta_1^{\frac{q+m+1}{q}}+\frac{d+1}{q+d+1}\delta_1^{-\frac{q+d+1}{d+1}}\right)\right. \\
&\left.-Ck_2^{-r}\left(1+\frac{\upsilon}{\upsilon+r+1}\delta_2^{\frac{\upsilon+r+1}{\upsilon}}+\frac{r+1}{\upsilon+r+1}\delta_2^{-\frac{\upsilon+r+1}{r+1}}\right)\right]\left(\|u\|_{2(\alpha+2)}^{2(\alpha+2)}+\|v\|_{2(\alpha+2)}^{2(\alpha+2)}\right),
\end{aligned} \tag{3.30}$$

where

$$\begin{aligned}
C_1 &= (\alpha+2)+\frac{(1+\frac{1}{\eta_1})(1-\mu_1)}{2}, \\
C_2 &= (\alpha+2)+\frac{(1+\frac{1}{\eta_2})(1-\mu_2)}{2}, \\
C_3 &= (\alpha+2)(1-\mu_1)+\frac{(1+\eta_1)(1-\mu_1)^2-1}{2}, \\
C_4 &= (\alpha+2)(1-\mu_2)+\frac{(1+\eta_2)(1-\mu_2)^2-1}{2}.
\end{aligned}$$

At this point, choosing $\eta_1 = \frac{\mu_1}{1-\mu_1}$, $\eta_2 = \frac{\mu_2}{1-\mu_2}$ and picking μ_1 and μ_2 small enough such that

$$(\alpha+2)(1-\mu_1) \geq \frac{\mu_1}{2} \text{ and } (\alpha+2)(1-\mu_2) \geq \frac{\mu_2}{2}.$$

$C_i, i = 1,2,3,4$ are positive constants and for large values of k_1 and k_2, we can reach $K_1 > 0$ and $K_2 > 0$ such that (3.30) reduce

$$\begin{aligned}
G'(t) &\geq [(1-\rho)-K\varepsilon] H^{-\rho}(t) H'(t)+\varepsilon(\alpha+3)\left(\|u_t\|^2+\|v_t\|^2\right) \\
&+\varepsilon C_1(\lambda_1 \diamond \Delta u)(t)+\varepsilon C_2(\lambda_2 \diamond \Delta v)(t)+\varepsilon C_3\|\Delta u\|_2^2+\varepsilon C_4\|\Delta v\|_2^2 \\
&+\varepsilon K_1 H(t)+\varepsilon K_2\left(\|u\|_{2(\alpha+2)}^{2(\alpha+2)}+\|v\|_{2(\alpha+2)}^{2(\alpha+2)}\right) \\
&\geq \beta\left(\|u_t\|^2+\|v_t\|^2+H(t)+\|\Delta u\|^2+\|\Delta v\|^2+\|u\|_{2(\alpha+2)}^{2(\alpha+2)}+\|v\|_{2(\alpha+2)}^{2(\alpha+2)}\right)
\end{aligned} \tag{3.31}$$

where

$$\beta = \min\{\varepsilon C_1,\ \varepsilon C_2, \varepsilon C_3,\ \varepsilon C_4,\ \varepsilon K_1,\ \varepsilon K_2, \varepsilon(\alpha+3)\}$$

and choosing ε small enough so that $(1-\rho)-K\varepsilon \geq 0$. As a result, we arrive at

$$G(t) \geq G(0) > 0,\ \forall t \geq 0. \tag{3.32}$$

In order to estimate $G(t)^{\frac{1}{1-\rho}}$, we use Hölder inequality and Young inequality, we have

$$\begin{aligned}\left|\int_\Omega u_t u dx + \int_\Omega v_t v dx\right|^{\frac{1}{1-\rho}} &\leq \|u_t\|^{\frac{1}{1-\rho}} \|u\|^{\frac{1}{1-\rho}} + \|v_t\|^{\frac{1}{1-\rho}} \|v\|^{\frac{1}{1-\rho}} \\ &\leq C(\|u_t\|^{\frac{\mu}{1-\rho}} + \|u\|_{2(\alpha+2)}^{\frac{\theta}{1-\rho}} + \|v_t\|^{\frac{\mu}{1-\rho}} + \|v\|_{2(\alpha+2)}^{\frac{\theta}{1-\rho}}),\end{aligned}$$

where $\frac{1}{\mu} + \frac{1}{\theta} = 1$. Noting (3.3), since $\mu = 2(1-\rho)$, then $\theta = \frac{2(1-\rho)}{1-2\rho}$ and using Lemma 1, we have

$$\left|\int_\Omega u_t u dx + \int_\Omega v_t v dx\right|^{\frac{1}{1-\rho}} \leq C(\|u_t\|^2 + \|u\|_{2(\alpha+2)}^{2(\alpha+2)} + \|v_t\|^2 + \|v\|_{2(\alpha+2)}^{2(\alpha+2)} + \|\Delta u\|^2 + \|\Delta v\|^2)$$

Thus,

$$\begin{aligned}G^{\frac{1}{1-\rho}}(t) &= \left[H^{1-\rho}(t) + \varepsilon\left(\int_\Omega u_t u dx + \int_\Omega v_t v dx\right)\right]^{\frac{1}{1-\rho}} \\ &\leq C\left(\|u_t\|^2 + \|v_t\|^2 + H(t) + \|u\|_{2(\alpha+2)}^{2(\alpha+2)} + \|v\|_{2(\alpha+2)}^{2(\alpha+2)} + \|\Delta u\|^2 + \|\Delta v\|^2\right). \qquad (3.33)\end{aligned}$$

From (3.31) and (3.33), we arrive at

$$G'(t) \geq \xi G^{\frac{1}{1-\rho}}(t), \qquad (3.34)$$

where ξ is a positive constant. A simple integration of (3.34) over $(0,t)$ yields $G^{\frac{\rho}{1-\rho}}(t) \geq \frac{1}{G^{-\frac{\rho}{1-\rho}}(0) - \frac{\xi\rho t}{1-\rho}}$, which implies that the solution blows up in a finite time T^*, with

$$T^* \leq \frac{1-\sigma}{\xi\sigma\Psi^{\frac{\sigma}{1-\sigma}}(0)}.$$

□

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GROWTH OF SOLUTIONS FOR A SYSTEM OF KIRCHHOFF-TYPE EQUATIONS WITH DEGENERATE DAMPING TERMS

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Abstract. In this study, we considered a coupled Kirchhoff-type equations with degenerate damping terms. We prove exponential growth of solutions.

1. Introduction

We considere the following initial-boundary value problem on domain $(x,t)\in\Omega\times(0,T)$,

$$(1.1)\qquad \begin{cases} |u_t|^s u_{tt} - M\left(\|\nabla u\|^2+\|\nabla v\|^2\right)\Delta u + \left(|u|^w+|v|^b\right)|u_t|^{\alpha-1}u_t \\ \qquad = g_1(u,v), \\ |v_t|^s v_{tt} - M\left(\|\nabla u\|^2+\|\nabla v\|^2\right)\Delta v + \left(|v|^c+|u|^d\right)|v_t|^{\beta-1}v_t \\ \qquad = g_2(u,v), \\ u(x,t)=v(x,t)=0, & (x,t)\in\partial\Omega\times(0,T), \\ u(x,0)=u_0(x),\ u_t(x,0)=u_1(x), & x\in\Omega, \\ v(x,0)=v_0(x),\ v_t(x,0)=v_1(x), & x\in\Omega, \end{cases}$$

where Ω is a bounded domain with smooth boundary $\partial\Omega$ in R^n $(n\geq 1)$; $\alpha,\beta\geq 1$, $s,w,b,c,d\geq 0$; $g_i(.\,,.):R^2\longrightarrow R$ are given functions to be specified later. $M(s)$ is a nonnegative C^1 function for $\theta\geq 0$ satisfying $M(\theta)=1+\theta^\lambda$, $\lambda>1$.

In the case of $s=0$ and $M(\theta)\equiv 1$ for problem (1.1) was investigated by Rammaha and Sakuntasathien [1] and Zennir et al. [2, 3]. Rammaha and Sakuntasathien obtained the global well posedness of the solution and Zennir et al. showed the blow up and growth result. Also, some authors studied the system with degenerate damping terms [4, 5, 6, 7].

Ye [8] considered the problem (1.1) when $w=b=c=d=0$ and obtained the global existence and energy decay results.

The rest of this study is organized as follows: In Section 2, we give some lemmas and assumption. In last section, we prove exponenetial growth of the solution.

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2. Preliminaries

Let $\|.\|$ and $\|.\|_p$ denote the usual $L^2(\Omega)$ norm and $L^p(\Omega)$ norm in this paper, respectively.

We require the following assumptions for our result.

(**A**) We assume that

$$\begin{cases} \alpha, \beta \geq 1 & \text{if } n = 1, 2, \\ 1 \leq \alpha, \beta \leq 5 & \text{if } n = 3. \end{cases}$$

We take $g_1(u, v)$ and $g_2(u, v)$ such that

$$\begin{aligned} g_1(u, v) &= k|u + v|^{2(\gamma+1)}(u + v) + l|u|^\gamma u|v|^{\gamma+2}, \\ g_2(u, v) &= k|u + v|^{2(\gamma+1)}(u + v) + l|v|^\gamma v|u|^{\gamma+2}, \end{aligned}$$

where k, l are positive constants and γ satisfies

$$\begin{cases} -1 < \gamma & \text{if } n = 1, 2, \\ -1 < \gamma \leq 1 & \text{if } n = 3. \end{cases} \tag{2.1}$$

The below equality can been easily verify that

$$ug_1(u, v) + vg_2(u, v) = 2(\gamma + 2)G(u, v), \ \forall (u, v) \in R^2, \tag{2.2}$$

where

$$G(u, v) = \frac{1}{2(\gamma + 2)}\left[k|u + v|^{2(\gamma+2)} + 2l|uv|^{\gamma+2}\right]. \tag{2.3}$$

The energy functional $E(t)$ of problem (1.1) such that

$$\begin{aligned} E(t) &= \frac{1}{s+2}\left(\|u_t\|_{s+2}^{s+2} + \|v_t\|_{s+2}^{s+2}\right) + \frac{1}{2}(\|\nabla u\|^2 + \|\nabla v\|^2) \\ &\quad + \frac{1}{2(\lambda+1)}(\|\nabla u\|^2 + \|\nabla v\|^2)^{\lambda+1} - \int_\Omega G(u, v)\,dx. \end{aligned} \tag{2.4}$$

Lemma 2.1. [9]. *There exist two positive constants c_0 and c_1 such that*

$$c_0\left(|u|^{2(\gamma+2)} + |v|^{2(\gamma+2)}\right) \leq 2(\gamma + 2)G(u, v) \leq c_1\left(|u|^{2(\gamma+2)} + |v|^{2(\gamma+2)}\right) \tag{2.5}$$

is satisfied.

Lemma 2.2. *$E(t)$ is a nonincreasing function for $t \geq 0$ and*

$$\frac{d}{dt}E(t) = -\int_\Omega \left(|u|^w + |v|^b\right)|u_t|^{\alpha+1}\,dx - \int_\Omega \left(|v|^c + |u|^d\right)|v_t|^{\beta+1}\,dx. \tag{2.6}$$

3. Growth of solutions

In this part, we show that the energy grow up as an exponential function as time as goes to infinity.

We take $k = l = 1$ for sake of simplicity and present the following:

$$B = \eta^{\frac{1}{2(\gamma+2)}}, \ \alpha_1 = B^{-\frac{\gamma+2}{\gamma+1}}, \ E_1 = \left(\frac{1}{2} - \frac{1}{2(\gamma + 2)}\right)\alpha_1^2. \tag{3.1}$$

The following lemma used firstly by Vitillaro [10] and is very important for our result.

Lemma 3.1. *Suppose that assumption (A) and (2.1) hold. Let* (u,v) *be a solution of (1.1). Moreover, suppose that* $E(0) < E_1$ *and*

$$\left(\|\nabla u_0\|^2 + \|\nabla v_0\|^2\right)^{\frac{1}{2}} > \alpha_1. \tag{3.2}$$

Then there exists a constant $\alpha_2 > \alpha_1$ *such that*

$$\left(\|\nabla u\|^2 + \|\nabla v\|^2 + \frac{1}{\lambda+1}(\|\nabla u\|^2 + \|\nabla v\|^2)^{\lambda+1}\right)^{\frac{1}{2}} > \alpha_2, \text{ for } t > 0, \tag{3.3}$$

$$\left(\|u+v\|_{2(\gamma+2)}^{2(\gamma+2)} + 2\|uv\|_{\gamma+2}^{\gamma+2}\right)^{\frac{1}{2(\gamma+2)}} \geq B\alpha_2, \text{ for } t > 0. \tag{3.4}$$

for all $t \in [0,T)$.

Theorem 3.2. *Suppose that (A1), (A2) and (2.1) hold. Assume further that*

$$2(\gamma+2) > \max\{ s+2,\ k+p+1,\ l+p+1,\ \theta+q+1,\ \varrho+q+1\}.$$

Then any the solution of the problem (1.1) with initial data satisfying

$$\left(\|\nabla u_0\|^2 + \|\nabla v_0\|^2\right)^{\frac{1}{2}} > \alpha_1,\ E(0) < E_1,$$

grows exponentially, where α_1 *and* E_1 *are defined in (3.1).*

Proof. We define as follows

$$H(t) = E_1 - E(t). \tag{3.5}$$

By applying (2.6) and (3.5), we get

$$H'(t) = -E'(t) \geq 0, \quad \forall t \geq 0. \tag{3.6}$$

Since $E'(t)$ is definitely continuous, we have

$$0 < E_1 - E(0) = H(0) \leq H(t). \tag{3.7}$$

We then define

$$\Psi(t) = H(t) + \frac{\varepsilon}{s+1}\left(\int_\Omega |u_t|^s u_t u dx + \int_\Omega |v_t|^s v_t v dx\right) \tag{3.8}$$

where ε small to be chosen later.

By derivating (3.8) and using Eq.(1.1), we get

$$\begin{aligned}
\Psi'(t) &= H'(t) + \frac{\varepsilon}{s+1}\left(\int_\Omega |u_t|^{s+2} dx + \int_\Omega |v_t|^{s+2} dx\right) \\
&\quad +\varepsilon\left(\int_\Omega |u_t|^s u_{tt} u dx + \int_\Omega |v_t|^s v_{tt} v dx\right) \\
&= H'(t) + \frac{\varepsilon}{s+1}(\|u_t\|_{s+2}^{s+2} + \|v_t\|_{s+2}^{s+2}) - \varepsilon(\|\nabla u\|^2 + \|\nabla v\|^2) \\
&\quad -\varepsilon(\|\nabla u\|^2 + \|\nabla v\|^2)^{\lambda+1} + 2\varepsilon(\gamma+2)\int_\Omega G(u,v)dx \\
&\quad -\varepsilon\left(\int_\Omega u\left(|u|^w + |v|^b\right) u_t |u_t|^{\alpha-1} dx + \int_\Omega v\left(|v|^c + |u|^d\right) v_t |v_t|^{\beta-1} dx\right)
\end{aligned} \tag{3.9}$$

From the definition of $H(t)$, we have

$$\begin{aligned}
&-(\|\nabla u\|^2+\|\nabla v\|^2)^{\lambda+1} \\
= \;& 2(\lambda+1)H(t)-2(\lambda+1)E_1+\frac{2(\lambda+1)}{s+2}\left(\|u_t\|_{s+2}^{s+2}+\|v_t\|_{s+2}^{s+2}\right) \\
&+(\lambda+1)\left(\|\nabla u\|^2+\|\nabla v\|^2\right)-2(\lambda+1)\int_\Omega G(u,v)\,dx
\end{aligned} \tag{3.10}$$

Inserting (3.10) into (3.9), we get

$$\begin{aligned}
\Psi'(t) \;=\; & H'(t)+\varepsilon\left(\frac{1}{s+1}+\frac{2(\lambda+1)}{s+2}\right)\left(\|u_t\|_{s+2}^{s+2}+\|v_t\|_{s+2}^{s+2}\right) \\
&+\varepsilon\lambda\left(\|\nabla u\|^2+\|\nabla v\|^2\right)+2\varepsilon(\lambda+1)H(t)-2\varepsilon(\lambda+1)E_1 \\
&+\varepsilon(1-\frac{\lambda+1}{\gamma+2})(\|u+v\|_{2(\gamma+2)}^{2(\gamma+2)}+2\|uv\|_{\gamma+2}^{\gamma+2}) \\
&-\varepsilon\left(\int_\Omega u\left(|u|^w+|v|^b\right)u_t|u_t|^{\alpha-1}\,dx+\int_\Omega v\left(|v|^c+|u|^d\right)v_t|v_t|^{\beta-1}\,dx\right).
\end{aligned}$$

Then using (3.4), we have

$$\begin{aligned}
\Psi'(t) \;\geq\; & H'(t)+\varepsilon\left(\frac{1}{s+1}+\frac{2(\lambda+1)}{s+2}\right)\left(\|u_t\|_{s+2}^{s+2}+\|v_t\|_{s+2}^{s+2}\right) \\
&+\varepsilon\lambda\left(\|\nabla u\|^2+\|\nabla v\|^2\right) \\
&+2(\lambda+1)\varepsilon H(t)+\varepsilon c'(\|u+v\|_{2(\gamma+2)}^{2(\gamma+2)}+2\|uv\|_{\gamma+2}^{\gamma+2}) \\
&-\varepsilon\left(\int_\Omega u\left(|u|^w+|v|^b\right)u_t|u_t|^{\alpha-1}\,dx+\int_\Omega v\left(|v|^c+|u|^d\right)v_t|v_t|^{\beta-1}\,dx\right)
\end{aligned} \tag{3.11}$$

where $c'=1-\frac{\lambda+1}{\gamma+2}-2(\lambda+1)E_1(B\alpha_2)^{-2(\gamma+2)}>0$, since $\alpha_2>B^{-\frac{\gamma+2}{\gamma+1}}$. For estimating the last two terms in (3.11) we will use the following Young inequality

$$AB\leq\frac{\eta^kA^k}{k}+\frac{\eta^{-l}B^l}{l},$$

where $A,B\geq0,\ \eta>0,\ k,l\in R^+$ such that $\frac{1}{k}+\frac{1}{l}=1$. Therefore, using the above inequality we obtain

$$\int_\Omega uu_t|u_t|^{\alpha-1}\,dx\leq\frac{\eta_1^{\alpha+1}}{\alpha+1}\|u\|_{\alpha+1}^{\alpha+1}+\frac{\alpha\eta_1^{-\frac{\alpha+1}{\alpha}}}{\alpha+1}\|u_t\|_{\alpha+1}^{\alpha+1},$$

and therefore

$$\begin{aligned}
\int_\Omega\left(|u|^w+|v|^b\right)uu_t|u_t|^{\alpha-1}\,dx \;\leq\; & \frac{\eta_1^{\alpha+1}}{\alpha+1}\int_\Omega\left(|u|^w+|v|^b\right)|u|^{\alpha+1}\,dx \\
&+\frac{\alpha\eta_1^{-\frac{\alpha+1}{\alpha}}}{\alpha+1}\int_\Omega\left(|u|^w+|v|^b\right)|u_t|^{\alpha+1}\,dx.
\end{aligned}$$

In the same way, we conclude that

$$\int_\Omega vv_t|v_t|^{\beta-1}\,dx\leq\frac{\eta_2^{\beta+1}}{\beta+1}\|v\|_{\beta+1}^{\beta+1}+\frac{\beta\eta_2^{-\frac{\beta+1}{\beta}}}{\beta+1}\|v_t\|_{\beta+1}^{\beta+1},$$

and therefore

$$\begin{aligned}\int_\Omega v\left(|v|^c+|u|^d\right)v_t|v_t|^{\beta-1}dx &\leq \frac{\eta_2^{\beta+1}}{\beta+1}\int_\Omega\left(|v|^c+|u|^d\right)|v|^{\beta+1}dx\\ &+\frac{\beta\eta_2^{-\frac{\beta+1}{\beta}}}{\beta+1}\int_\Omega\left(|v|^c+|u|^d\right)|v_t|^{\beta+1}dx,\end{aligned}$$

where $\eta_1,\ \eta_2$ are constants depending on the time t and specified later. Consequently, (3.11) reduce

$$\begin{aligned}\Psi'(t) &\geq H'(t)+\varepsilon\left(\frac{1}{s+1}+\frac{2(\lambda+1)}{s+2}\right)\left(\|u_t\|_{s+2}^{s+2}+\|v_t\|_{s+2}^{s+2}\right)\\ &+\varepsilon\lambda\left(\|\nabla u\|^2+\|\nabla v\|^2\right)+2\varepsilon(\lambda+1)H(t)+\varepsilon c'\left(\|u+v\|_{2(\gamma+2)}^{2(\gamma+2)}+2\|uv\|_{\gamma+2}^{\gamma+2}\right)\\ &-\varepsilon\frac{\eta_1^{\alpha+1}}{\alpha+1}\int_\Omega\left(|u|^w+|v|^b\right)|u|^{\alpha+1}dx-\varepsilon\frac{\alpha\eta_1^{-\frac{\alpha+1}{\alpha}}}{\alpha+1}\int_\Omega\left(|u|^w+|v|^b\right)|u_t|^{\alpha+1}dx\\ &-\varepsilon\frac{\eta_2^{\beta+1}}{\beta+1}\int_\Omega\left(|v|^c+|u|^d\right)|v|^{q+1}dx-\varepsilon\frac{\beta\eta_2^{-\frac{\beta+1}{\beta}}}{\beta+1}\int_\Omega\left(|v|^c+|u|^d\right)|v_t|^{\beta+1}dx.\end{aligned} \tag{3.12}$$

By using Young's inequality, we have

$$\begin{aligned}\int_\Omega\left(|u|^w+|v|^b\right)|u|^{\alpha+1}dx &\leq \int_\Omega|u|^{w+\alpha+1}dx+\int_\Omega|v|^b|u|^{\alpha+1}dx\\ &\leq \int_\Omega|u|^{w+\alpha+1}dx+\frac{b}{b+\alpha+1}\chi_1^{\frac{b+\alpha+1}{b}}\int_\Omega|v|^{b+\alpha+1}dx\\ &+\frac{\alpha+1}{b+\alpha+1}\chi_1^{-\frac{b+\alpha+1}{\alpha+1}}\int_\Omega|u|^{b+\alpha+1}dx\\ &= \|u\|_{w+\alpha+1}^{w+\alpha+1}+\frac{b}{b+\alpha+1}\chi_1^{\frac{b+\alpha+1}{b}}\|v\|_{b+\alpha+1}^{b+\alpha+1}\\ &+\frac{\alpha+1}{b+\alpha+1}\chi_1^{-\frac{b+\alpha+1}{\alpha+1}}\|u\|_{b+\alpha+1}^{b+\alpha+1}.\end{aligned} \tag{3.13}$$

Similarly

$$\begin{aligned}\int_\Omega\left(|v|^c+|u|^d\right)|v|^{\beta+1}dx &\leq \|v\|_{c+\beta+1}^{c+\beta+1}+\frac{d}{d+\beta+1}\chi_2^{\frac{\beta+d+1}{d}}\|u\|_{d+\beta+1}^{d+\beta+1}\\ &+\frac{\beta+1}{d+\beta+1}\chi_2^{-\frac{d+\beta+1}{\beta+1}}\|v\|_{d+\beta+1}^{d+\beta+1}.\end{aligned} \tag{3.14}$$

Inserting (3.14) and (3.13) into (3.12) and using Lemma 1, we conclude that

$$\begin{aligned}\Psi'(t) \geq\ & H'(t) + \varepsilon\left(\frac{1}{s+1} + \frac{2(\lambda+1)}{s+2}\right)\left(\|u_t\|_{s+2}^{s+2} + \|v_t\|_{s+2}^{s+2}\right) \\ & + \varepsilon\lambda\left(\|\nabla u\|^2 + \|\nabla v\|^2\right) + 2\varepsilon(\lambda+1)H(t) + \varepsilon c'\left(\|u\|_{2(\gamma+2)}^{2(\gamma+2)} + \|v\|_{2(\gamma+2)}^{2(\gamma+2)}\right) \\ & - \varepsilon\frac{\eta_1^{\alpha+1}}{\alpha+1}\left(\|u\|_{w+\alpha+1}^{w+\alpha+1} + \frac{b}{b+\alpha+1}\chi_1^{\frac{b+\alpha+1}{b}}\|v\|_{b+\alpha+1}^{b+\alpha+1} + \frac{\alpha+1}{b+\alpha+1}\chi_1^{-\frac{b+\alpha+1}{\alpha+1}}\|u\|_{b+\alpha+1}^{b+\alpha+1}\right) \\ & - \varepsilon\frac{\eta_2^{\beta+1}}{\beta+1}\left(\|v\|_{c+\beta+1}^{c+\beta+1} + \frac{d}{d+\beta+1}\chi_2^{\frac{d+\beta+1}{d}}\|u\|_{d+\beta+1}^{d+\beta+1} + \frac{\beta+1}{d+\beta+1}\chi_2^{-\frac{d+\beta+1}{\beta+1}}\|v\|_{d+\beta+1}^{d+\beta+1}\right) \\ & - \varepsilon\frac{\alpha\eta_1^{-\frac{\alpha+1}{\alpha}}}{\alpha+1}\int_\Omega\left(|u|^w + |v|^b\right)|u_t|^{\alpha+1}dx - \varepsilon\frac{\beta\eta_2^{-\frac{\beta+1}{\beta}}}{\beta+1}\int_\Omega\left(|v|^c + |u|^d\right)|v_t|^{\beta+1}dx.\end{aligned} \tag{3.15}$$

Since

$$2(\gamma+2) > \max\{w+\alpha+1, b+\alpha+1, c+\beta+1, d+\beta+1\},$$

and applying the following algebraic inequality

$$x^\upsilon \leq x + 1 \leq \left(1 + \frac{1}{\delta}\right)(x+\delta), \quad \forall x \geq 0,\ 0 < \upsilon \leq 1,\ \delta > 0,$$

we have, for all $t \geq 0$,

$$\begin{aligned}\|u\|_{w+\alpha+1}^{w+\alpha+1} &\leq c_1\|u\|_{2(\gamma+2)}^{w+\alpha+1} \leq C\left(\|u\|_{2(\gamma+2)}^{2(\gamma+2)} + H(0)\right) \\ &\leq C\left(\|u\|_{2(\gamma+2)}^{2(\gamma+2)} + H(t)\right),\end{aligned} \tag{3.16}$$

$$\|v\|_{c+\beta+1}^{c+\beta+1} \leq c_2\|v\|_{2(\gamma+2)}^{c+\beta+1} \leq C\left(\|v\|_{2(\gamma+2)}^{2(\gamma+2)} + H(t)\right) \tag{3.17}$$

where $C = 1 + \frac{1}{H(0)}$. Likewise

$$\|u\|_{d+\beta+1}^{d+\beta+1} \leq c_3\|u\|_{2(\gamma+2)}^{d+\beta+1} \leq C\left(\|u\|_{2(\gamma+2)}^{2(\gamma+2)} + H(t)\right), \tag{3.18}$$

$$\|v\|_{b+\alpha+1}^{b+\alpha+1} \leq c_4\|v\|_{2(\gamma+2)}^{b+\alpha+1} \leq C\left(\|v\|_{2(\gamma+2)}^{2(\gamma+2)} + H(t)\right), \tag{3.19}$$

$$\|v\|_{d+\beta+1}^{d+\beta+1} \leq c_5\|v\|_{2(\gamma+2)}^{d+\beta+1} \leq C\left(\|v\|_{2(\gamma+2)}^{2(\gamma+2)} + H(t)\right), \tag{3.20}$$

$$\|u\|_{b+\alpha+1}^{b+\alpha+1} \leq c_6\|u\|_{2(\gamma+2)}^{b+\alpha+1} \leq C\left(\|u\|_{2(\gamma+2)}^{2(\gamma+2)} + H(t)\right). \tag{3.21}$$

Choosing K_1, K_2, K_3, K_4 and K_5 such that

$$K_1 = \frac{\alpha\eta_1^{-(\alpha+1)/\alpha}}{\alpha+1}, \quad K_2 = \frac{\beta\eta_1^{-(\beta+1)/\beta}}{\beta+1}, \tag{3.22}$$

$$K_3 = \frac{\eta_1^{\alpha+1}}{\alpha+1}\left(1 + \frac{\alpha+1}{b+\alpha+1}\chi_1^{-\frac{b+\alpha+1}{\alpha+1}}\right) + \left(\frac{\eta_2^{\beta+1}}{\beta+1}\frac{d}{d+\beta+1}\chi_2^{\frac{d+\beta+1}{d}}\right), \tag{3.23}$$

$$K_4 = \frac{\eta_2^{\beta+1}}{\beta+1}\left(1 + \frac{\beta+1}{d+\beta+1}\chi_2^{-\frac{d+\beta+1}{\beta+1}}\right) + \left(\frac{\eta_1^{\alpha+1}}{\alpha+1}\frac{b}{b+\alpha+1}\chi_1^{\frac{b+\alpha+1}{b}}\right), \tag{3.24}$$

and

$$
\begin{aligned}
K_5 &= \frac{\eta_1^{\alpha+1}}{\alpha+1}(1+\frac{b}{b+\alpha+1}\chi_1^{\frac{b+\alpha+1}{b}}+\frac{\alpha+1}{b+\alpha+1}\chi_1^{-\frac{b+\alpha+1}{\alpha+1}}) \\
&\quad +\frac{\eta_2^{\beta+1}}{\beta+1}(1+\frac{\beta+1}{d+\beta+1}\chi_2^{-\frac{d+\beta+1}{\beta+1}}+\frac{d}{d+\beta+1}\chi_2^{\frac{d+\beta+1}{d}}). \qquad (3.25)
\end{aligned}
$$

From (3.16)-(3.25), (3.15) becomes such that

$$
\begin{aligned}
\Psi'(t) &\geq H'(t)+\varepsilon\left(\frac{1}{s+1}+\frac{2(\lambda+1)}{s+2}\right)\left(\|u_t\|_{s+2}^{s+2}+\|v_t\|_{s+2}^{s+2}\right) \\
&\quad +\varepsilon\lambda\left(\|\nabla u\|^2+\|\nabla v\|^2\right)+\varepsilon\left[2(\lambda+1)-CK_5\right]H(t) \\
&\quad -\varepsilon K_1\int_\Omega\left(|u|^w+|v|^b\right)|u_t|^{\alpha+1}dx-\varepsilon K_2\int_\Omega\left(|v|^c+|u|^d\right)|v_t|^{\beta+1}dx \\
&\quad +\varepsilon\left[c'-CK_3\right]\|u\|_{2(\gamma+2)}^{2(\gamma+2)}+\varepsilon\left[c'-CK_4\right]\|v\|_{2(\gamma+2)}^{2(\gamma+2)}. \qquad (3.26)
\end{aligned}
$$

At this point, we can find positive constats M_1 , M_2, M_3 and K_6 such that (3.26) becomes

$$
\begin{aligned}
\Psi'(t) &\geq (1-\varepsilon K_6)H'(t)+\varepsilon\left(\frac{1}{s+1}+\frac{2(\lambda+1)}{s+2}\right)\left(\|u_t\|_{s+2}^{s+2}+\|v_t\|_{s+2}^{s+2}\right) \\
&\quad +\varepsilon\gamma\left(\|\nabla u\|^2+\|\nabla v\|^2\right)+\varepsilon M_1H(t)+\varepsilon M_2\|u\|_{2(\gamma+2)}^{2(\gamma+2)}+\varepsilon M_3\|v\|_{2(\gamma+2)}^{2(\gamma+2)} \\
&\geq \beta\left(\|u_t\|_{s+2}^{s+2}+\|v_t\|_{s+2}^{s+2}+H(t)+\|\nabla u\|^2+\|\nabla v\|^2+\|u\|_{2(\gamma+2)}^{2(\gamma+2)}+\|v\|_{2(\gamma+2)}^{2(\gamma+2)}\right) \qquad (3.27)
\end{aligned}
$$

where $\beta=\min\left\{\varepsilon\left(\frac{1}{s+1}+\frac{2(\lambda+1)}{s+2}\right),\varepsilon M_1,\varepsilon M_2,\varepsilon M_3\right\}$ and we pick ε small enough so that $1-\varepsilon K_6\geq 0$. Consequently we have

$$\Psi(t)\geq\Psi(0)>0,\ \forall t\geq 0. \qquad (3.28)$$

We now estimate $\Psi(t)$. By using Hölder inequality and Young inequality, we find

$$
\begin{aligned}
\left|\int_\Omega |u_t|^s u_t u dx\right| &\leq \|u_t\|_{s+2}^{s+1}\|u\|_{s+2} \\
&\leq C|\Omega|^{\frac{1}{s+2}-\frac{1}{2(\gamma+2)}}\left(\|u_t\|_{s+2}^{s+2}+\|u\|_{2(\gamma+2)}^{s+2}\right) \\
&\leq C|\Omega|^{\frac{1}{s+2}-\frac{1}{2(\gamma+2)}}\left(\|u_t\|_{s+2}^{s+2}+\|u\|_{2(\gamma+2)}^{2(\gamma+2)}\right). \qquad (3.29)
\end{aligned}
$$

Similarly

$$\left|\int_\Omega |v_t|^s v_t v dx\right|\leq C|\Omega|^{\frac{1}{s+2}-\frac{1}{2(\gamma+2)}}\left(\|v_t\|_{s+2}^{s+2}+\|v\|_{2(\gamma+2)}^{2(\gamma+2)}\right). \qquad (3.30)$$

By (3.7), it yields

$$
\begin{aligned}
\Psi(t) &= H(t)+\frac{\varepsilon}{s+1}\left(\int_\Omega |u_t|^s u_t u dx+\int_\Omega |v_t|^s v_t v dx\right) \\
&\leq C\left(\|u_t\|_{s+2}^{s+2}+\|v_t\|_{s+2}^{s+2}+H(t)+\|u\|_{2(\gamma+2)}^{2(\gamma+2)}+\|v\|_{2(\gamma+2)}^{2(\gamma+2)}+\|\nabla u\|^2+\|\nabla v\|^2\right). \qquad (3.31)
\end{aligned}
$$

By unifacation of (3.27) and (3.31) we reach at for $\xi>0$ constant

$$\Psi'(t)\geq\xi\Psi(t). \qquad (3.32)$$

A simple integration of (3.32) over $(0,t)$ yields $\Psi(t) \geq \Psi(0)\exp(\xi t)$ .We showed the desired result. □

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GLOBAL EXISTENCE OF SOLUTIONS TO EQUATION WITH DEGENERATE DAMPING

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Abstract. This study deal with the strongly damped equation with degenerate damping has the initial-boundary value. We establish global existence of weak solution by potential well theory.

1. Introduction and Preliminaries

In this work, we focus on the global existence of solution for the following problem

$$\left\{ \begin{array}{ll} u_{tt} + \Delta^2 u - \Delta u_t - \Delta u_{tt} + \left(|u|^{\varrho} u\right)_t = |u|^p u & \text{in } \Omega \times (0,T), \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), & x \in \Omega, \\ u(x,t) = \frac{\partial}{\partial n} u(x,t) = 0, & x \in \partial\Omega, \end{array} \right. \tag{1.1}$$

n is the outer normal and Ω is a bounded domain in R^n with a smooth boundary $\partial\Omega$.

This type of problem with degenerate damping was firstly examined by Levine and Serrin [2] and studied the blow up properties for negative initial energy. Then, Pitts and Rammaha [3] obtained global and local existence. In additional, the authors obtained blow up solutions for negative initial energy.

There are numerous study has degenerate damping terms (see [1, 4, 5, 6, 7, 8]).

Now, we present some preliminary material which will be helpful in the proof of our result. Throughout this paper, we denote the standart $L^2(\Omega)$ norm by $\|.\| = \|.\|_{L^2(\Omega)}$ and $L^q(\Omega)$ norm $\|.\|_q = \|.\|_{L^q(\Omega)}$.

We present the following functionalls:

$$I(t) = I(u) = \|\Delta u\|^2 - \|u\|_{p+2}^{p+2},$$

$$J(t) = J(u) = \frac{1}{2}\|\Delta u\|^2 - \frac{1}{p+2}\|u\|_{p+2}^{p+2},$$

$$E(t) = E(u) = \frac{1}{2}\left[\|u_t\|^2 + \|\nabla u_t\|^2\right] + J(u), \tag{1.2}$$

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and as in [12], the potential well depth such that

$$d = \inf_{u \in H_0^2(\Omega)\setminus\{0\}} \max_{\lambda \geq 0} j(\gamma u).$$

By multiplying Eq.(1.1) by u_t, integrating over Ω, and using integration by parts, we get

$$E'(t) = -\|\nabla u_t\|^2 - (\varrho+1)\int_\Omega |u|^\varrho |u_t|^2 \, dx \leq 0, \text{ for } t \geq 0.$$

Thus,

$$E(t) \leq E(0).$$

We can now define the stable set [9, 10, 11]

$$\Sigma = \left\{ u \in H_0^2 \,|\, I(u) > 0, J(u) < d \right\}$$

and otherwise the stable set can be defined by

$$\Sigma = \{(\gamma, E) \in [0, +\infty) \times R : 0 < h(\gamma) \leq E < d, 0 < \gamma < \gamma_0\},$$

in which $h(\gamma) = \frac{1}{2}\gamma^2 - C_*^{q+2}\frac{\gamma^{p+2}}{p+2}$, h attains its absolute maximum point for $\gamma_0 = C_*^{-\frac{p+2}{p}}$, and finally $d = h(\gamma_0) = \left(\frac{1}{2} - \frac{1}{p+2}\right)\gamma_0^2 > 0$.

Now, we give main result in the next section.

2. Global existence of solutions

Lemma 2.1. *Assume that u is solution of problem (1.1), and $u_0, u_1 \in H_0^2(\Omega)$, if $u_0, u_1 \in \Sigma$ and $E(0) < d$, then $u(t)$ remains inside the set $\in \Sigma$ for any $t \geq 0$.*

The proof is similiar to that of Lemma 2.2 in [10], thus we omit it.

Theorem 2.2. *Let $\varrho > 1$, if $n = 1, 2$; $\varrho < \frac{2n}{n-2}$, if $n \geq 3$; $u_0, u_1 \in H_0^2(\Omega)$, suppose that $\varrho > q$, $E(0) < d$ and $u_0 \in \Sigma$, then the problem (1.1) is bounded and global in time. Moreover a global weak solution u and $u(.) \in \Sigma$ for $t \geq 0$.*

Proof. From Lemma 1, we get $u(t) \in \Sigma$ for all $t \in [0, T_0)$, then $I(u) > 0$, $J(u) < d$ for all $t \in [0, T_0)$. Therefore,

$$\left(\frac{1}{2} - \frac{1}{p+2}\right)\|u\|_{p+2}^{p+2} = \frac{1}{2}\|\Delta u\|^2 - \frac{1}{p+2}\|u\|_{p+2}^{p+2} - \frac{1}{2}I(u) \leq J(u) < d, \tag{2.1}$$

then

$$\|u\|_{p+2}^{p+2} < d.$$

By the energy equation (1.2), by definition of $J(u)$ and (2.1), we reach

$$\frac{1}{2}\left[\|u_t\|^2 + \|\nabla u_t\|^2 + \|\Delta u\|^2\right] \leq E(0) + \frac{1}{p+2}\|u\|_{p+2}^{p+2} \leq Cd, \text{ for } 0 \leq t < T_0, \tag{2.2}$$

That is, u is a global solution. Lastly from Lemma 1 we have $u \in \Sigma$ for $t \in [0, \infty)$. ∎

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BIVARIATE MAX-PRODUCT BERNSTEIN CHLODOWSKY OPERATORS

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Abstract. In the approximation theory, polynomials are particularly positive linear operators. Nonlinear positive operators by means of maximum and product were introduced by B. Bede. In this study, nonlinear maximum product type Bivariate Bernstein Chlodowsky operators are defined and approximation properties are investigated with the help new definitions.

1. Introduction

The main topic in the classical approximation theory is approximating a continuous function $f : [a, b] \to \mathbb{R}$ with more elementary functions such as polynomials, trigonometric functions, etc.. The well-known Korovkin's theorem, which gives a simple proof of Weierstrass theorem, is based on the approximation of functions by linear and positive operators. The underlying algebraic structure of these mentioned operators is linear over $\mathbb{R}$ and they are also linear operators. In 2006, Bede et.al [1] asked whether they could change the underlying algebraic structure to more general structures. In this sense they presented nonlinear Shepard-type operators by replacing the operations sum and product by max and product. They proved Weierstrass-type uniform approximation theorem and obtained error estimates in terms of the modulus of continuity. Following this paper Bede et. al. [2] defined and studied pseudo linear approximation operators. Several authors introduced the nonlinear versions of the stated operators and studied order of approximation [1-4]. Also see [2] for the collected papers.In addition, in the book published in 2016, Bede et al. [2] gave the definition of Bivariate Max-Product Bernstein Operators and examined various approximation properties. In this study, we will give the definition of Bivariate Max-Product Bernstein Chlodowsky Operators and examine various approach features.

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2. Preliminaries

Two bivariate max-product Bernstein operators was defined in [2] (B. Bede, L. Coroianu, and S. G. Gal ,2016) as

$$B_{n,m}^{(M)}(f)(x,y) = \frac{\bigvee_{i=0}^{n}\bigvee_{j=0}^{m} P_{n,i}(x)\, P_{m,i}(y)\, f\left(\frac{i}{n},\frac{j}{m}\right)}{\bigvee_{i=0}^{n}\bigvee_{j=0}^{m} P_{n,i}(x)\, P_{m,i}(y)}$$

$$= \frac{\bigvee_{i=0}^{n}\bigvee_{j=0}^{m} P_{n,i}(x)\, P_{m,i}(y)\, f\left(\frac{i}{n},\frac{j}{m}\right)}{\bigvee_{i=0}^{n} P_{n,i}(x) \,.\, \bigvee_{j=0}^{m} P_{m,i}(y)}, (x,y) \in [0,1]^2, n,m \in \mathbb{N},$$

where $f : [0,1] \times [0,1] \to \mathbb{R}$ and $P_{n,i}(x) = \binom{n}{i} x^i (1-x)^{n-i}$. and

$$T_{n,m}^{(M)}(f)(x) = \frac{\bigvee_{i=0}^{n}\bigvee_{j=0}^{n-i} \binom{n}{i}\binom{n-i}{j} x^i y^j (1-x-y)^{n-i-j} f\left(\frac{i}{n},\frac{j}{n}\right)}{\bigvee_{i=0}^{n}\bigvee_{j=0}^{n-i} \binom{n}{i}\binom{n-i}{j} x^i y^j (1-x-y)^{n-i-j}}, (x,y) \in \Delta, n \in \mathbb{N},$$

where $f : \Delta \to \mathbb{R}^+$,$\Delta = (x,y)\,; x \geq 0, y \geq 0, x+y \leq 1$, [2].

Remarks. 1) Since we have $\bigvee_{i=0}^{n} P_{n,i}(x) \,.\, \bigvee_{j=0}^{m} P_{m,i}(y) > 0$ for all $x,y \in [0,1]$ and by Lemma 2.1.7in [2] in the univariate case, we explicitly can write

$$\bigvee_{i=0}^{n} P_{m,i}(x) \,.\, \bigvee_{j=0}^{m} P_{m,i}(y) = P_{n,r}(x) \,.P_{m,s}(y),$$

for all $(x,y) \in \left[\frac{r}{n+1}, \frac{r+1}{n+1}\right] \times \left[\frac{s}{m+1}, \frac{s+1}{m+1}\right], r = 0,...,n, s = 0,...,m,$ it follows that $B_{n,m}^{(M)}(f)(x,y)$ is well defined on $[0,1] \times [0,1]$ and a continuous functions of (x,y) in $[0,1]^2$.

Also

$$A_{i,n,r}(x) = \frac{P_{n,i}(x)}{P_{n,r}(x)} = \frac{\binom{n}{i}}{\binom{n}{r}} \left(\frac{x}{1-x}\right)^{i-r},$$

$$A_{j,m,s}(y) = \frac{P_{m,j}(y)}{P_{m,s}(y)} = \frac{\binom{m}{j}}{\binom{m}{s}} \left(\frac{y}{1-y}\right)^{j-s}$$

and

$$A_{i,n,r,j,m,s}(x,y) = A_{i,n,r}(x) \,.A_{j,m,s}(y)$$

write the following formula which is useful in proving approximate results,

$$B_{n,m}^{(M)}(f)(x,y) = \bigvee_{i=0}^{n}\bigvee_{j=0}^{m} A_{i,n,r,j,m,s}(x,y)\, f\left(\frac{i}{n},\frac{j}{m}\right)$$

for all $(x,y) \in \left[\frac{r}{n+1}, \frac{r+1}{n+1}\right] \times \left[\frac{s}{m+1}, \frac{s+1}{m+1}\right], r = 0,...,n, s = 0,...,m.$[2](B. Bede, L. Coroianu, and S. G. Gal ,2016)

2) It easily can be followed

$$B_{n,m}^{(M)}(f)(x,y) = B_{n,x}^{(M)}[B_{m,y}^{(M)}(f)](x,y),$$

where, if $G = G(x,y)$ then the notations $B_{n,x}^{(M)}(G)$ means that the univariate max-product Bernstein operator $B_n^{(M)}(G)$ is applied to G considered as function of x, while $B_{n,y}^{(M)}(G)$ means that the univariate max-product Bernstein operator $B_n^{(M)}(G)$ is applied to Gconsidered as function of y. In other words, the bivariate max-product Bernstein operators are tensor products of the univariate max product Bernstein operators. [2]

In order to obtain the shape-preserving properties, as in the univariate case, several shape concepts are needed in the bivariate case, and some of them are obtained by using the "tensor product" method.

Definition 2.1. [2]Let$f : [0,1] \times [0,1] \to \mathbb{R}$.

(i) We say that $f(x,y)$ is increasing (decreasing) with respect to x on $[0,1] \times [0,1]$,if

$$f(x+h,y) - f(x,y) \geq 0, (\leq 0), \forall y \in [0,1], \forall x, x+h \in [0,1], h > 0.$$

(ii) We say that $f(x,y)$ is increasing (decreasing) with respect to y on $[0,1] \times [0,1]$,if

$$f(x,y+h) - f(x,y) \geq 0, (\leq 0), \forall x \in [0,1], \forall y, y+h \in [0,1], h > 0.$$

(iii) We say that $f(x,y)$ is upper (lower) bidimensional monotone on $[0,1] \times [0,1]$ if

$$\Delta_2 f(x,y) = f(x+h,y+k) - f(x,y+k) - f(x+h,y) + f(x,y) \geq 0 \, (\leq 0),$$

for all $x, x+h \in [0,1], y, y+k \in [0,1], h \geq 0, k \geq 0$.

There are a few more items.(See the book B. Bede, L. Coroianu, and S. G. Gal ,2016.[2])

3. Construction of the Operators

The aim of this study is to introduce bivariate max-product Bernstein Chlodowsky operators and some properties of the this operators.

Definition 3.1. The bivariate maximum product Bernstein Chlodowsky operators are defined as

$$C_{n,m}^{(M)}(f)(x,y) = \frac{\bigvee_{i=0}^{n} \bigvee_{j=0}^{m} S_{n,i}(x) S_{m,j}(y) f\left(\frac{i}{n} b_n, \frac{j}{m} b_m\right)}{\bigvee_{i=0}^{n} S_{n,i}(x) . \bigvee_{j=0}^{m} S_{m,i}(y)}, (x,y) \in [0,b_n] \times [0,b_m], n,m \in \mathbb{N},$$

where $f : [0,b_n] \times [0,b_m] \to \mathbb{R}$ and $S_{n,i}(x) = \binom{n}{i} \left(\frac{x}{b_n}\right)^i \left(1 - \frac{x}{b_n}\right)^{n-i}, S_{m,j}(y) = \binom{m}{j} \left(\frac{y}{b_m}\right)^j \left(1 - \frac{y}{b_m}\right)^{m-j},$

$\lim_{n\to\infty} b_n = \infty$, $\lim_{m\to\infty} b_m = \infty$ and $\lim_{n\to\infty} \dfrac{b_n}{n} = 0, \lim_{m\to\infty} \dfrac{b_m}{m} = 0$.

Güngör et. al [4] investigated maximum product Bernstein Chlodowsky operators and they examined the various notations and approximation of these operators. The following notations and Theorem are needed to examine the future approximation properties of bivariate maximum product Bernstein Chlodowsky operators.

Theorem 3.2. *Denoting* $S_{n,i}(x) = \binom{n}{i}\left(\frac{x}{b_n}\right)^i\left(1-\frac{x}{b_n}\right)^{n-i}$,

$S_{m,j}(y) = \binom{m}{j}\left(\frac{y}{b_m}\right)^j\left(1-\frac{y}{b_m}\right)^{m-j}$, we have

$$\bigvee_{i=0}^{n} S_{m,i}(x) . \bigvee_{j=0}^{m} S_{m,j}(y) = S_{n,r}(x) . S_{m,s}(y),$$

for all $(x,y) \in \left[\frac{rb_n}{n+1}, \frac{(r+1)b_n}{n+1}\right] \times \left[\frac{sb_m}{m+1}, \frac{(s+1)b_m}{m+1}\right], r = 0,...,n, s = 0,...,m.$

Proof. Firsly , for $n \in \mathbb{N}$ and $0 \le r < r+1 \le n$, $0 \le s < s+1 \le m$,

$$\begin{aligned} 0 &\le S_{n,r+1}(x) \le S_{n,r}(x) \\ 0 &\le S_{m,s+1}(x) \le S_{m,s}(x) \end{aligned}$$

where $x \in \left[0, \frac{(r+1)b_n}{n+1}\right], y \in \left[0, \frac{(s+1)b_m}{m+1}\right]$. We have,

$$\begin{aligned} 0 &\le \binom{n}{r+1}\left(\frac{x}{b_n}\right)^{r+1}\left(1-\frac{x}{b_n}\right)^{n-(r+1)} \le \binom{n}{r}\left(\frac{x}{b_n}\right)^{r}\left(1-\frac{x}{b_n}\right)^{n-r} \\ 0 &\le \binom{m}{s+1}\left(\frac{y}{b_m}\right)^{s+1}\left(1-\frac{y}{b_m}\right)^{n-(s+1)} \le \binom{m}{s}\left(\frac{y}{b_m}\right)^{s}\left(1-\frac{y}{b_m}\right)^{n-s} \end{aligned}$$

and

$$\begin{aligned} 0 &\le \frac{x}{b_n}\left[\binom{n}{r+1} + \binom{n}{r}\right] \le \binom{n}{r} \\ 0 &\le \frac{y}{b_m}\left[\binom{m}{s+1} + \binom{m}{s}\right] \le \binom{m}{s} \end{aligned}$$

Since $\binom{n}{k+1} + \binom{n}{k} = \binom{n+1}{k+1}$,we get $0 \le x \le \frac{(r+1)b_n}{n+1}$, $0 \le y \le \frac{(s+1)b_m}{m+1}$.

Also

$$\begin{aligned} B_{i,n,r}(x) &= \frac{S_{n,i}(x)}{S_{n,r}(x)} = \frac{\binom{n}{i}}{\binom{n}{r}}\left(\frac{\frac{x}{b_n}}{1-\frac{x}{b_n}}\right)^{i-r}, \\ B_{j,m,s}(y) &= \frac{S_{m,j}(y)}{S_{m,s}(y)} = \frac{\binom{m}{j}}{\binom{m}{s}}\left(\frac{\frac{y}{b_m}}{1-\frac{y}{b_m}}\right)^{j-s} \end{aligned}$$

and

$$B_{i,n,r,j,m,s}(x,y) = B_{i,n,r}(x) . B_{j,m,s}(y)$$

write the following formula which is useful in proving approximate results,

$$C_{n,m}^{(M)}(f)(x,y) = \bigvee_{i=0}^{n}\bigvee_{j=0}^{m} B_{i,n,r,j,m,s}(x,y) f\left(\frac{ib_n}{n}, \frac{jb_m}{m}\right)$$

for all $(x,y) \in \left[\frac{rb_n}{n+1}, \frac{(r+1)b_n}{n+1}\right] \times \left[\frac{sb_m}{m+1}, \frac{(s+1)b_m}{m+1}\right], r = 0,...,n, s = 0,...,m.$ □

It easily can be followed

$$C_{n,m}^{(M)}(f)(x,y) = C_{n,x}^{(M)}[C_{m,y}^{(M)}(f)](x,y),$$

where, if $H = H(x,y)$ then the notations $C_{n,x}^{(M)}(H)$ means that the univariate max-product Bernstein Chlodowsky operator $C_n^{(M)}(H)$ is applied to H considered as function of x, while $C_{n,y}^{(M)}(G)$ means that the univariate max-product Bernstein Chlodowsky operator $C_n^{(M)}(H)$ is applied to H considered as function of y. In other words, the bivariate max-product Bernstein Chlodowsky operators are tensor products of the univariate max product Bernstein Chlodowsky operators.

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SOME RESULTS ON THE WHITEHEAD ASPHERICITY PROBLEM

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ABSTRACT. Given a group presentation $\mathcal{P} = \mathcal{GP}(\mathbf{x}, \mathbf{r})$, we consider the free $FG(\mathbf{x})$-crossed module $(\mathcal{G}(\Upsilon), \tilde{\theta}, FG(\mathbf{x}))$ on the set $Y^{\pm 1}$ of symbols $({}^u r)^\varepsilon$ ($\varepsilon = \pm 1$) with $r \in \mathbf{r}$. In terms of $\mathcal{G}(\Upsilon)$ we prove that if $d = (a_1, ..., a_n)$ is an identity Y-sequence over $\mathcal{P}$, then d is Peiffer equivalent to the empty sequence if and only if, the image of d in $\mathcal{G}(\Upsilon)$ belongs to the subgroup $\hat{\mathfrak{U}}$ of $\mathcal{G}(\Upsilon)$ generated by the images of aa^{-1} with $a \in Y \cup Y^{-1}$. We use this to prove a necessary and sufficient condition under which a subpresentation of an aspherical group presentation is aspherical. We also consider the pair of presentations $\mathcal{P} = \mathcal{GP}(\mathbf{x} \cup z, \mathbf{r}_1 \cup \{r_0\})$ and $\mathcal{P}_1 = \mathcal{GP}(\mathbf{x}, \mathbf{r}_1)$ where $\mathcal{P}$ represents the trivial group and $r_0 \notin \mathbf{r}_1$. If we let N_0 be the normal closure of r_0 in the free group F of $\mathbf{x}$, then we prove that if the presentation $\mathcal{P} = \mathcal{GP}(\mathbf{x} \cup z, \mathbf{r}_1 \cup \{r_0\})$ is aspherical, then the structure map $\tilde{\vartheta}_1$ of the free crossed module $(\tilde{\mathcal{C}}_1, F/N_0, \tilde{\vartheta}_1)$ on $\mathbf{r}_1$ over N/N_0 is injective.

1. INTRODUCTION

The Whitehead asphericity conjecture, raised as a problem in [15], asks whether any subcomplex of an aspherical 2-complex is also aspherical. In group theoretic terms it can be rephrased as follows: *given an aspherical presentation $\mathcal{P} = \mathcal{GP}(\mathbf{x}, \mathbf{r})$ of a group G, is it true that every subpresentation $\mathcal{P}' = \mathcal{GP}(\mathbf{x}', \mathbf{r}')$ of the first is also aspherical*? The aim of this paper is to give a necessary and sufficient condition under which a subpresentation of an aspherical group presentation is aspherical. To achieve this we use among other things, some results from the theory of monoid acts. In this section we give a rough idea of how monoid acts come into play. First, we recall that a group presentation $\mathcal{P} = \mathcal{GP}(\mathbf{x}, \mathbf{r})$ is aspherical if its geometric realisation $K(\mathcal{P})$ is an aspherical 2-complex, that is $\pi_2(K(\mathcal{P})) = 0$. In [2] Brown and Huebschmann have proved several key results about aspherical group presentation one of which is their proposition 14 that gives sufficient and necessary conditions under which a group presentation $\mathcal{P} = \mathcal{GP}(\mathbf{x}, \mathbf{r})$ is aspherical. As we use two of

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them in particular, we will state them here and explain their meanings. One of these conditions states that *the relation module $\mathcal{N}(\mathcal{P})$ is a free $\mathbb{Z}G$ module.* We give below the definition of $\mathcal{N}(\mathcal{P})$ and afterwards introduce its basis when $\mathcal{P}$ is aspherical. If $\mathcal{P} = \mathcal{GP}(\mathbf{x}, \mathbf{r})$ is a presentation for a group G, we denote by $FG(\mathbf{x})$ the free group on $\mathbf{x}$ and let $\alpha : FG(\mathbf{x}) \to G$ and $\beta : N \to N/[N, N]$ be the canonical homomorphisms where N is the normal closure of $\mathbf{r}$ in $FG(\mathbf{x})$ and $[N, N]$ its commutator subgroup. There is a well defined G-action on $\mathcal{N}(\mathcal{P}) = N/[N, N]$ given by

$$w^\alpha \cdot s^\beta = (w^{-1}sw)^\beta$$

for every $w \in FG(\mathbf{x})$ and $s \in N$. This action extends to an action of $\mathbb{Z}G$ over $\mathcal{N}(\mathcal{P})$ by setting

$$(w_1^\alpha \pm w_2^\alpha) \cdot s^\beta = (w_1^{-1}sw_1w_2^{-1}s^{\pm 1}w_2)^\beta.$$

Now the bases of $\mathcal{N}(\mathcal{P})$ as a free $\mathbb{Z}G$ module is the set of elements r^β with $r \in \mathbf{r}$. The other condition of proposition 14 states that *any identity Y-sequence for $\mathcal{P}$ is Peiffer equivalent to the empty sequence.* Related to the given data, it is denoted by H the free group on the set Y of symbols r^u where $r \in \mathbf{r}$ and $u \in FG(\mathbf{x})$. The group homomorphism $\theta : H \to FG(\mathbf{x})$ defined by $\theta(^u r) = uru^{-1}$ has kernel E the set of *identities among the relations for $\mathcal{P}$*. Besides H it is considered the free monoid on the set $Y \cup Y^{-1}$ consisting of strings $(a_1, ..., a_n)$ where $n \geq 0$ and each $a_i \in Y \cup Y^{-1}$. The elements of this monoid are usually called Y-sequences and a string $(a_1, ..., a_n)$ for which $\theta(a_1) \cdots \theta(a_n) = 1$ in $FG(\mathbf{x})$ is called an *identity Y-sequences for $\mathcal{P}$*. Of a particular importance is the concept of Peiffer operations on Y-sequences.

(i) An *elementary Peiffer exchange* replaces an adjacent pair (a, b) in a Y-sequence by either $(b, ^{\theta(b^{-1})}a)$ or $(^{\theta(a)}b, a)$.
(ii) A *Peiffer deletion* deletes an adjacent pair (a, a^{-1}) in a Y-sequence.
(iii) A *Peiffer insertion* is the inverse of the Peiffer deletion.

The equivalence relation on the set of Y-sequences generated by the above operations is called *Peiffer equivalence.* In the next section we will see that, when it comes for the study of aspherical group presentations, Peiffer operations on Y-sequences can be better understood within the framework of the theory of monoid actions. For the benefit of the reader not familiar with monoid actions we will list below some basic notions and results that are used in the paper. For further results on the subject the reader may consult the monograph [7]. If S is a monoid with identity element 1 and X a nonempty set, we say that X is a *left S-system* if there is an action $(s, x) \mapsto sx$ from $S \times X$ into X with the properties

$$(st)x = s(tx) \text{ for all } s, t \in S \text{ and } x \in X,$$
$$1x = x \text{ for all } x \in X.$$

Right S-systems are defined dually in the obvious way. If S and T are (not necessarily different) monoids, we say that X is an *(S,T)-bisystem* if it is a left S-system, a right T-system, and if

$$(sx)t = s(xt) \text{ for all } s \in S, t \in T \text{ and } x \in X.$$

If X and Y are both left S-systems, then an *S-morphism* or *S-map* is a map $\phi : X \to Y$ such that

$$\phi(sx) = s\phi(x) \text{ for all } s \in S \text{ and } x \in X.$$

Morphisms of right S-systems and of (S,T)-bisystems are defined in an analogue way. If we are given a left T-system X and a right S-system Y, then we can give the cartesian product $X \times Y$ the structure of an (T,S)-bisystem by setting

$$t(x,y) = (tx,y) \text{ and } (x,y)s = (x,ys).$$

Let now A be an (T,U)-bisystem, B an (U,S)-bisystem and C an (T,S)-bisystem. As explained above, we can give to $A\times B$ the structure of an (T,S)-bisystem. With this in mind we say that a (T,S)-map $\beta : A \times B \to C$ is a *bimap* if

$$\beta(au,b) = \beta(a,ub) \text{ for all } a \in A, b \in B \text{ and } u \in U.$$

A pair $(A\otimes_U B, \psi)$ consisting of a (T,S)-bisystem $A\otimes_U B$ and a bimap $\psi : A\times B \to A\otimes_U B$ will be called a *tensor product of A and B over U* if for every (T,S)-bisystem C and every bimap $\beta : A\times B \to C$, there exists a unique (T,S)-map $\bar{\beta} : A\otimes_U B \to C$ such that the diagram

$$\begin{array}{ccc} A \times B & \xrightarrow{\ \psi\ } & A \otimes_U B \\ {\scriptstyle\beta}\big\downarrow & \swarrow{\scriptstyle\bar{\beta}} & \\ C & & \end{array}$$

commutes. It is proved that $A \otimes_U B$ exists and is unique up to isomorphism. The existence theorem reveals that $A \otimes_U B = (A \times B)/\tau$ where τ is the equivalence on $A \times B$ generated by the relation

$$T = \{((au,b),(a,ub)) : a \in A, b \in B, u \in U\}.$$

The equivalence class of a pair (a,b) is usually denoted by $a \otimes_U b$. To us is of interest the situation when $A = S = B$ where S is a monoid and U is a submonoid of S. Here A is clearly regarded as an (S,U)-bisystem with U acting on the right of A by multiplication, and B as an (U,S)-bisystem where U acts on the left of B by multiplication. Another concept that is important to our approach is that of the dominion. If U is a submonoid of a monoid S, then we say that a is in the *dominion of U in S*, written as $a \in \mathrm{Dom}_S(U)$, if for all monoids T and all monoid homomorphisms $f,g : S \to T$ that agree on U, we have that $f(a) = g(a)$. Related to dominions there is the well know zigzag theorem of Isbell. We will present here the Stenstrom version of it which reads. *Let U be a submonoid of a monoid S and let $d \in S$. Then, $d \in Dom_S(U)$ if and only if $d \otimes_U 1 = 1 \otimes_U d$ in the tensor product* $A = S \otimes_U S$. We mention here that this result holds true if S turns out to be a group and U a subgroup, both regarded as monoids. A key result that is used to prove our main theorem in the next section is the fact that any inverse semigroup U is absolutely closed in the sense that for every semigroup S containing U as a subsemigroup, $\mathrm{Dom}_S(U) = U$. It is obvious that groups are absolutely closed as special cases of inverse monoids (see [8]).

The monoids involved in our approach are the following. The first one is the monoid Υ defined by the monoid presentation $\mathcal{MP}(Y \cup Y^{-1}, P)$ where Y^{-1} is the set of group inverses of the elements of Y and P consists of all pairs $(ab, {}^{\theta(a)}ba)$ where $a,b \in Y \cup Y^{-1}$. The second one is the group $\mathcal{G}(\Upsilon)$ given by the group presentation $\mathcal{GP}(Y \cup Y^{-1}, \hat{P})$ where $\hat{P}$ is the set of all words $ab\iota(a)\iota({}^{\theta(a)}b)$ where by $\iota(c)$ we denote the inverse of c in the free group over $Y \cup Y^{-1}$. Before we introduce the next two monoids and the respective monoid actions, we stop to explain that Υ and $\mathcal{G}(\Upsilon)$ are special cases of a more general situation. If a monoid S is given by the monoid presentation $\mathcal{MP}(X,R)$, then its *universal enveloping group* $\mathcal{G}(S)$ (see [1] and [4]) is defined to be the group given by the group presentation $\mathcal{GP}(X,\hat{R})$ where $\hat{R}$ consists of all words $u\iota(v)$ whenever $(u,v)\in R$ where $\iota(v)$ is the inverse of v in the free group over X. We let for future use $\sigma : FM(X)\to S$ be the respective canonical homomorphism where $FM(X)$ is the free monoid on X. It is easy to see that there is a monoid homomorphism $\mu_S : S\to\mathcal{G}(S)$ which satisfies the following universal property. For every group G and monoid homomorphism $f : S\to G$, there is a unique group homomorphism $\hat{f} : \mathcal{G}(S)\to G$ such that $\hat{f}\mu_S = f$. This universal property is indication of an adjoint situation. Specifically, the functor $\mathcal{G} : \mathbf{Mon}\to\mathbf{Grp}$ which maps every monoid to its universal group, is a left adjoint to the forgetful functor $U : \mathbf{Grp}\to\mathbf{Mon}$. This ensures that $\mathcal{G}(S)$ is an invariant of the presentation of S.

The third monoid we consider is the submonoid $\mathfrak{U}$ of Υ, having the same unit as Υ, and is generated from all the elements of the form $\sigma(a)\sigma(a^{-1})$ with $a\in Y\cup Y^{-1}$. This monoid, acts on the left and on the right of Υ by the multiplication in Υ. The last monoid considered is the subgroup $\hat{\mathfrak{U}}$ of $\mathcal{G}(\Upsilon)$ generated by $\mu(\mathfrak{U})$. Similarly to above, $\hat{\mathfrak{U}}$ acts on $\mathcal{G}(\Upsilon)$ by multiplication.

In the next section we will see that an identity Y-sequence $(a_1,...,a_n)$ is Peiffer equivalent to the empty sequence if and only if for the element $a=\mu(\sigma(a_1)...\sigma(a_n))$ of $\mathcal{G}(\Upsilon)$ we have $a\otimes_{\hat{\mathfrak{U}}}1 = 1\otimes_{\hat{\mathfrak{U}}}a$ in the tensor product $\mathcal{G}(\Upsilon)\otimes_{\hat{\mathfrak{U}}}\mathcal{G}(\Upsilon)$. From the zigzag theorem of Isbell the last equality is equivalent to assuming that $a\in \mathrm{Dom}_{\mathcal{G}(\Upsilon)}(\hat{\mathfrak{U}})$, where $\mathrm{Dom}_{\mathcal{G}(\Upsilon)}(\hat{\mathfrak{U}})$ is the dominion of $\hat{\mathfrak{U}}$ in $\mathcal{G}(\Upsilon)$. Recalling that the group $\hat{\mathfrak{U}}$ is absolutely closed we infer that an identity Y-sequence $(a_1,...,a_n)$ is Peiffer equivalent to the empty sequence if and only if $a=\mu(\sigma(a_1)...\sigma(a_n))\in\hat{\mathfrak{U}}$. Having proved this it is not to difficult to prove our theorem 2.6 which gives a necessary and sufficient condition under which a subpresentation of an aspherical group presentation is itself aspherical.

In the second part of the paper we are interested for pairs of presentations $\mathcal{P}=\mathcal{GP}(\mathbf{x},\mathbf{r}_1\cup\mathbf{r}_0)$ and $\mathcal{P}_1=\mathcal{GP}(\mathbf{x},\mathbf{r}_1)$ where $\mathbf{r}_1\cap\mathbf{r}_0=\emptyset$, $\mathbf{r}_0=\{r_0\}$ is a singleton and $\mathcal{P}$ is an aspherical presentation of the trivial group. This situation is of a particular interest due to a result of Ivanov from [9] which states that *if the Whitehead conjecture is false, then there is an aspherical presentation $E=\langle\mathcal{A},\mathcal{R}\cup z\rangle$ of the trivial group E, where the alphabet $\mathcal{A}$ is finite or countably infinite and $z\in\mathcal{A}$, such that its subpresentation $\langle\mathcal{A},\mathcal{R}\rangle$ is not aspherical.* In virtue of this, we see that the conjecture is true if and only if it is true for subpresentations that differs from the given aspherical presentation by a single defining relation. This problem is very difficult and no answer is know, but if we relativize the problem then the answer is affirmative. To make this term precise, we denote by N the normal closure of $\mathbf{r}_1\cup\mathbf{r}_0$ in the free group F which coincides to the latter in our case, and by N_0 the normal closure of $\mathbf{r}_0$ in F, and let $(\tilde{\mathcal{C}}_1, N/N_0,\tilde{\vartheta}_1)$ be the free crossed module over N/N_0 on $\mathbf{r}_1$. Then we prove in theorem 3.1 that if $\mathcal{P}$ is aspherical, then the map $\tilde{\vartheta}_1$ is injective.

To prove our theorem 3.1 we need from [10] the notion of the semidirect product of two crossed modules. We give below the ingredients which make possible the definition. For a crossed module (T,G,∂) we denote by $\mathrm{Aut}(T,G,\partial)$ the group of automorphisms of (T,G,∂) and by $\mathrm{Der}(T,G,\partial)$ the set of all derivations from G to

T, that is maps $d : G \to T$ such that for all $x, y \in G$,

$$d(xy) = d(x)\ {}^{x}d(y).$$

Each such derivation d defines automorphisms σ and τ of G and T respectively given by

$$\sigma(x) = \partial d(x)x \text{ and } \tau(t) = d\partial(t)t.$$

$\mathrm{Der}(T, G, \partial)$ becomes a monoid (with identity the trivial derivation) if we define the product $d_1 \circ d_2 = d$ where

$$d(x) = d_1(\sigma_2(x))d_2(x) = \tau_1(d_2(x))d_1(x).$$

We let $\mathrm{D}(T, G, \partial)$ the group of units of $\mathrm{Der}(T, G, \partial)$ whose derivations are called the regular derivations and sometimes $\mathrm{D}(T, G, \partial)$ is called the Whitehead groups. There is a homomorphism $\Delta : \mathrm{D}(G, T) \to \mathrm{Aut}(T, G)$ defined by $\Delta(d) = \langle \tau, \sigma \rangle$ and there is an action of $\mathrm{Aut}(T, G)$ on $\mathrm{D}(G, T)$ defined by ${}^{\langle \sigma, \phi \rangle}d = \alpha d \phi^{-1}$. With this action, $(\mathrm{D}(G, T), \mathrm{Aut}(T, G), \Delta)$ becomes a crossed module which is called the actor crossed module $\mathcal{A}(T, G, \partial)$ of (T, G, ∂). Suppose now that we are given two crossed modules (M, P, μ) and (T, G, ∂) and that the first acts on the second which means that there is a morphism of crossed modules (a commutative diagram of groups)

$$\begin{array}{ccc} M & \xrightarrow{\ \mu\ } & P \\ \Big\downarrow{\scriptstyle \eta} & & \Big\downarrow{\scriptstyle \rho} \\ \mathrm{D}(G, T) & \xrightarrow[\Delta]{} & \mathrm{Aut}(T, G, \partial) \end{array}$$

Suppose that ρ has components $\rho_1 : P \to \mathrm{Aut}(T)$ and $\rho_2 : T \to \mathrm{Aut}(G)$, that is to say $\rho(p) = \langle \rho_1(p), \rho_2(p) \rangle$ for all $p \in P$. Then M acts on T via $\rho_1\mu$ and with this action we can form the semidirect product of groups $T \rtimes M$. Likewise, since P acts on G via ρ_2, we can form the semidirect product $G \rtimes P$. With these data one can define an action of $G \rtimes P$ on $T \rtimes M$ by

$${}^{(g,p)}(t, m) = \left({}^{g}({}^{p}t)(\eta({}^{p}m)g)^{-1},\ {}^{p}m\right).$$

The map $\pi : T \rtimes M \to G \rtimes P$ given by $(t, m) \mapsto (\partial(t), \mu(m))$ is a homomorphism. With the action just defined the triple $(T \rtimes M, G \rtimes P, \pi)$ is a crossed module called the semidirect product crossed module relative to $\langle \eta, \rho \rangle$ and denoted by $(T, G, \partial) \rtimes_{\langle \eta, \rho \rangle} (M, P, \mu)$.

Finally we mention that results related to ours can be found in [3], [5], [6] and [14]. Also a good account on the Whitehead asphericity problem can be found in [12].

2. Peiffer operations and monoid actions

If $\alpha = (a_1, ..., a_n)$ is any Y-sequence over the group presentation $\mathcal{P} = \langle \mathbf{x}, \mathbf{r} \rangle$, then performing an elementary Peiffer operation on α can be interpreted in a simple way in terms of monoids Υ and $\mathfrak{U}$ defined in the introduction. In what follows we will denote by $\sigma(\alpha)$ the element $\sigma(a_1) \cdots \sigma(a_n) \in \Upsilon$. If $\beta = (b_1, ..., b_n)$ is obtained from $\alpha = (a_1, ..., a_n)$ by performing an elementary Peiffer exchange, then from the definition of Υ, $\sigma(\alpha) = \sigma(\beta)$, therefore an elementary Peiffer exchange or a finite sequence of such has no effect on the element $\sigma(a_1) \cdots \sigma(a_n) \in \Upsilon$. Before we see the effect that a Peiffer insertion in α has on $\sigma(\alpha)$ we need the first claim of the following.

Lemma 2.1. *The elements of* $\mathfrak{U}$ *are central in* Υ *and those of* $\hat{\mathfrak{U}}$ *are central in* $\mathcal{G}(\Upsilon)$.

Proof. We see that for every a and $b \in Y \cup Y^{-1}$, $\sigma(a)\sigma(a^{-1})\sigma(b) = \sigma(b)\sigma(a)\sigma(a^{-1})$. Indeed,

$$\begin{aligned}\sigma(b)\sigma(a)\sigma(a^{-1}) = \sigma(baa^{-1}) &= \\ &= \sigma(ab^{\theta a}a^{-1}) = \sigma(aa^{-1}(b^{\theta a})^{\theta a^{-1}}) \\ &= \sigma(a)\sigma(a^{-1})\sigma(b).\end{aligned}$$

Since elements $\sigma(b)$ and $\sigma(a)\sigma(a^{-1})$ are generators of Υ and $\mathfrak{U}$ respectively, then the first claim holds true. The second claim follows easily. □

If we insert (a, a^{-1}) at some point in $\alpha = (a_1, ..., a_n)$ to obtain $\alpha' = (a_1, ..., a, a^{-1}, ..., a_n)$, then from lemma 2.1,

$$\sigma(\alpha') = \sigma(\alpha) \cdot (\sigma(a)\sigma(a^{-1})),$$

which means that inserting (a, a^{-1}) inside a Y-sequence α has the same effect as multiplying the corresponding $\sigma(\alpha)$ in Υ by the element $\sigma(a)\sigma(a^{-1})$ of $\mathfrak{U}$ and conversely. Of course the deletion has the obvious interpretation in our semigroup theoretic terms as the inverse of the above process. We retain the same names for our semigroup operations, that is insertion for multiplication by $\sigma(a)\sigma(a^{-1})$ and deletion for its inverse. Related these operations on the elements of Υ we make the following definition.

Definition 2.2. We denote by $\sim_{\mathfrak{U}}$ the equivalence relation in Υ generated by all pairs $(\sigma(\alpha), \sigma(\alpha) \cdot \sigma(a)\sigma(a^{-1}))$ where $\alpha \in \mathrm{FM}(Y \cup Y^{-1})$ and $a \in Y \cup Y^{-1}$. We say that two elements $\sigma(a_1) \cdots \sigma(a_n)$ and $\sigma(b_1) \cdots \sigma(b_m)$ where $m, n \geq 0$ are *Peiffer equivalent in* Υ if they fall in the same $\sim_{\mathfrak{U}}$-class.

It is obvious that two Y-sequences α and β are Peiffer equivalent in the usual sense if and only if $\sigma(\alpha) \sim_{\mathfrak{U}} \sigma(\beta)$, but it should be mentioned that the study of $\sim_{\mathfrak{U}}$ might be as hard as the study of Peiffer operations on Y-sequences, and at this point it seems we have not made any progress at all. In fact this definition will become useful latter in this section and yet we have to prove a few more things before we utilize it.

The process of inserting and deleting generators of $\mathfrak{U}$ in an element of Υ is related to the following new concept. If in general U is a submonoid of a monoid S and $d \in S$, then we say that d belongs to the *weak dominion of* U, shortly written as $d \in \mathrm{WDom}_S(U)$, if for every group G and every monoid homomorphisms $f, g : S \to G$ such that $f(u) = g(u)$ for every $u \in U$, then $f(d) = g(d)$. An analogue of Stenström version of Isbell theorem (theorem 8.3.3 of [7]) for weak dominion holds true. The proof of the if part of its analogue is similar to that of Isbell theorem apart from some minor differences that reflect the fact that we are working with $WDom$ rather than Dom and that will become clear along the proof, while the converse relies on the universal property of $\mu : S \to \mathcal{G}(S)$.

Proposition 1. Let S be a monoid, U a submonoid and let $\hat{U}$ be the subgroup of $\mathcal{G}(S)$ generated by elements $\mu(u)$ with $u \in U$. Then $d \in \mathrm{WDom}_S(U)$ if and only if $\mu(d) \in \hat{U}$.

Proof. The set $\hat{A} = \mathcal{G}(S) \otimes_{\hat{U}} \mathcal{G}(S)$ has an obvious $(\mathcal{G}(S), \mathcal{G}(S))$-bisystem structure. The free abelian group $\mathbb{Z}\hat{A}$ on $\hat{A}$ inherits a $(\mathcal{G}(S), \mathcal{G}(S))$-bisystem structure if we define

$$g \cdot \sum z_i(g_i \otimes_{\hat{U}} h_i) = \sum z_i(gg_i \otimes_{\hat{U}} h_i),$$

and

$$\Big(\sum z_i(g_i \otimes_{\hat{U}} h_i)\Big) \cdot g = \sum z_i(g_i \otimes_{\hat{U}} h_i g).$$

The set $\mathcal{G}(S) \times \mathbb{Z}\hat{A}$ becomes a group by defining

$$(g, \sum z_i g_i \otimes_{\hat{U}} h_i) \cdot (g', \sum z'_i g'_i \otimes_{\hat{U}} h'_i) =$$
$$(gg', \sum z_i g_i \otimes_{\hat{U}} h_i g' + \sum z'_i g g'_i \otimes_{\hat{U}} h'_i).$$

The associativity is proved easily. The unit element is $(1, 0)$ and for every $(g, \sum z_i g_i \otimes_{\hat{U}} h_i)$ its inverse is the element $(g^{-1}, -\sum z_i g^{-1} g_i \otimes_{\hat{U}} h_i g^{-1})$. Let now define

$$\beta : S \to \mathcal{G}(S) \times \mathbb{Z}\hat{A} \text{ by } s \mapsto (\mu(s), 0),$$

which is clearly a monoid homomorphism, and

$$\gamma : S \to \mathcal{G}(S) \times \mathbb{Z}\hat{A} \text{ by } s \mapsto (\mu(s), \mu(s) \otimes_{\hat{U}} 1 - 1 \otimes_{\hat{U}} \mu(s)),$$

which is again seen to be a monoid homomorphism. These two coincide on U since for every $u \in U$

$$\gamma(u) = (\mu(u), \mu(u) \otimes_{\hat{U}} 1 - 1 \otimes_{\hat{U}} \mu(u)) = (\mu(u), 0) = \beta(u).$$

The last equality and the assumption that $d \in \mathrm{WDom}_S(U)$ imply that $\beta(d) = \gamma(d)$, therefore

$$(\mu(d), 0) = (\mu(d), \mu(d) \otimes_{\hat{U}} 1 - 1 \otimes_{\hat{U}} \mu(d)),$$

which shows that $\mu(d) \otimes_{\hat{U}} 1 = 1 \otimes_{\hat{U}} \mu(d)$ in the tensor product $\mathcal{G}(S) \otimes_{\hat{U}} \mathcal{G}(S)$ and therefore theorem 8.3.3, [7], applied for monoids $\mathcal{G}(S)$ and $\hat{U}$, implies that $\mu(d) \in \mathrm{Dom}_{\mathcal{G}(S)}(\hat{U})$. But $\mathrm{Dom}_{\mathcal{G}(S)}(\hat{U}) = \hat{U}$ as from theorem 8.3.6, [7] every inverse semigroup is absolutely closed, whence $\mu(d) \in \hat{U}$.

Conversely, suppose that $\mu(d) \in \hat{U}$ and want to show that $d \in \mathrm{WDom}_S(U)$. Let G be a group and $f, g : S \to G$ two monoid homomorphisms that coincide in U, therefore the group homomorphisms $\hat{f}, \hat{g} : \mathcal{G}(S) \to G$ of the universal property of μ coincide in $\hat{U}$ which, from our assumption, implies that $\hat{f}(\mu(d)) = \hat{g}(\mu(d))$, and then $f(d) = g(d)$ proving that $d \in \mathrm{WDom}_S(U)$. □

Before we reveal the connection between Peiffer deletions (insertions) and weak dominion, we need a few more technical result. Let

$$\Psi : FG(Y \cup Y^{-1}) \to \Upsilon$$

be the map defined as follows.

$$\Psi(u) = \sigma(u)$$

if the reduced word u does not contain any $\iota(a)$ with $a \in Y \cup Y^{-1}$, otherwise if u has occurrences of $\iota(a)$ with $a \in Y \cup Y^{-1}$, then

$$\Psi(u) = \sigma(u')$$

where u' is obtained from u by replacing any $\iota(a)$ by a^{-1}. Let $u, v, x \in FG(Y \cup Y^{-1})$ be irreducibles such that $u = u_1 x$, $v = \iota(x) v_1$ where $\iota(x)$ is the inverse of x and $u_1, v_1, u_1 v_1$ are irreducibles. It is easy to see that

$$\Psi(uv) = \Psi(u_1 v_1) = \Psi(u_1)\Psi(v_1),$$

and that

$$\begin{aligned}\Psi(u)\Psi(v) &= \Psi(u_1)\Psi(x)\Psi(\iota(x))\Psi(v_1)\\ &= \Psi(u_1)\Psi(v_1)\Psi(x)\Psi(\iota(x))\\ &= \Psi(uv)[u, v],\end{aligned}$$

where $[u, v]$ is $\Psi(x)\Psi(\iota(x))$ for short. In this way we have proved that for any irreducibles $u, v \in FG(Y \cup Y^{-1})$, there is $[u, v] \in \mathfrak{U}$ such that $\Psi(uv)[u, v] = \Psi(u)\Psi(v)$.

Lemma 2.3. *Let ρ be any defining relation of $\mathcal{G}(\Upsilon)$ or its inverse and $\xi\rho\iota(\xi)$ any conjugate of ρ in $FG(Y \cup Y^{-1})$. Then there is $u \in \mathfrak{U}$ such that $\Psi(\xi\rho\iota(\xi)) \sim_{\mathfrak{U}} u$.*

Proof. First we see that for any defining relation ρ of $\mathcal{G}(\Upsilon)$ we have that $\Psi(\rho) \in \mathfrak{U}$. Indeed, if $\rho = ab\iota(a^{\theta b})\iota(b)$, then

$$\begin{aligned}\Psi(ab\iota(a^{\theta b})\iota(b)) &= \sigma(a)\sigma(b)\sigma((a^{\theta b})^{-1})\sigma(b^{-1})\\ &= \sigma(b)\sigma(a^{\theta b})\sigma((a^{\theta b})^{-1})\sigma(b^{-1})\\ &= \sigma(b)\sigma(b^{-1})\sigma(a^{\theta b})\sigma((a^{\theta b})^{-1}) \in \mathfrak{U}.\end{aligned}$$

The proof for the second type of relations is similar. In the same way one can show that for every defining relation ρ, $\Psi(\iota(\rho)) \in \mathfrak{U}$. Finally, if $\xi\rho\iota(\xi)$ is a conjugate of a defining relation or its inverse, then $\Psi(\xi\rho\iota(\xi))$ is Peiffer equivalent in Υ to an element $\mathfrak{U}$. Indeed,

$$\begin{aligned}\Psi(\xi)\Psi(\rho)\Psi(\iota(\xi)) &= \Psi(\xi)\Psi(\iota(\xi))\varepsilon && \text{with } \varepsilon = \Psi(\rho)\\ &= [\xi, \iota(\xi)]\Psi(\xi\iota(\xi))\varepsilon\\ &= [\xi, \iota(\xi)]\varepsilon \in \mathfrak{U}.\end{aligned}$$

On the other had,

$$\begin{aligned}\Psi(\xi)\Psi(\rho)\Psi(\iota(\xi)) &= [\xi, \rho]\Psi(\xi\rho)\Psi(\iota(\xi))\\ &= [\xi, \rho][\xi\rho, \iota(\xi)]\Psi(\xi\rho\iota(\xi)).\end{aligned}$$

Since $[\xi, \rho][\xi\rho, \iota(\xi)] \in \mathfrak{U}$, and from above $\Psi(\xi)\Psi(\rho)\Psi(\iota(\xi)) \in \mathfrak{U}$, then we have that $\Psi(\xi\rho\iota(\xi)) \sim_{\mathfrak{U}} u$ where $u \in \mathfrak{U}$. □

The reason why we had to define the map Ψ will become apparent shortly. It is obvious that when $A \in FM(Y \cup Y^{-1})$, then $\Psi(A)$ is nothing but $\sigma(A)$. The following lemma shows that if two words which contain letters from $Y \cup Y^{-1}$ but not inverses in $FG(Y \cup Y^{-1})$ represent the same element in $\mathcal{G}(\Upsilon)$, then seen as elements of Υ, they are $\sim_{\mathfrak{U}}$ equivalent.

Lemma 2.4. *If $A, B \in FM(Y \cup Y^{-1})$ such that $\hat{\sigma}(A) = \hat{\sigma}(B)$ in $\mathcal{G}(\Upsilon)$, then $\sigma(A) \sim_{\mathfrak{U}} \sigma(B)$.*

Proof. Suppose that $A = (\xi_1\rho_1\iota(\xi_1)) \cdots (\xi_n\rho_n\iota(\xi_n))B$ and want to prove that $\sigma(A) \sim_{\mathfrak{U}} \sigma(B)$. For every $1 \leq i \leq n-1$ make the following notations

$$\varepsilon_i = [\xi_i\rho_i\iota(\xi_i), (\xi_{i+1}\rho_{i+1}\iota(\xi_{i+1})) \cdots (\xi_n\rho_n\iota(\xi_n)) \cdot B].$$

Also set

$$\varepsilon_n = [\xi_n\rho_n\iota(\xi_n), B].$$

The following hold true

$$\Psi(A)\cdot\varepsilon_1 = \Psi(\xi_1\rho_1\iota(\xi_1))\cdot\Psi((\xi_2\rho_2\iota(\xi_2))\cdot\cdot\cdot$$
$$(\xi_n\rho_n\iota(\xi_n))\cdot B)$$
$$\Psi(A)\cdot\varepsilon_1\cdot\varepsilon_2 = \Psi(\xi_1\rho_1\iota(\xi_1))\cdot\Psi(\xi_2\rho_2\iota(\xi_2))\cdot$$
$$\Psi((\xi_3\rho_3\iota(\xi_3))\cdots(\xi_n\rho_n\iota(\xi_n))\cdot B)$$
$$\cdot\cdot\cdot$$
$$\Psi(A)\cdot\varepsilon_1\cdots\varepsilon_{n-1} = \Psi(\xi_1\rho_1\iota(\xi_1))$$
$$\cdots\Psi(\xi_{n-1}\rho_{n-1}\iota(\xi_{n-1}))\cdot\Psi((\xi_n\rho_n\iota(\xi_n))\cdot B)$$
$$\Psi(A)\cdot\varepsilon_1\cdots\varepsilon_{n-1}\cdot\varepsilon_n =$$
$$\Psi(\xi_1\rho_1\iota(\xi_1))\cdots\Psi(\xi_n\rho_n\iota(\xi_n))\cdot\Psi(B).$$

Since from the proceeding lemma, each $\Psi(\xi_i\rho_i\iota(\xi_i)) \sim_{\mathfrak{U}} u_i$ with $u_i \in \mathfrak{U}$ and since every $\varepsilon_i \in \mathfrak{U}$, one can easily see that $\Psi(A) \sim_{\mathfrak{U}} \Psi(B)$, hence $\sigma(A) \sim_{\mathfrak{U}} \sigma(B)$. □

The relation between insertion (deletion) and the weak dominion is now revealed from the following.

Theorem 2.5. *Let $d \in \Upsilon$, then $d \sim_{\mathfrak{U}} 1$ if and only if $d \in WDom_{\Upsilon}(\mathfrak{U})$.*

Proof. Let G be any group and $f, g : \Upsilon \to G$ two monoid homomorphisms that coincide in $\mathfrak{U}$ and want to show that $f(d) = g(d)$. The proof will be done by induction on the minimal number $h(d)$ of insertions and deletions needed to transform $d = \sigma(a_1)\cdots\sigma(a_n)$ to 1. If $h(d) = 1$, then $d \in \mathfrak{U}$ and $f(d) = g(d)$. Suppose that $h(d) = n > 1$ and let τ be the first operation performed on d in a series of operations of minimal length. After τ is performed on d, it is obtained an element d' with $h(d') = n - 1$. By induction hypothesis, $f(d') = g(d')$ and want to prove that $f(d) = g(d)$. There are two possible cases for τ. First, τ is an insertion and let $u = \sigma(a)\sigma(a^{-1}) \in \mathfrak{U}$ be the element inserted. It follows that $f(d') = f(d)f(u)$ and $g(d') = g(d)g(u)$, but $f(u) = g(u)$, therefore from cancellation law in the group G we get $f(d) = g(d)$. Second, τ is a deletion and let $u = \sigma(a)\sigma(a^{-1}) \in \mathfrak{U}$ be the element deleted, that is $d = d'u$. It follows immediately from the assumptions that $f(d) = g(d)$.

Conversely, assume that $d \in \mathrm{WDom}_{\Upsilon}(\mathfrak{U})$ and want to prove that $d \sim_{\mathfrak{U}} 1$. From proposition 1, $\mu(d) \in \hat{\mathfrak{U}}$ and let $u_1, ..., u_n$ be group generators from $\hat{\mathfrak{U}}$ such that $\mu(d) = u_1 \cdots u_n$. For $i = 1, ..., n$ define

$$\omega_{u_i} = \begin{cases} (\sigma(a)\sigma(a^{-1}))^2 & \text{if} \quad u_i = \iota(\mu\sigma(a)\mu\sigma(a^{-1})) \\ 1 & \text{if} \quad u_i \text{ is not an inverse} \end{cases}$$

We may now write

$$\mu(\omega_{u_1}\cdots\omega_{u_n}d) = \mu(\omega_{u_1})u_1\cdots\mu(\omega_{u_n})u_n,$$

where the right hand side belongs to $\mu(\mathfrak{U})$ and let $u \in \mathfrak{U}$ be such that

$$\mu(\omega_{u_1}\cdots\omega_{u_n}d) = \mu(u).$$

Lemma 2.4 implies that $\omega_{u_1}\cdots\omega_{u_n}d \sim_{\mathfrak{U}} u$. Since each ω_{u_i} is either 1 or square of a generator from $\mathfrak{U}$ and since $u \sim_{\mathfrak{U}} 1$, we infer that $d \sim_{\mathfrak{U}} 1$ concluding the proof. □

Let $\mathcal{P} = \mathcal{GP}(\mathbf{x}, \mathbf{r})$ be an aspherical group presentation and $\mathcal{P}_1 = \mathcal{GP}(\mathbf{x}, \mathbf{r}_1)$ a subpresentation of the first where $\mathbf{r}_1 = \mathbf{r} \setminus \{r_0\}$ and $r_0 \in \mathbf{r}$ is a fixed relation. We denote by $\Upsilon_1, \mathfrak{U}_1$ monoids associated with $\mathcal{P}_1$ and by $\mathcal{G}(\Upsilon_1)$ and $\hat{\mathfrak{U}}_1$ their respective groups. Also we consider $\hat{\mathfrak{A}}_1$ the subgroup of $\hat{\mathfrak{U}}_1$ generated by all $\mu_1\sigma_1(bb^{-1})$ where $b \in Y_1 \cup Y_1^{-1}$. Finally note that the monomorphism $f : \Upsilon_1 \to \Upsilon$ induced by the map $\sigma_1(a) \to \sigma(a)$ induces a homomorphism $\hat{\phi} : \mathcal{G}(\Upsilon_1) \to \mathcal{G}(\Upsilon)$. With the above notation we have the following.

Theorem 2.6. *The subpresentation $\mathcal{P}_1 = \mathcal{GP}(\mathbf{x}, \mathbf{r}_1)$ is aspherical if and only if $\hat{\phi}^{-1}(\hat{\mathfrak{A}}_1) = \hat{\mathfrak{U}}_1$.*

Proof. Suppose that $(a_1, ..., a_n)$ is an identity Y_1-sequence. Since it is also an identity Y-sequence and $\mathcal{P} = \mathcal{GP}(\mathbf{x}, \mathbf{r})$ is aspherical, then from [2] $(a_1, ..., a_n)$ is Peiffer equivalent in $\mathcal{P}$ to the empty sequence. The latter is equivalent to assuming that $d = (\sigma(a_1) \cdots \sigma(a_n)) \sim_{\mathfrak{U}} 1$, and then theorem 2.5 and proposition 1 imply that $\mu(d) \in \hat{\mathfrak{U}}$. We claim that $\mu(d) \in \hat{\mathfrak{A}}_1$. To see this we first let

$$\begin{aligned}\mu(d) =& (\mu\sigma(b_1b_1^{-1}) \cdots \mu\sigma(b_sb_s^{-1}))\cdot \\ & (\iota(\mu\sigma(b_{s+1}b_{s+1}^{-1})) \cdots \iota(\mu\sigma(b_rb_r^{-1}))) \\ & (\mu\sigma(c_1c_1^{-1}) \cdots \mu\sigma(c_tc_t^{-1}))\cdot \\ & (\iota(\mu\sigma(d_1d_1^{-1})) \cdots \iota(\mu\sigma(d_kd_k^{-1}))),\end{aligned}$$

where the first half involves elements from $Y_1 \cup Y_1^{-1}$ and the second one is

$$\mu\sigma(C)\iota(\mu\sigma(D))$$

with

$$C = c_1c_1^{-1} \cdots c_tc_t^{-1} \text{ and } D = d_1d_1^{-1} \cdots d_kd_k^{-1},$$

where C and D involve only elements of the form $(r_0^u)^\varepsilon$ with $\varepsilon = \pm 1$. Define

$$\psi : FM(Y \cup Y^{-1}) \to \mathcal{N}(\mathcal{P})$$

on free generators as follows

$$(r^u)^\varepsilon \mapsto (u^{-1}ru)^\beta.$$

It is easy to see that ψ is compatible with the defining relations of Υ, hence there is $g : \Upsilon \to \mathcal{N}(\mathcal{P})$ and then the universal property of μ implies the existence of $\hat{g} : \mathcal{G}(\Upsilon) \to \mathcal{N}(\mathcal{P})$ such that $\hat{g}\mu = g$. Recalling from above that in $\mathcal{G}(\Upsilon)$ we have

$$\begin{aligned}&\mu\sigma((a_1 \cdots a_n)\cdot \\ &((b_{s+1}b_{s+1}^{-1}) \cdots (b_rb_r^{-1})) \cdot ((d_1d_1^{-1}) \cdots (d_kd_k^{-1}))) \\ &= \mu\sigma(((b_1b_1^{-1}) \cdots (b_sb_s^{-1})) \cdot ((c_1c_1^{-1}) \cdots (c_tc_t^{-1}))),\end{aligned}$$

we can apply $\hat{g}$ on both sides and get

$$\begin{aligned}&g\sigma((a_1 \cdots a_n)\cdot \\ &((b_{s+1}b_{s+1}^{-1}) \cdots (b_rb_r^{-1})) \cdot ((d_1d_1^{-1}) \cdots (d_kd_k^{-1}))) \\ &= g\sigma(((b_1b_1^{-1}) \cdots (b_sb_s^{-1})) \cdot ((c_1c_1^{-1}) \cdots (c_tc_t^{-1}))).\end{aligned}$$

If we now write each $c_i = (r_0^{u_i})^{\varepsilon_i}$ and each $d_j = (r_0^{v_j})^{\delta_j}$ where ε_i and $\delta_j = \pm 1$, while we write each $a_\ell = (r_\ell^{w_\ell})^{\gamma_\ell}$ and each $b_p = (\rho_p^{\eta_p})^{\epsilon_p}$ where all r_ℓ and ρ_p belong to $\mathbf{r}_1$ and $\gamma_\ell, \epsilon_p = \pm 1$, then the definition of g yields

$$\begin{aligned} &(w_1^\alpha \cdot r_1^\beta + \cdots + w_n^\alpha \cdot r_n^\beta) + \\ (2\eta_{s+1}^\alpha \cdot \rho_{s+1}^\beta + \cdots 2\eta_r^\alpha \cdot \rho_r^\beta) &+ (2v_1^\alpha + \cdots + 2v_k^\alpha) \cdot r_0^\beta \\ = (2\eta_1^\alpha \cdot \rho_1^\beta + \cdots 2\eta_s^\alpha \cdot \rho_s^\beta) &+ (2u_1^\alpha + \cdots + 2u_t^\alpha) \cdot r_0^\beta \end{aligned}$$

The freeness of $\mathcal{N}(\mathcal{P})$ on the set of elements r^β implies in particular that

$$(2v_1^\alpha + \cdots + 2v_k^\alpha) \cdot r_0^\beta = (2u_1^\alpha + \cdots + 2u_t^\alpha) \cdot r_0^\beta$$

from which we see that $k = t$, and after a rearrangement of terms $u_i^\alpha = v_i^\alpha$ for $i = 1, ..., k$. One can see that in general if $v = u \cdot \prod_{i=1}^s w_i^{-1} r_i^{\lambda_i} w_i$ in $FG(\mathbf{x})$ where $\lambda_i = \pm 1$ and $r_i \in \mathbf{r}$, then in $\mathcal{G}(\Upsilon)$ we have

$$\begin{aligned} \mu\sigma((r_0^v)^\delta) = \iota\left(\prod_{i=1}^s \mu\sigma(r_i^{w_i})^{\lambda_i}\right) &\cdot \mu\sigma((r_0^u)^\delta) \\ &\cdot \left(\prod_{i=1}^s \mu\sigma(r_i^{w_i})^{\lambda_i}\right). \end{aligned}$$

Using this it is easy to see that

$$\mu\sigma((r_0^v)^\delta (r_0^v)^{-\delta}) = \mu\sigma((r_0^u)^\delta (r_0^u)^{-\delta}).$$

The easily verified fact that in $\mathcal{G}(\Upsilon)$, $\mu\sigma(aa^{-1}) = \mu\sigma(a^{-1}a)$, implies

$$\mu\sigma((r_0^v)^\delta (r_0^v)^{-\delta}) = \mu\sigma((r_0^u)^\varepsilon (r_0^u)^{-\varepsilon}).$$

If we apply the latter to pairs (c_i, d_i) for which $u_i^\alpha = v_i^\alpha$, we get that $\mu\sigma(C)\iota(\mu\sigma(D)) = 1$ which shows that $\mu\sigma(a_1 \cdots a_n) \in \hat{\mathfrak{A}}_1$. If we are now given that $\hat{\phi}^{-1}(\hat{\mathfrak{A}}_1) = \hat{\mathfrak{U}}_1$, then $\mu_1\sigma_1(a_1 \cdots a_n) \in \hat{\mathfrak{U}}_1$. Proposition 1 and theorem 2.5 imply that $\sigma_1(a_1 \cdots a_n) \sim_{\mathfrak{U}_1} 1$ proving that $\mathcal{P}_1$ is aspherical. For the converse, assume that $\hat{\mathfrak{U}}_1 \neq \hat{\phi}^{-1}(\hat{\mathfrak{A}}_1)$. It follows that there is an identity Y_1-sequence $(a_1, ..., a_n)$ such that $\mu_1\sigma_1(a_1 \cdots a_n) \in \hat{\phi}^{-1}(\hat{\mathfrak{A}}_1) \setminus \hat{\mathfrak{U}}_1$ contrary to the assumption of the asphericity for $\mathcal{P}_1$. □

3. Relativizing the problem

The special case we deal with in this section is that of the pair of presentations $\mathcal{P} = \mathcal{GP}(\mathbf{x}, \mathbf{r}_1 \cup \mathbf{r}_0)$ and $\mathcal{P}_1 = \mathcal{GP}(\mathbf{x}, \mathbf{r}_1)$ where $\mathbf{r}_1 \cap \mathbf{r}_0 = \emptyset$, $\mathbf{r}_0 = \{r_0\}$ is a singleton and $\mathcal{P}$ is an aspherical presentation of the trivial group. If we denote by N_0, N_1 and by N the normal closures of $\mathbf{r}_0$, $\mathbf{r}_1$ and $\mathbf{r}_0 \cup \mathbf{r}_1$ respectively in $F = FG(\mathbf{x})$, then $N = F$, and as we know from [5] $N_0 \cap N_1 = [N_0, N_1]$. We write for short $K = [N_0, N_1]$. Observe that for N we have the isomorphism $N/K \cong N_0/K \rtimes N_1/K$ where N_1/K acts on N_0/K by conjugation. To see this we first note that every $n \in N$ decomposes (non uniquely) as $n = n_0 n_1$ where $n_0 \in N_0$ and $n_1 \in N_1$. Now we define

$$f : N/K \to N_0/K \rtimes N_1/K \text{ by } nK \mapsto (n_0K, n_1K)$$

and show that it is well defined and a homomorphism. Indeed, if $n_0n_1K = m_0m_1K$, then $n_0n_1 = m_0m_1k$ where $k \in K$, hence $m_0^{-1}n_0 = m_1kn_1^{-1} = k_1 \in K$. It follows that $n_0 = m_0k_1$ and $n_1 = k_1^{-1}m_1k$ and then $n_0K = m_0K$ and $n_1K = m_1K$. To see that f is a homomorphism we let $n = n_0n_1$ and $m = m_0m_1$ from N. Then,

$$\begin{aligned} f(nK \cdot mK) &= f(n_0n_1m_0m_1K) \\ &= f(n_0n_1m_0n_1^{-1} \cdot n_1m_1K) \\ &= (n_0n_1m_0n_1^{-1}K, n_1m_1K) \\ &= (n_0K, n_1K) \cdot (m_0K, m_1K) \\ &= f(nK) \cdot f(mK). \end{aligned}$$

Next we define

$$g : N_0/K \rtimes N_1/K \to N/K \text{ by } (n_0K, n_1K) \mapsto n_0n_1K.$$

This is obviously a well defined homomorphism and inverse to f. Finally, we remark that $N_1/K \cong N/N_0$. Indeed, let

$$h : N_1/K \to N/N_0 \text{ such that } n_1K \mapsto n_1N_0.$$

This is well defined since $K \subseteq N_0$, and a homomorphism. Its inverse is the map

$$j : N/N_0 \to N_1/K \text{ defined by } n_1n_0N_0 \mapsto n_1K,$$

which is well defined since if $n_1n_0N_0 = m_1m_0N_0$, then $n_1^{-1}m_1 \in N_0 \cap N_1 = K$, hence $m_1 = n_1k$ where $k \in K$ and then $m_1K = n_1kK = n_1K$. That j is a homomorphism and inverse of h, this is straightforward.

Now we define three crossed modules that will be needed to state and prove the next theorem. The first one is $(\tilde{C}_1, N_1/K, \tilde{\theta}_1)$ the free crossed modules over N/K on $\mathbf{r}_1$ with codomain restricted to N_1/K. This can be also seen as being obtained from the free crossed module $(H_1/P_1, N_1, \theta_1)$ associated with $\mathcal{P}_1 = \mathcal{GP}(\mathbf{x}, \mathbf{r}_1)$ (with codomain restricted to N_1) by factoring out elements ${}^{u}r_i({}^{ku}r_i)^{-1}P_1$ for $k \in K$ in the domain and the whole K in the codomain, and let $\alpha_1 : H_1/P_1 \to \tilde{C}_1$ and $\beta_1 : N_1 \to N_1/K$ be the respective quotient maps. From the previous remark, $N_1/K \cong N/N_0$ and so the free crossed module $(\tilde{C}_1, N_1/K, \tilde{\theta}_1)$ is isomorphic to $(\tilde{\mathcal{C}}_1, N/N_0, \tilde{\vartheta}_1)$ which stands for the free crossed module over N/N_0 on $\mathbf{r}_1$. An implication of this is that $\tilde{\theta}_1$ is injective if and only if $\tilde{\vartheta}_1$ is. The second crossed module is $(\tilde{C}_0, N_0/K, \tilde{\theta}_0)$ obtained from the free crossed module over N/K on $\mathbf{r}_0$ with codomain restricted to N_0/K. Again, this can be seen as being obtained from the free crossed module $(H_0/P_0, N_0, \theta_0)$ associated with $\mathcal{P}_0 = \mathcal{GP}(\mathbf{x}, \mathbf{r}_0)$ by factoring out elements ${}^{u}r_0({}^{ku}r_0)^{-1}P_0$ for $k \in K$ in the domain and K in the codomain, and let $\alpha_0 : H_0/P_0 \to \tilde{C}_0$ and $\beta_0 : N_0 \to N_0/K$ be the respective quotient maps. The last crossed module is $(\tilde{C}, N/K, \tilde{\theta})$ the free crossed module over N/K with base $\mathbf{r}_1 \cup r_0$ and let $\alpha : H/P \to \tilde{C}$ and $\beta : N \to N/K$ be the respective quotient map.

Theorem 3.1. *If $\mathcal{P} = \mathcal{GP}(\mathbf{x}, \mathbf{r}_1 \cup r_0)$ is an aspherical presentation of the trivial group, then $\tilde{\vartheta}_1$ is an injection.*

Proof. From the comment above it suffices to prove that $\tilde{\theta}_1$ is injective, therefore we use in our proof the free crossed module $(\tilde{C}_1, N_1/K, \tilde{\theta}_1)$. Since $\mathcal{P}_0$ is aspherical, then there exists an F-homomorphism

$$a : N_0 \to H_0/P_0$$

such that

$$\prod_{i\in I} u_i r_0^{\varepsilon_i} u_i^{-1} \mapsto \prod_{i\in I} ({}^{u_i} r_0)^{\varepsilon_i} P_0.$$

The existence of a comes for free from theorem 3.1 of [11] and the definition shows that it is an F-map to. The homomorphism a induces $\tilde{a} : N_0/K \to \tilde{C}_0$ by the rule

$$\tilde{a}(\beta_0(n_0)) = \alpha_0(a(n_0)), \tag{3.1}$$

for if ${}^{n_1} n_0 n_0^{-1} \in K$ where $n_0 = \prod_{i\in I} u_i r_0^{\varepsilon_i} u_i^{-1}$, then

$$\alpha_0 a({}^{n_1} n_0 n_0^{-1}) = \alpha_0 a \left(\prod_{i\in I} n_1 u_i r_0^{\varepsilon_i} u_i^{-1} n_1^{-1} \right) \cdot$$
$$\alpha_0 a \left(\prod_{i\in I} u_i r_0^{\varepsilon_i} u_i^{-1} \right)^{-1} = \alpha_0 \left(\prod_{i\in I} ({}^{n_1 u_i} r_0)^{\varepsilon_i} P_0 \right) \cdot$$
$$\alpha_0 \left(\prod_{i\in I} ({}^{u_i} r_0)^{\varepsilon_i} P_0 \right)^{-1} = 1.$$

Also $\tilde{a}$ is an N/K map. Indeed,

$$\begin{aligned} \tilde{a}({}^{\beta(n)} \beta_0(n_0)) &= \tilde{a}(\beta_0(n n_0 n^{-1})) \\ &= \alpha_0(a(n n_0 n^{-1})) \\ &= \alpha_0({}^{n} a(n_0)) \\ &= {}^{\beta(n)} \alpha_0(a(n_0)) \\ &= {}^{\beta(n)} \tilde{a}(\beta_0(n_0)). \end{aligned}$$

Using (3.1) one can directly check that for every $u_0 \in N_0$, $\tilde{\theta}_0 \tilde{a} \beta_0(u_0) = \beta_0(u_0)$.

For every $u \in N_1$, $r \in \mathbf{r}_1$ and $\varepsilon = \pm 1$ we define

$$\eta(\alpha_1(({}^{u} r)^{\varepsilon} P_1)) : N_0/K \to \tilde{C}_0$$

by

$$\beta_0(n_0) \mapsto {}^{\beta(u r^{\varepsilon} u^{-1})} (\tilde{a}\beta_0(n_0))(\tilde{a}\beta_0(n_0))^{-1}.$$

$\eta(\alpha_1(({}^{u} r)^{\varepsilon} P_1))$ is a derivation. Indeed, for every $\beta_0(n_0), \beta_0(m_0) \in N_0/K$ we have

$$\eta(\alpha_1(({}^{u} r)^{\varepsilon} P_1))(\beta_0(n_0)\beta_0(m_0)) =$$
$${}^{\beta(u r^{\varepsilon} u^{-1})} (\tilde{a}(\beta_0(n_0)\beta_0(m_0)))(\tilde{a}(\beta_0(n_0)\beta_0(m_0)))^{-1} =$$
$$({}^{\beta(u r^{\varepsilon} u^{-1})} \tilde{a}(\beta_0(n_0)))({}^{\beta(u r^{\varepsilon} u^{-1})} \tilde{a}(\beta_0(m_0)))(\tilde{a}(\beta_0(m_0)))^{-1}$$
$$(\tilde{a}(\beta_0(n_0)))^{-1} = \left({}^{\beta(u r^{\varepsilon} u^{-1})} \tilde{a}(\beta_0(n_0))(\tilde{a}(\beta_0(n_0)))^{-1} \right)$$
$${}^{\beta_0(n_0)} \left({}^{\beta(u r^{\varepsilon} u^{-1})} \tilde{a}(\beta_0(m_0))(\tilde{a}(\beta_0(m_0)))^{-1} \right) =$$
$$(\eta(({}^{u} r)^{\varepsilon} P_1)(\beta_0(n_0))) \ {}^{\beta_0(n_0)} (\eta(({}^{u} r)^{\varepsilon} P_1)(\beta_0(m_0))).$$

Also $\eta(\alpha_1((\ ^{u}r)^{\varepsilon}P_1))$ is regular for if $\beta_0(n_0) \in N_0/K$ we see that

$$\begin{aligned}
&(\eta(\alpha_1((\ ^{u}r)^{\varepsilon}P_1))) \circ \eta(\alpha_1((\ ^{u}r)^{-\varepsilon}P_1))(\beta_0(n_0)) = \\
&\eta(\alpha_1((\ ^{u}r)^{\varepsilon}P_1)) \left(\tilde{\theta}_0 \left(\eta((\ ^{u}r)^{-\varepsilon}P_1)(\beta_0(n_0))\right) \beta_0(n_0)\right) \\
&\eta(\alpha_1((\ ^{u}r)^{-\varepsilon}P_1))(\beta_0(n_0))) = \\
&\eta(\alpha_1((\ ^{u}r)^{\varepsilon}P_1))(\beta_0(ur^{-\varepsilon}u^{-1}n_0ur^{\varepsilon}u^{-1})) \\
&(\ ^{\beta(ur^{-\varepsilon}u^{-1})}\tilde{a}(\beta_0(n_0))(\tilde{a}(\beta_0((n_0))^{-1}) = \\
&\ ^{\beta(ur^{\varepsilon}u^{-1})}\tilde{a}(\beta_0(ur^{-\varepsilon}u^{-1}n_0ur^{\varepsilon}u^{-1})) \\
&(\tilde{a}\beta_0(ur^{-\varepsilon}u^{-1}n_0ur^{\varepsilon}u^{-1})))^{-1}(\ ^{\beta(ur^{-\varepsilon}u^{-1})}\tilde{a}(\beta_0(n_0)) \\
&(\tilde{a}(\beta_0(n_0))^{-1}) = \ ^{\beta(ur^{\varepsilon}u^{-1})}\tilde{a}(\ ^{\beta(ur^{-\varepsilon}u^{-1})}\beta_0(n_0))) \\
&(\tilde{a}(\ ^{\beta(ur^{-\varepsilon}u^{-1})}\beta_0(n_0)))^{-1} \\
&(\ ^{\beta(ur^{-\varepsilon}u^{-1})}\tilde{a}(\beta_0(n_0))(\tilde{a}(\beta_0(n_0))^{-1}) = 1.
\end{aligned}$$

So far we have defined a map η from the generators of $\tilde{C}_1$ to $\mathrm{D}(N_0/K, \tilde{C}_0)$ and show that it extends to a homomorphism

$$\eta : \tilde{C}_1 \to \mathrm{D}(N_0/K, \tilde{C}_0).$$

For this we need to show that for every $\beta_0(n_0) \in N_0/K$ and $\alpha_1(\ ^{u}rP_1), \alpha_1(\ ^{v}sP_1) \in \tilde{C}_1$ we have that $\eta(\alpha_1(\ ^{u}r\ ^{v}s)P_1)(\beta_0(n_0)) = \eta(\alpha_1(\ ^{uru^{-1}v}s\ ^{u}rP_1))(\beta_0(n_0))$. Indeed, on the one hand we have that

$$\begin{aligned}
&\eta(\alpha_1(\ ^{u}r\ ^{v}s)P_1)(\beta_0(n_0)) = \\
&(\eta(\alpha_1(\ ^{u}rP_1)) \circ \eta(\alpha_1(\ ^{v}sP_1)))(\beta_0(n_0)) = \\
&\eta(\alpha_1(\ ^{u}r)P_1)(\sigma_{\eta(\alpha_1(\ ^{v}sP_1))}(\beta_0(n_0)))\cdot \\
&(\eta(\alpha_1(\ ^{v}s)P_1)(\beta_0(n_0))) = \\
&\eta(\alpha_1(\ ^{u}r)P_1)\left(\beta_0(vsv^{-1}n_0vs^{-1}v^{-1})\right)\cdot \\
&\left(\ ^{\beta(vsv^{-1})}\tilde{a}\beta_0(n_0))(\tilde{a}\beta_0((n_0)))^{-1}\right) = \\
&\ ^{\beta(uru^{-1}\cdot vsv^{-1})}\tilde{a}\beta_0((n_0))(\tilde{a}\beta_0((n_0)))^{-1},
\end{aligned}$$

and on the other hand we have

$$\begin{aligned}
&\eta(\alpha_1(\ ^{uru^{-1}v}s\ ^{u}r)P_1)(\beta_0(n_0)) = \\
&(\eta(\alpha_1(\ ^{uru^{-1}v}sP_1)) \circ \eta(\alpha_1(\ ^{u}rP_1)))(\beta_0(n_0)) = \\
&\ ^{\beta(uru^{-1}\cdot vsv^{-1})}\tilde{a}(\beta_0(n_0))(\tilde{a}(\beta_0(n_0)))^{-1},
\end{aligned}$$

which shows that

$$\eta(\alpha_1(\ ^{u}r\ ^{v}s)P_1)(\beta_0(n_0)) = \eta(\alpha_1(\ ^{uru^{-1}v}s\ ^{u}rP_1))(\beta_0(n_0)).$$

Further we define

$$\rho : N_1/K \to \mathrm{Aut}(\tilde{C}_0, N_0/K)$$

by

$$\beta_1(n_1) \mapsto (\rho_1(\beta_1(n_1)), \rho_2(\beta_1(n_1))),$$

where

$$\rho_1(\beta_1(n_1)) : \tilde{C}_0 \to \tilde{C}_0$$

is defined on generators by

$$\alpha_0({}^{w}r_0P_0) \mapsto \alpha_0({}^{n_1w}r_0P_0),$$

and similarly

$$\rho_2(\beta_1(n_1)) : N_0/K \to N_0/K \text{ by } \beta_0(n_0) \mapsto \beta_0(n_1n_0n_1^{-1}).$$

It is easy to see that both $\rho_1(\beta_1(n_1))$ and $\rho_2(\beta_1(n_1))$ are automorphisms that make the following diagram commutative

$$\begin{array}{ccc} \tilde{C}_0 & \xrightarrow{\tilde{\theta}_0} & N_0/K \\ {\scriptstyle \rho_1(\beta_1(n_1))}\downarrow & & \downarrow{\scriptstyle \rho_2(\beta_1(n_1))} \\ \tilde{C}_0 & \xrightarrow[\tilde{\theta}_0]{} & N_0/K \end{array}$$

and that ρ is itself a homomorphism.

Further we check that homomorphisms η and ρ make the following diagram commutative.

$$\begin{array}{ccc} \tilde{C}_1 & \xrightarrow{\tilde{\theta}_1} & N_1/K \\ {\scriptstyle \eta}\downarrow & & \downarrow{\scriptstyle \rho} \\ \mathrm{D}(N_0/K, \tilde{C}_0) & \xrightarrow[\Delta]{} & \mathrm{Aut}(\tilde{C}_0, N_0/K) \end{array}$$

Indeed, on the one hand we have that

$$\Delta(\eta(\alpha_1({}^{u}rP_1))) = (\tau_{\eta(\alpha_1({}^{u}rP_1))}, \sigma_{\eta(\alpha_1({}^{u}rP_1))}),$$

and on the other hand that

$$\rho\tilde{\theta}_1(\alpha_1({}^{u}rP_1)) = (\rho_1(\beta_1(uru^{-1})), \rho_2(\beta_1(uru^{-1}))),$$

and see that

$$\begin{aligned} &\tau_{\eta(\alpha_1({}^{u}rP_1))}(\alpha_0({}^{w}r_0P_0)) \\ &= \eta(\alpha_1({}^{u}rP_1))(\beta_0(wr_0w^{-1})) \cdot (\alpha_0({}^{w}r_0P_0)) \\ &= \alpha_0({}^{uru^{-1}w}r_0P_0) \cdot (\alpha_0({}^{w}r_0P_0))^{-1} \cdot \alpha_0({}^{w}r_0P_0) \\ &= \alpha_0({}^{uru^{-1}w}r_0P_0) \\ &= \rho_1(\beta_1(uru^{-1}))(\alpha_0({}^{w}r_0P_0)), \end{aligned}$$

and similarly that

$$\begin{aligned} &\sigma_{\eta(\alpha_1({}^{u}rP_1))}(\beta_0(wr_0w^{-1})) \\ &= \rho_2(\beta_1(uru^{-1}))(\beta_0(wr_0w^{-1})), \end{aligned}$$

showing that $\Delta\eta = \rho\theta_1$.

So far we have proved that there exists the semidirect product $(\tilde{C}_0, N_0/K, \tilde{\theta}_0) \rtimes_{\langle\eta,\rho\rangle} (\tilde{C}_1, N_1/K, \tilde{\theta}_1)$. Further we will check that the triple $(\tilde{C}_0 \rtimes_{\langle\eta,\rho\rangle} \tilde{C}_1, N/K, g \circ \pi)$ is a crossed module where $g : N_0/K \rtimes N_1/K \to N/K$ is the isomorphism established earlier by setting $(n_0K, n_1K) \mapsto n_0n_1K$ and was the inverse of $f : N/K \to N_0/K \rtimes N_1/K$ which maps $\beta(u) \in N/K$ to $(\beta_0(u_0), \beta_1(u_1))$ where $u = u_0u_1$ is any decomposition of u and so $\beta(u) \in N/K$ is identified with $f(\beta(u))$. With this in mind we define for every $\beta(u) \in N/K$ and $(a_0, a_1) \in (\tilde{C}_0, \tilde{C}_1)$

$$^{\beta(u)}(a_0, a_1) = {}^{f(\beta(u))}(a_0, a_1),$$

which establishes a left action of N/K on $\tilde{C}_0 \rtimes_{\langle \eta,\rho \rangle} \tilde{C}_1$. Let check the conditions for $(\tilde{C}_0 \rtimes_{\langle \eta,\rho \rangle} \tilde{C}_1, N/K, g \circ \pi)$ to be a crossed module. First,

$$\begin{aligned}(a_0, a_1)(b_0, b_1) &= {}^{\pi((a_0,a_1))}(b_0, b_1)(a_0, a_1)\\ &= {}^{(\tilde{\theta}_0(a_0),\tilde{\theta}_1(a_1))}(b_0, b_1)(a_0, a_1)\\ &= {}^{fg(\tilde{\theta}_0(a_0),\tilde{\theta}_1(a_1))}(b_0, b_1)(a_0, a_1)\\ &= {}^{g(\tilde{\theta}_0(a_0),\tilde{\theta}_1(a_1))}(b_0, b_1)(a_0, a_1)\\ &= {}^{(g\circ\pi)((a_0,a_1))}(b_0, b_1)(a_0, a_1).\end{aligned}$$

Second, if $u \in N$ is decomposed as u_0u_1 with $u_0 \in N_0$ and $u_1 \in N_1$, then

$$\begin{aligned}&(g \circ \pi)\left({}^{\beta(u)}(a_0, a_1)\right) =\\ &(g \circ \pi)\left({}^{(\beta_0(u_0),\beta_1(u_1))}(a_0, a_1)\right) =\\ &(g \circ \pi)\left({}^{\beta(u)}a_0 \left(\eta({}^{\beta_1(u_1)}a_1)(\beta_0(u_0))\right)^{-1},\ {}^{\beta_1(u_1)}a_1 \right) =\\ &\beta(u)\tilde{\theta}_0(a_0)\beta(u)^{-1}\cdot\\ &\tilde{\theta}_0\left(\tilde{a}(\beta_0(u_0))\left({}^{\beta_1(u_1)\tilde{\theta}_1(a_1)\beta_1(u_1)^{-1}}\tilde{a}(\beta_0(u_0))\right)^{-1}\right)\cdot\\ &\beta_1(u_1)\tilde{\theta}_1(a_1)\beta_1(u_1)^{-1} =\\ &\beta(u)\tilde{\theta}_0(a_0)\tilde{\theta}_1(a_1)\beta_1(u_1)^{-1}\beta_0(u_0)^{-1} =\\ &\beta(u)(g \circ \pi)(a_0, a_1)\beta(u)^{-1}.\end{aligned}$$

Now we show that there is a morphism ψ from the free crossed module $(\tilde{C}, N/K, \tilde{\theta})$ to the crossed module $(\tilde{C}_0 \rtimes_{\langle \eta,\rho \rangle} \tilde{C}_1, N/K, g \circ \pi)$. For this we define a map

$$w : \mathbf{r}_1 \cup r_0 \to \tilde{C}_0 \rtimes_{\langle \eta,\rho \rangle} \tilde{C}_1$$

such that

$$w(s) = \begin{cases} (\alpha_0(r_0P_0), 1) & \text{if} \quad s = r_0 \\ (1, \alpha_1(rP_1)) & \text{if} \quad s = r \in \mathbf{r}_1 \end{cases}$$

Obviously, $\tilde{\theta}(\alpha(sP)) = ((g \circ \pi) \circ w)(s)$ for every $s \in \mathbf{r}_1 \cup r_0$, therefore the freeness of $\tilde{C}$ implies the existence of the desired ψ.

Finally we prove that $\tilde{\theta}_1$ is injective. Let $\prod_{i\in I} \alpha_1({}^{u_i}r_iP_1)^{\varepsilon_i} \in \mathrm{Ker}\tilde{\theta}_1$ where each u_i is regarded as an element of N_1. It follows that $\prod_{i\in I} u_ir_i^{\varepsilon_i}u_i^{-1} \in K \subseteq N_0$ and let $\prod_{j\in J} v_jr_0^{\delta_j}v_j^{-1} \in N_0$ such that $\prod_{i\in I} u_ir_i^{\varepsilon_i}u_i^{-1} = \prod_{j\in J} v_jr_0^{\delta_j}v_j^{-1}$. The asphericity of $\mathcal{P}$ implies that

$$d = \prod_{j\in J}({}^{v_j}r_0P)^{-\delta_j} \cdot \prod_{i\in I}({}^{u_i}r_iP)^{\varepsilon_i} = 1$$

in H/P, hence in $\tilde{C}$ we have

$$1 = \alpha(d) = \prod_{j\in J} \alpha(({}^{v_j} r_0 P))^{-\delta_j} \cdot \prod_{i\in I} \alpha(({}^{u_i} r_i P))^{\varepsilon_i}.$$

Applying ψ on $\alpha(d)$ yields

$$\begin{aligned}
&(1,1) = \psi(\alpha(d)) \\
&= \prod_{j\in J} ({}^{(f\circ\beta)(v_j)} (\alpha_0(r_0P_0), 1))^{-\delta_j} \cdot \prod_{i\in I} ({}^{(f\circ\beta)(u_i)} (1, \alpha_1(r_iP_1)))^{\varepsilon_i} \\
&= \left(\prod_{j\in J} ({}^{\beta(v_j)} \alpha_0(r_0P_0))^{-\delta_j}, 1 \right) \cdot \left(1, \prod_{i\in I} ({}^{\beta_1(u_i)} \alpha_1(r_iP_1))^{\varepsilon_i} \right) \\
&= \left(\prod_{j\in J} ({}^{\beta(v_j)} \alpha_0(r_0P_0))^{-\delta_j}, 1 \right) \cdot \left(1, \prod_{i\in I} \alpha_1({}^{u_i} r_iP_1)^{\varepsilon_i} \right) \\
&= \left(\prod_{j\in J} ({}^{\beta(v_j)} \alpha_0(r_0P_0))^{-\delta_j}, \prod_{i\in I} \alpha_1({}^{u_i} r_iP_1)^{\varepsilon_i} \right),
\end{aligned}$$

hence $\prod_{i\in I} \alpha_1({}^{u_i} r_iP_1)^{\varepsilon_i} = 1$ proving that $\tilde{\theta}_1$ is injective. □

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A WAGNER-PRESTON REPRESENTATION THEOREM FOR CLIFFORD SEMIGROUPS

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Abstract. We prove in this paper an analogue of the Wagner-Preston theorem for Clifford semigroups. The role of the symmetric inverse semigroup I_X on a set X is played in our theorem by what we define here as the symmetric Clifford semigroup $\mathscr{C}(S)$ on the semilattice of ideals of a semigroup $(S, \cdot)$, which consists of all partial bijections of the underlying set S with domain and codomain an ideal of $(S, \cdot)$ and that preserve all ideals of $(S, \cdot)$ which include in the domain. Our theorem then states that every Clifford semigroup $(S, \cdot)$ embeds into its symmetric Clifford semigroup $\mathscr{C}(S)$.

1. Introduction and preliminaries

It is shown in [5] that the construction of the symmetric inverse semigroup I_X on a fixed set X is an aspect of a more general construction that can be carried out in every small monosetting. We will give below a few details on monosettings in general, and than stop to the monosettings associated with an object X in **Set** to see how the symmetric inverse semigroup I_X can be constructed in that case. As it is emphasized in [5], such construction is still possible if **Set** is replaced by any well powered category $\mathbf{K}$ having finite intersections. The reason we pursue this path is our intention to prove an analogue of the Wagner-Preston theorem for Clifford semigroups which would first require the definition of a Clifford semigroup analogue of the symmetric inverse semigroup on a set X. This definition is made in this paper following the new conceptual framework of monosettings.

Let $\mathbf{K}$ be a category and let X be an object there. The monocontext of X in $\mathbf{K}$ is the pair $(M(X), X)$ where $M(X)$ denotes the subcategory of $\mathbf{K}$ of all monomorphisms between all objects A of $\mathbf{K}$ for which a monomorphism $\alpha : A \to X$ exists. If it happens that $M(X)$ has finite intersections, then $(M(X), X)$ is called a monosetting.

Given any monosetting $(\mathbf{M}, X)$, we can define an inverse semigroup $I(\mathbf{M}, X)$ in the following fashion. Consider parallel pairs of morphisms $(\alpha, \alpha') : A \to X$ with A

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varying in $\mathbf{M}$. We say that two such pairs (α, α') and (β, β') are equivalent if there is an isomorphism $\mu \in \mathbf{M}$ such that $\beta = \alpha\mu$ and $\beta' = \alpha'\mu$. A fractional morphism is the equivalence class of such a pair, with the class of (α, α') denoted by $[\alpha, \alpha']$. We let $I(\mathbf{M}, X)$ be the set of all such classes, and endow it with a multiplication defined by setting

$$[\alpha, \alpha'][\beta, \beta'] = [\alpha\kappa, \beta'\lambda],$$

where κ and λ arise from taking the intersection of the middle pair, α' and β in $\mathbf{M}$: $\alpha' \cap \beta = \alpha'\kappa = \beta\lambda$. This multiplication turns out to be independent on κ and λ, and on the representatives (α, α') and (β, β'). Furthermore $I(\mathbf{M}, X)$ is an inverse monoid. In example 1.4 of [5] it is shown how this construction gives the symmetric inverse monoid on X when the monocontext of X in **Set** is made of all non empty subsets of X with morphisms being precisely the inclusions between such subsets. In the next section we will modify the monosetting in such a way that, following the recipe provided in example 1.4 of [5], we obtain an inverse semigroup with central idempotents, aka a Clifford semigroup. Further, we prove that every Clifford semigroups embeds into a Clifford semigroup constructed as above, which is our representation theorem.

Finally, we give below a few detail regarding the structure of Clifford semigroups. The following is theorem 4.2.1 of [3].

Theorem 1.1. *Let S be a semigroup and E the set of its idempotents. Then the following are equivalent:*

1. *S is a Clifford semigroup;*
2. *S is a semilattice of groups;*
3. *S is a strong semilattice of groups;*
4. *S is regular, and the idempotents are central;*
5. *S is regular, and $\mathcal{D}^S \cap (E \times E) = 1_E$*

If E is a semilattice, then a strong semilattice of groups is a collection of groups $\{H_e \text{ with } e \in E\}$ together with the group homomorphisms $\varphi_{e,e\prime} : H_e \to H_{e\prime}$ for every $e \geq e'$ satisfying the following conditions.

1. For every $e \in E$, $\varphi_{e,e}$ is the identical homomorphism of H_e
2. For every $e_1 \geq e_2 \geq e_3$, $\varphi_{e_2,e_3}\varphi_{e_1,e_2} = \varphi_{e_1,e_3}$

It turns out that if $e \geq e\prime$ and $f \in E$ such that $e\prime = ef$, then $\varphi_{e,e\prime} : H_e \to H_{e\prime}$ maps every $x \in H_e$ to xf. In other words we have that $\varphi_{e,e\prime} = \rho_f \mid H_e$, where ρ_f is the right translation of S by f. We can express the multiplication in Clifford semigroup S in terms of mappings $\varphi_{e,e\prime}$ and the multiplication in each group H_e in the following way: for every $e_1, e_2 \in E$ and every $x_1 \in H_{e_1}$ and $x_2 \in H_{e_2}$, we have: $x_1x_2 = \varphi_{e_1,e_1e_2}(x_1)\varphi_{e_2,e_1e_2}(x_2) = (x_1e_2)(x_2e_1)$ where the multiplication of the right hand side is the multiplication of the group $H_{e_1e_2}$. In the particular case when $x \in H_e$, $y \in H_f$ and $e \leq f$, then the product xy equals to $\varphi_{e,f}(x)y$.

2. The representation theorem

Let **Sgrp** be the category of semigroups and $\mathcal{U} : \mathbf{Sgrp} \to \mathbf{Set}$ be the forgetful functor. For a fixed semigroup S, we consider the subcategory $A(\mathcal{U}(S))$ of **Set** with objects all $\mathcal{U}(I)$ where I is an ideal of S, and morphisms all bijective maps $\alpha : \mathcal{U}(I) \to \mathcal{U}(I)$ such that for every ideal J of S such that $J \subseteq I$, $\alpha(\mathcal{U}(J)) = \mathcal{U}(J)$. We say that α preserves ideals of S that include in I. The letter A of $A(\mathcal{U}(S))$ stands for automorphism and is chosen to reflect the fact that a morphism of $A(\mathcal{U}(S))$

with domain the underlying set of some ideal I fixes the semilattice of ideals which include in I. It is a routine matter to prove that $A(\mathcal{U}(S))$ is a category with finite intersections. The pair $(A(\mathcal{U}(S)),\mathcal{U}(S))$ is called the $\mathfrak{I}$-isosetting of $\mathcal{U}(S)$ in **Set**. Consider now the set of all pairs of parallel morphisms (α,α') with domain and codomain some $\mathcal{U}(I)$. Two such pairs (α,α') and (β,β') with respective domains $\mathcal{U}(I)$ and $\mathcal{U}(J)$ are called equivalent if there is a bijection $\mu:\mathcal{U}(J)\to\mathcal{U}(I)$ such that $\beta=\alpha\mu$ and $\beta'=\alpha'\mu$. From this it follows directly that $\mathcal{U}(I)=\mathcal{U}(J)$ and that μ is a morphism in $A(\mathcal{U}(S))$. We call a fractional morphism the equivalence class of (α,α') and denote it by $[\alpha,\alpha']$. The set of all such classes is denoted by $\mathfrak{I}(S)$. Beside the $\mathfrak{I}$-isosetting $(A(\mathcal{U}(S)),\mathcal{U}(S))$ defined above, we consider the monosetting $(M(\mathcal{U}(S)),\mathcal{U}(S))$ and the inverse semigroup $I_{\mathcal{U}(S)}$ whose elements are the equivalence classes of pairs (α,α') of parallel morphisms with domain a subset $D\subseteq\mathcal{U}(S)$ and codomain $\mathcal{U}(S)$. There is an injective map $\Theta:\mathfrak{I}(S)\to I_{\mathcal{U}(S)}$ which sends $[\alpha,\alpha']$ to $[\iota\alpha,\iota\alpha']$ where ι embeds the image of α into $\mathcal{U}(S)$. We use this injection to define a multiplication in $\mathfrak{I}(S)$ in terms of the multiplication in $I_{\mathcal{U}(S)}$. Let $[\alpha,\alpha'],[\beta,\beta']$ be two fractional morphisms in $\mathfrak{I}(S)$ where the domain of α is $\mathcal{U}(I)$ and that of β is $\mathcal{U}(J)$. Let now $[\iota\alpha,\iota\alpha']=\Theta([\alpha,\alpha'])$ and $[\eta\beta,\eta\beta']=\Theta([\beta,\beta'])$. As it is explained in Example 1.4 of [5], $[\iota\alpha,\iota\alpha']=[\iota\alpha(\alpha')^{-1},\iota]$ and similarly $[\eta\beta,\eta\beta']=[\eta\beta(\beta')^{-1},\eta]$. The product $[\iota\alpha,\iota\alpha']\circ[\eta\beta,\eta\beta']$ is the class $[\iota\alpha(\alpha')^{-1}\kappa,\eta\lambda]$ where $\kappa=\beta(\beta')^{-1}|C:C\to\mathcal{U}(I)$ is the restriction of $\beta(\beta')^{-1}$ on $C=(\beta(\beta')^{-1})^{-1}(\mathcal{U}(I)\cap\mathcal{U}(J))$, and $\lambda:C\subseteq\mathcal{U}(J)$ is the inclusion. Since the preimage under Θ of $[\iota\alpha(\alpha')^{-1},\iota]$ is $[\alpha(\alpha')^{-1},id_{\mathcal{U}(I)}]$ and similarly the preimage of $[\eta\beta(\beta')^{-1},\eta]$ is $[\beta(\beta')^{-1},id_{\mathcal{U}(J)}]$, we can now define the product in $\mathfrak{I}(S)$ by setting

$$[\alpha,\alpha']\circ[\beta,\beta']=[\alpha(\alpha')^{-1}\kappa,id_{\mathcal{U}(I)\cap\mathcal{U}(J)}].$$

Summarizing, we have defined a semigroup $(\mathfrak{I}(S),\circ)$ where the elements of $\mathfrak{I}(S)$ are classes $[\alpha,id_{\mathcal{U}(I)}]$ where $\mathcal{U}(I)$ is the underlying set of an ideal I of S, $\alpha:\mathcal{U}(I)\to\mathcal{U}(I)$ is an ideal preserving bijection, and the multiplication of two such classes $[\alpha,id_{\mathcal{U}(I)}]$ and $[\beta,id_{\mathcal{U}(J)}]$ is given by

$$[\alpha,id_{\mathcal{U}(I)}]\circ[\beta,id_{\mathcal{U}(J)}]=[\alpha\kappa,id_{\mathcal{U}(I)\cap\mathcal{U}(J)}],$$

where κ is the restriction of β in $\beta^{-1}(\mathcal{U}(I)\cap\mathcal{U}(J))=\mathcal{U}(I)\cap\mathcal{U}(J)=\mathcal{U}(I\cap J)$. We can think of the composition $\alpha\kappa$ as the composition $\alpha\beta$ restricted in $\mathcal{U}(I\cap J)$. It is more suitable for our purpose to give the whole thing a more semigroup theoretic flavor. Before we do so, we give the following.

Definition 2.1. Let $(S,\cdot)$ be an ordinary semigroup. We denote by $\mathscr{C}(S)$ the set of all partial bijections α of the underlying set S such that:

(i) $dom(\alpha)=im(\alpha)=I$ where I is the underlying set of an ideal of S;
(ii) If $J\subseteq I$ is the underlying set of an ideal of S, then $\alpha(J)=J$. We say that α preserves the ideals J of S which include in I.

Remark 2.2. The set $\mathscr{C}(S)$ can never be empty because for any ideal I, the identity map on I satisfies property (ii) of the definition.

We will make $\mathscr{C}(S)$ into a Clifford semigroup in the following way. Let $\alpha:I\to I$ and $\beta:J\to J$ be two elements of $\mathscr{C}(S)$ and let $K=I\cap J$ which is again an ideal of S. Define now $\alpha\circ\beta:K\to K$ such that $(\alpha\circ\beta)(x)=\alpha(\beta(x))$ for all $x\in K$. The composition $\alpha\beta$ is a bijection which preserves the ideals of S that are contained in K, since both α and β do so. This shows that $\alpha\circ\beta\in\mathscr{C}(S)$.

Theorem 2.3. *$(\mathscr{C}(S), \circ)$ is a Clifford semigroup.*

Proof. It is obvious that $\circ$ is associative. On the other hand, $(\mathscr{C}(S), \circ)$ is regular for if $\alpha : I \to I$ is an element of $\mathscr{C}(S)$, its inverse map $\alpha^{-1} : I \to I$ is from definition 2.1 again in $\mathscr{C}(S)$. We remark now that an idempotent $\varepsilon \in \mathscr{C}(S)$ is nothing but the identity map on some ordered ideal I of S since for every $x \in I$, $(\varepsilon\varepsilon)(x) = \varepsilon(x)$ implies that

$$\varepsilon(x) = (\varepsilon^{-1}(\varepsilon\varepsilon))(x) = (\varepsilon^{-1}\varepsilon)(x) = x.$$

Now we prove that idempotents are central. So let $\varepsilon : J \to J$ be any idempotent and $\alpha : I \to I$ any element of $\mathscr{C}(S)$. Write $K = I \cap J$, and then from the definition of $\circ$ we have that $\varepsilon \circ \alpha$ and $\alpha \circ \varepsilon$ are both in $\mathscr{C}(S)$ with domain K. We prove that they in fact coincide. Indeed, for every $x \in K$ we have

$$(\varepsilon \circ \alpha)(x) = (\varepsilon\alpha)(x) = \alpha(x),$$

and

$$(\alpha \circ \varepsilon)(x) = \alpha(\varepsilon(x)) = \alpha(x),$$

proving the equality $\varepsilon \circ \alpha = \alpha \circ \varepsilon$. □

Definition 2.4. For every semigroup $(S, \cdot)$ we call $(\mathscr{C}(S), \circ)$ the symmetric Clifford semigroup on the semilattice of ideals of $(S, \cdot)$.

Now we return to $(\mathfrak{I}(S), \circ)$ to see how it is related with $(\mathscr{C}(S), \circ)$.

Proposition 1. The two semigroups $(\mathfrak{I}(S), \circ)$ and $(\mathscr{C}(S), \circ)$ are isomorphic. In particular, $(\mathfrak{I}(S), \circ)$ is a Clifford semigroup.

Proof. Define $\Omega : \mathscr{C}(S) \to \mathfrak{I}(S)$ by sending each ideal preserving bijection $\alpha : I \to I$ to $[\alpha, id_{\mathcal{U}(I)}]$. This map is clearly bijective, and a homomorphism since for every two ideal preserving bijections $\alpha : I \to I$ and $\beta : J \to J$ we have that

$$\Omega(\alpha \circ \beta) = [\alpha\beta, id_{\mathcal{U}(I \cap J)}] = [\alpha, id_{\mathcal{U}(I)}] \circ [\beta, id_{\mathcal{U}(J)}] = \Omega(\alpha) \circ \Omega(\beta),$$

where $\alpha \circ \beta$ is regarded as the usual composition $\alpha\beta$ but restricted in $I \cap J$. □

Before we prove our main theorem, we prove a preliminary result.

Lemma 2.5. *For every Clifford semigroup $(S, \cdot)$, the intersection of two principal ideals aS and bS is the principal ideal abS.*

Proof. Let aS and bS be two principal ideals of S, and want to prove that $aS \cap bS = abS$. First we note that for every $a \in S$, $aS = aa^{-1}S$. Indeed, since $a = aa^{-1}a \in aa^{-1}S$, then $aS \subseteq aa^{-1}S$. Conversely, $aa^{-1}S \subseteq aS$ is trivial. Finally we want to prove that $aS \cap bS = abS$. It is obvious that on the one hand $abS \subseteq aS$, and on the other hand $abS = Sab \subseteq Sb = bS$, therefore $abS \subseteq aS \cap bS$. For the converse, let $ax = by \in aS \cap bS$. Since $aS = aa^{-1}S$, then $ax = aa^{-1}x'$, and since $bS = Sb$, then $by = y'b$. So our element in the intersection now is $aa^{-1}x' = y'b$. Since idempotents are central, $aa^{-1}x = xaa^{-1}$, hence $x'aa^{-1} = y'b$. Multiplying both sides with the idempotent aa^{-1} we have $x'aa^{-1} = y'baa^{-1}$. Using the fact that $aa^{-1} = a^{-1}a$ is central, we obtain $x'aa^{-1} = y'a^{-1}ab \in Sab = abS$, which proves that $ax \in abS$. □

For every Clifford semigroup $(S, \cdot)$ we can consider the symmetric Clifford semigroup $(\mathscr{C}(S), \circ)$ associated with $(S, \cdot)$. The following is the analogue of the Vagner Preston theorem for Clifford semigroups.

Theorem 2.6. *Every Clifford semigroup* $(S, \cdot)$ *embeds into* $(\mathscr{C}(S), \circ)$.

Proof. Define $\phi : S \to \mathscr{C}(S)$ by sending every $a \in S$ to $\phi(a) : aS \to aS$ such that $\phi(a)(x) = ax$ for every $x \in aS$. We prove first that ϕ is correct which amounts to saying that $\phi(a)$ is indeed in $\mathscr{C}(S)$, which in turn means that aS is an ideal, $\phi(a)$ is bijective, and $\phi(a)$ preserves all ideals $J \subseteq I$ of S. To prove that $\phi(a)$ is injective, we recall first that $aS = a^{-1}S$. Let now $a^{-1}s$ and $a^{-1}t$ be two elements of $a^{-1}S$ such that $\phi(a)(a^{-1}s) = \phi(a)(a^{-1}t)$. Hence, $aa^{-1}s = aa^{-1}s$, and after multiplying on the left by a^{-1}, we obtain $a^{-1}s = a^{-1}t$. Also $\phi(a)$ is surjective since for every $ay \in aS$, $\phi(a)(a^{-1}ay) = a(a^{-1}ay) = ay$. It remains to prove that $\phi(a)$ preserves all ideals $J \subseteq I$ of $(S, \cdot)$. This is an obvious implication of the weaker statement that $\phi(a)$ preserves all principal ideals $bS \subseteq aS$. To prove that $\phi(a)(bS) = bS$, we recall first that bS is a disjoint union of $\mathcal{H}$-classes H_γ where γ is an idempotent such that $\gamma \leq \alpha$ where $\alpha = aa^{-1}$ is the idempotent of the $\mathcal{H}$-class H_a. Consequently, for every $y \in H_\gamma$, we have that $ay = \varphi_{\alpha,\gamma}(a)y$. Letting E be the semilattice of the idempotents of S, we can now write

$$\begin{aligned}\phi(a)(bS) &= \bigcup_{\gamma \in E \cap bS} \varphi_{\alpha,\gamma}(a)H_\gamma \\ &= \bigcup_{\gamma \in E \cap bS} H_\gamma = bS, \qquad\qquad (\text{since } \varphi_{\alpha,\gamma}(a) \in H_\gamma,)\end{aligned}$$

which proves that $\phi(a)(bS) = bS$. Next we prove that ϕ is injective and a homomorphism. Indeed, if there are $a, b \in S$ such that $\phi(a) = \phi(b)$, then $aS = bS$ and as a result $a^{-1}a = b^{-1}b$. Now we can write

$$a = a(a^{-1}a) = \phi(a)(a^{-1}a) = \phi(b)(a^{-1}a) = \phi(b)(b^{-1}b) = b(b^{-1}b) = b,$$

which proves the injectivity. To prove that ϕ is a homomorphism, let $a, b \in S$, then, on the one hand $\phi(ab) : abS \to abS$ is the left translation by ab, and on the other hand $\phi(a) \circ \phi(b) : (aS \cap bS) \to (aS \cap bS)$ is the composition $\phi(a) \circ \phi(b)$ of the restrictions of $\phi(a)$ and $\phi(b)$ on $aS \cap bS$ which from lemma 2.5 equals abS. This composition sends every $b^{-1}a^{-1}s \in abS$ to

$$\begin{aligned}(\phi(a) \circ \phi(b))(b^{-1}a^{-1}s) &= \phi(a)(\phi(b)(b^{-1}a^{-1}s)) \\ &= \phi(a)(bb^{-1}a^{-1}s) \\ &= \phi(a)(a^{-1}bb^{-1}s) \\ &= aa^{-1}bb^{-1}s \\ &= abb^{-1}a^{-1}s \\ &= \phi(ab)(b^{-1}a^{-1}s),\end{aligned}$$

which proves that $\phi(ab) = \phi(a) \circ \phi(b)$. □

Remark 2.7. The benefit of considering monosettings to define the symmetric inverse monoid on a set X, is that it makes it possible to define the symmetric Clifford semigroups by restricting in the appropriate subcategory. This restriction shows also that the symmetric Clifford semigroup $\mathscr{C}(S)$ we define is a subsemigroup the symmetric inverse semigroup I_S.

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INDEPENDENCE VERSUS UNCORRELATEDNESS: FROM EARLY STUDIES TO CURRENT RESEARCHES

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Abstract. The paper presents a review of the results related to the interconnectedness between the two fundamental notions of probability theory and mathematical statistics, namely, the independence and uncorrelatedness of random variables. Both classical results and recent researches will be discussed. Two open problems are formulated.

1. Introduction

Independence as a concept is vital in Probability Theory, Mathematical Statistics, and their different applications. It is commonly known that the condition of independence or dependence are crucial conditions in the great majority of probabilistic results. Due to a high degree of importance of the independence concept, various generalizations of independence have been introduced and studied. One of the earliest and most useful generalizations is uncorrelatedness of random variables. This paper presents an overview of results related to extensions of the uncorrelatedness property to sets of n random variables along with their powers.

To begin with, let us recall the following basic definition.

Definition 1.1. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space. Random events $A_1, A_2, \dots, A_n$ are *independent* if, for every selection $1 \leq j_1 < j_2 < \dots < j_k \leq n$, the following equality holds:

$$\mathbf{P}\left(A_{j_1} \cap A_{j_2} \cap \dots \cap A_{j_k}\right) = \mathbf{P}\left(A_{j_1}\right) \cdot \mathbf{P}\left(A_{j_2}\right) \dots \mathbf{P}\left(A_{j_k}\right) \ k = 2, \dots, n. \tag{1.1}$$

That is, n random events $A_1, A_2, \dots, A_n$ are *independent* if the $2^n - n - 1$ product rules hold. If at least on the the equalities fail to be true, random events are said to be *dependent*. Despite the fact that this definition is presented in all texts in probability theory and statistics, the following question is often remains in the shade: do we need to check all of these equalities or it suffices to verify only some of them?

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The examples for $n = 3$ events show that it is not possible to decrease the number of conditions.

First, let us show that there exist 3 random events A, B, and C such that $\mathbf{P}(A \cap B \cap C) = \mathbf{P}(A)\mathbf{P}(B)\mathbf{P}(C)$, while $\mathbf{P}(A \cap B) \neq \mathbf{P}(A)\mathbf{P}(B)$, $\mathbf{P}(A \cap C) \neq \mathbf{P}(A)\mathbf{P}(C)$, and $\mathbf{P}(B \cap C) \neq \mathbf{P}(B)\mathbf{P}(C)$.

Example 1.2. Let $\Omega = \{1,2,3,4,5,6,7,8\}$, where each outcome has probability $1/8$. Consider the random events $A = \{1,2,3,4\}, B = \{1,3,4,5\}, C = \{1,6,7,8\}$, each of them has probability $1/2$. Since $A \cap B \cap C = \{1\}$, it follows that

$$\mathbf{P}(A \cap B \cap C) = \mathbf{P}(A)\mathbf{P}(B)\mathbf{P}(C).\checkmark$$

At the same time,

$$\begin{array}{lll} A \cap B = \{1,3,4\} & \Rightarrow & \mathbf{P}(A \cap B) = 3/8 \quad \neq \mathbf{P}(A)\mathbf{P}(B) = 1/4; \\ A \cap C = \{1\} & \Rightarrow & \mathbf{P}(A \cap C) = 1/8 \quad \neq \mathbf{P}(A)\mathbf{P}(C) = 1/4; \\ B \cap C = \{1\} & \Rightarrow & \mathbf{P}(B \cap C) = 1/8 \quad \neq \mathbf{P}(B)\mathbf{P}(C) = 1/4. \end{array}$$

The following important examples showing that 3 pairwise independent random events may not be independent are due to G. Bohlmann and S. N. Bernstein. They can be found, for example in [7]. The history of the problem is presented thoroughly in [4].

Example 1.3 (Georg Bohlmann, 1908)**.** There exist 3 random events, which are *pairwise* independent, but not mutually independent.

Let
$$\begin{aligned} \Omega = \{&(111),(111),(111),(100),(100),(100),(110),(101) \\ &(010),(010),(010),(001),(001),(001),(011),(000)\} \end{aligned}$$

with equal probability $1/16$ for all outcomes. Consider $A_i = \{\text{all outcomes having 1 at the } i^{th} \text{ place}\},\ i = 1,2,3$. Then $\mathbf{P}(A_i) = 1/2$, $i = 1,2,3$ and $\mathbf{P}(A_i \cap A_j) = 1/4 = \mathbf{P}(A_i)\mathbf{P}(A_j), i \neq j.\checkmark$ Therefore, A_1, A_2, and A_3 are pairwise independent. Meanwhile, we have:

$$\mathbf{P}(A_1 \cap A_2 \cap A_3) = 3/16 \neq 1/8 = \mathbf{P}(A_1)\mathbf{P}(A_2)\mathbf{P}(A_3)$$

Example 1.4 (S. N. Bernstein, 1928)**.** There exist 3 random events, which are *pairwise* independent, but not mutually independent.

Let $\Omega = \{1,2,3,4\}$ with equal probability $1/4$ for all outcomes. Consider $A_1 = \{1,2\}, A_2 = \{1,3\}, A_3 = \{1,4\}$. Obviously, all events $A_i, i = 1,2,3$ have probabilities $1/2$, their pairwise intersections as well as the intersection of the three events consist only of outcome $\{1\}$ and hence these intersections have probabilities $1/4$.

Consequently, $\mathbf{P}(A_i \cap A_j) = 1/4 = \mathbf{P}(A_i)\mathbf{P}(A_j), i \neq j.\checkmark$ Therefore, A_1, A_2, and A_3 are pairwise independent. Meanwhile, we have:

$$\mathbf{P}(A_1 \cap A_2 \cap A_3) = \frac{1}{4} \neq \frac{1}{8} = \mathbf{P}(A_1)\mathbf{P}(A_2)\mathbf{P}(A_3)$$

Although random events $A_1, A_2, \ldots, A_n$ for which at least one of equalities (1.1) is violated are said to be dependent, daily life experience proves that dependence among events may vary in terms of strength, ranging from mild 'influence' to strong 'cause-effect' connections. It seems practical, therefore, to distinguish the different types of independence or dependence, and to introduce appropriate notions regarding partial independence. Some of such notions will be discussed in the consequent sections.

2. Levels of independence and the Italian problem

The notion was introduced and examined by Jordan Stoyanov, the author of the "Counterexamples in Probability", see [7]. It provides a far-reaching generalization of the classical examples related to the independence conditions of n random events.

Definition 2.1. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space. Random events $A_1, A_2, \ldots, A_n$ are *independent at level* $k, 2 \leq k \leq n$ if, for every k-tuple $1 \leq j_1 < j_2 < \cdots < j_k \leq n$, the following equality holds:

$$\mathbf{P}\left(A_{j_1} \cap A_{j_2} \cap \cdots \cap A_{j_k}\right) = \mathbf{P}\left(A_{j_1}\right) \cdot \mathbf{P}\left(A_{j_2}\right) \ldots \mathbf{P}\left(A_{j_k}\right).$$

Otherwise, these random events are said to be *dependent at level* k.

Clearly random events are independent if and only if they are independent at all levels $2, 3, \ldots, n$. Independence at level 2 is just the pairwise independence.

The following observation is crucial for understanding the notion of independence:

The independence at one level does not imply the independence at any other level - either higher or lower.

A detailed proof of this statement can be found, for example, in [8].

J. Stoyanov introduced another important notion related to the independence properties of a collection of random events.

Definition 2.2. Given a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and a collection of random events $A_1, A_2, \ldots, A_n$. The *independence structure* of these random events is a finite sequence $(i_2, i_3, \ldots, i_n)$, where $0 \leq i_k \leq \binom{n}{k}$ is the number of k-tuples among $A_1, A_2, \ldots, A_n$ for which the product rule holds.

It is obvious that, for any collection of n random events, its independence structure can be uniquely determined. Yet, a more challenging problem is the next one, proposed and eventually solved by J. Stoyanov.

The Italian Problem. Let us have a sequence of integers $(i_2, i_3, \ldots, i_n)$ with $0 \leq i_k \leq \binom{n}{k}$. In this case, does there exist a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and a collection of n random events $A_1, A_2, \ldots, A_n$, such that $(i_2, i_3, \ldots, i_n)$ is exactly their independence structure?

J. Stoyanov obtained the affirmative answer to this question in the same paper [8]. Actually, he proved a more general statement. To formulate his result, we introduce the following notation. For random events $A_1, \ldots, A_n$ denote

$$J_k := \{(j_1, \ldots, j_k) : 1 \leq j_1 < \cdots < j_k \leq n$$
$$\text{and } P(\bigcap_{l=1}^{k} A_{j_l}) = \prod_{l=1}^{k} P(A_{j_l})\},\ k = 2, \ldots, n.$$

The sets $J_k (k = 2, \ldots, n)$ list those k- tuples for which the multiplication rule holds. Obviously, $i_k =\mid J_k \mid\ (k = 2, \ldots n)$. We call the finite sequence $(J_2, \ldots, J_n)$ the *independence characteristic* of the set of random events $A_1, \ldots, A_n$. Evidently, the independence characteristic provides more detailed information than the independence structure.

The following generalization of the Italian problem holds.

Theorem 2.3 ([8]). *Let* $(J_2, \ldots, J_n)$, *where*

$$J_k \subseteq \{(j_1, \ldots, j_k) : 1 \leq j_1 < \cdots < j_k \leq n\},\ k = 2, \ldots, n$$

be a given finite sequence.

Then there exists a probability space $(\Omega, \mathcal{F}, P)$ *and a collection of random events* $A_1, \dots, A_n \in \mathcal{F}$ *such that* $(J_2, \dots, J_n)$ *is the independence characteristic of* $A_1, \dots, A_n$.

That is, we may prescribe not only the number of k-tuples for which the multiplication rule holds, but also specify in advance k-tuples themselves.

3. Independence, uncorrelatedness and uncorrelatedness sets of random variables

For the sake of simplicity of presentation we discuss only the case of two random variables, though all results below can be generalized for $n \geq 2$ random variables.

Definition 3.1. Random variables X_1 and X_2 are *independent* if

$$\mathbf{P}\{X_1 \in E_1 \cap X_2 \in E_2\} = \mathbf{P}\{X_1 \in E_1\} \cdot \mathbf{P}\{X_2 \in E_2\} \text{ for all } E_1, E_2 \subset \mathbb{R}.$$

Although this definition is presented in all probability and statistic courses, in applications it is often confused by weaker conditions indicating the weak or no relationship between the variables.

The most popular condition to be used in place of independence is *uncorrelatedness* of random variables.

Definition 3.2. Random variables X_1 and X_2 are *uncorrelated* if

$$\mathbf{E}\,(X_1 X_2) = \mathbf{E}\,(X_1)\,\mathbf{E}\,(X_2)\,,$$

provided that all of the expected values exist.

It is commonly known that independent random variables are uncorrelated. However, random variables may be uncorrelated without being independent.

Example 3.3. Let $\Omega = [0, 2\pi]$ be a sample space with the probability $\mathbf{P}(A) = \frac{1}{2\pi}\text{length}(A)$, and let X_1, X_2 be random variables on Ω given by: $X_1(x) = \sin x$ and $X_2(x) = \cos x$. The expected values of these random variables can be found easily: $\mathbf{E}\,(X_1) = \mathbf{E}\,(X_2) = 0$. Also, $\mathbf{E}\,(X_1 X_2) = 0$, whence we see that X_1 and X_2 are *uncorrelated*. However they cannot be independent because they are connected with the well-known identity: $\sin^2 x + \cos^2 x = 1$.

Uncorrelatedness is measured with the help of a *correlation coefficient*

$$\rho = \frac{\mathbf{E}(X_1 X_2) - \mathbf{E}(X_1)\mathbf{E}(X_2)}{\sigma_{X_1}\sigma_{X_2}}$$

taking values from -1 to 1 with $\rho = 0$ if and only if random variables are uncorrelated, while $\rho = \pm 1$ indicates a linear dependence between X_1 and X_2.

Notice that not only $\rho = 0$ does not imply independence, but uncorrelated random variables can be even functionally dependent (but not linearly).

In the interesting article [2], the connection between the condition of independence and the lack of correlation is investigated from the historical perspective.

The aim of this paper is to discuss the uncorrelatedness of positive integer powers of random variables.

Regardless of the fact that many different approaches on measures of independence have been developed, in distinction from the uncorrelatedness, there is no universal way of measuring whether random variables are "more independent" or "less independent." Here, we make one more attempt to compare the degrees of relationship between random variables based on the next definition.

Definition 3.4. Let X_1 and X_2 be random variables with finite moments of all orders. The collection of pairs $(j, l) \in \mathbb{N}^2$ so that X_1^j and X_2^l are uncorrelated is called an *uncorrelatedness set* of X_1 and X_2.

We denote an uncorrelatedness set of X_1 and X_2 by $U(X_1, X_2)$. The definition above means that

$$(j,l) \in U(X_1, X_2) \Leftrightarrow \mathbf{E}\left(X_1^j X_2^l\right) = \mathbf{E}\left(X_1^j\right)\mathbf{E}\left(X_2^l\right).$$

Random variables X_1 and X_2 are uncorrelated in the usual sense if and only if $(1, 1) \in U(X_1, X_2)$.

Uncorrelatedness sets give us a *partial* order of "independencies": we may think that the wider an uncorrelatedness set is, the more independent random variables are. However, sometimes we cannot compare degrees of independence for different random variables with this approach. Obviously, for independent random variables $U = \mathbb{N}^2$.

Remark 3.5. Note that $U(X_1, X_2) = \mathbb{N}^2$ does not imply the independence of X_1 and X_2, as it was proved in [5]

Which sets in $\mathbb{N}^2$ can be uncorrelatedness sets?

Theorem 3.6. *([6]) Let a subset U of $\mathbb{N}^2$ be given. There exist random variables X_1 and X_2 such that U is their uncorrelatedness set.*

In other words, for an arbitrary subset U of $\mathbb{N}^2$, there exist random variables X_1 and X_2 such that

$$\mathbf{E}(X_1^j X_2^l) = \mathbf{E}(X_1^j)\mathbf{E}(X_2^l) \text{ for all } (j,l) \in U,$$

while

$$\mathbf{E}(X_1^j X_2^l) \neq \mathbf{E}(X_1^j)\mathbf{E}(X_2^l) \text{ for all } (j,l) \notin U.$$

4. Uncorrelatedness sets for random variables with given distributions

Despite the general result on an arbitrary uncorrelatedness set, the statement cannot be true for for random variables with predetermined distributions.

For example, two *binary* random variables are independent if and only if they are uncorrelated. In other words, for such random variables $U(X_1, X_2) \ni (1, 1) \Leftrightarrow U(X_1, X_2) = \mathbb{N}^2$.

Even more generally, if X_1 and X_2 are discrete random variables taking two values, then the uncorrelatedness implies independence, that is, again $U(X_1, X_2) \ni (1, 1) \Leftrightarrow U(X_1, X_2) = \mathbb{N}^2$.

As it turns out, we obtain the challenging problem of describing possible uncorrelatedness sets for random variables with given distributions.

First, consider some simple results in this directions.

Example 4.1 (D. Yıldırım)**.** If random variables X_1 and X_2 are such that $X_1 \in \{a, b\}, X_2 \in \{c, d\}$, $a < b, c < d$, then:

- If for both random variables the sets of values are *not* symmetric, then either $U(X_1, X_2) = \mathbb{N}^2$ (and random variables are independent) or $U(X_1, X_2) = \emptyset$.
- If just one set of values is symmetric, say, $X_1 \in \{-a, a\}$, then either $U(X_1, X_2) = \mathbb{N}^2$ (and random variables are independent) or $U(X_1, X_2) = 2\mathbb{N} \times \mathbb{N}$.

- If both of them sets of values are symmetric, then either $U(X_1, X_2) = \mathbb{N}^2$ (and random variables are independent) or $U(X_1, X_2) = 2\mathbb{N} \times 2\mathbb{N}$.

The next problems on the uncorrelatedness sets turn out to be very challenging:

- Describe all possible uncorrelatedness sets of discrete random variables taking three values: $X_1 \in \{a_1, b_1, c_1\},\ X_2 \in \{a_2, b_2, c_2\}$.
- Describe all possible uncorrelatedness sets of two normal random variables: variables taking three values: $X_1 \sim \mathcal{N}(a_1, \sigma_1\},\ X_2 \sim \mathcal{N}(a_2, \sigma_2)$.

Both problems are still open, only some special cases have been considered. In the proceeding section, we consider partail solution to the first one.

5. Uncorrelatedness sets for random variables taking 3 values

The problem was investigated in the paper [9]. We consider random variables X and Y uniformly distributed on the set $\{a, b, c\}, 0 < a < b < c$.

Theorem 5.1 ([9]). *The following possibilities exist for $U(X, Y)$:*

- $U(X, Y) = \emptyset$;
- $U(X, Y) = (j_0, l_0)$ *any given* $(j_0, l_0) \in \mathbb{N}^2$;
- $U(X, Y) = \{(j_1, l_1), (j_2, l_2)\}$, *where* $j_1 \neq j_2$ *and* $l_1 \neq l_2$.
- *If* $(j_1, l_1), (j_2, l_2) \in U(X, Y)$ *and* $j_1 = j_2$, *then* $\{j_1\} \times \mathbb{N} \in U(X, Y)$. *Likewise for* $l_1 = l_2$. *That is, two points on the same vertical/horizontal line cannot form an uncorrelatedness set. Meanwhile any vertical/horizontal line can be an uncorrelatedness set.*
- *The line* $j = l$ *may be an uncorrelatedness set.*

Corollary 5.2. *For random variables taking 3 values uncorrelatedness does not imply independence.*

Theorem 5.3 ([9]). *There exist random variables X and Y uniformly distributed on 3 values with uncorrelatedness set of any given size $n \in \mathbb{N}_0$.*

Namely, if $\{a, b, c\} = \{\alpha, \alpha\beta, \alpha\beta^2\}$, then every straight line $j + l = n$ is an uncorrelatedness set (of size $n - 1$).

6. A scale of degrees of independence

Uncorrelatedness sets may be used to construct not only partial but also linear order for the degrees of independence. Ona of the approaches uses the definition of k-independence given in [1, 5].

Definition 6.1. . Let $k \geq 2$ be a positive integer. We say that random variables X_1 and X_2 are *k-uncorrelated* if

$$\mathbf{E}(X_1^j X_2^l) = \mathbf{E}(X_1^j)\mathbf{E}(X_2^l) \text{ for } \{(j, l) \in \mathbb{N}^2 : j + l \leq k\}.$$

Obviously, 2-uncorrelatedness coincides with uncorrelatedness in the usual sense, and independent random variables are k-uncorrelated for all $k = 2, 3, \ldots$ As we have already mentioned, k-uncorrelatedness for all $k = 2, 3, \ldots$ does not imply independence.

In terms of uncorrelatedness sets, we may say that X_1 and X_2 are *k-uncorrelated* if and only if $\Delta_k \subset U(X_1, X_2)$, where Δ_k is the triangle $j + l \leq k$.

Clearly, $(k+1)$-uncorrelated random variables are k-uncorrelated. It is proved that converse is not true, that is $(k+1)$-uncorrelatedness is a strictly stronger condition than k-uncorrelatedness.

As a result, we obtain the following scale of independence: 2-uncorrelatedness (uncorrelatedness), 3-uncorrelatedness, 4-uncorrelatedness, ... , k-uncorrelatedness for all $k = 2, 3, \ldots$, convolutional independence and, finally, independence.

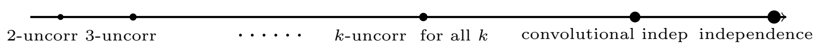

Figure 1

7. Conclusions

Since probabilistic methods play a profound role in modern theoretical and applied research, there is no doubt that the summary of results on the underlying fundamental concepts of probability theory is beneficial for the purpose of providing the educators and researches with with an up-to-date background to the subject.

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BLOW-UP OF THE PHENOMENON FOR SEMILINEAR PARABOLIC PROBLEMS WITH VARIABLE SOURCES

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Abstract. In this paper, we establish some sufficient conditions to guarantee the existence of non-global solutions to the model for any $\eta(0)$ and also derive the upper bounds for the blow-up time and a criterion for blow-up.

1. Introduction

In this paper, we study the following parabolic problem

$$\begin{cases} u_t = \Delta u + f(u), \; x \in \Omega, t > 0, \\ u(x,0) = u_0(x), \; x \in \Omega, \\ u(x,t) = 0, \; x \in \partial\Omega, t > 0, \end{cases} \tag{1.1}$$

where $\Omega \in \mathbb{R}^N$ is a bounded domain with a smooth boundary $\partial\Omega$ and the source term is of the form

$$f(x,u) = \beta u^{p(x)} \text{ or } f(u) = \beta \int_\Omega u^{q(y)}\,(y,t)\,dy,$$

and $\beta > 1$ is a parametr. We impose the following conditions on the variable sources functions p, $q : \Omega \to (1, +\infty)$ such that

$$1 < p^- < p\,(x) < p^+ < +\infty \text{ a.e. } x \in \Omega, \tag{1.2}$$

and

$$1 < q^- < q\,(x) < q^+ < +\infty \text{ a.e. } x \in \Omega. \tag{1.3}$$

Equation (1.1) describes the diffusion of concentration of some Newtonian fluids through porous medium or the density of some biological species in many physical phenomena and biological species theories (see [5, 8]).

Under certain conditions on the initial data and certain ranges of exponents, the existence, uniqueness, blow up and other qualitative properties of solutions for

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parabolic equations with constant and variable nonlinearity have been studied by many authors (see [1, 2, 4, 6, 7, 9, 11, 12, 14] and references therein).

In [13], the author studied the blow up in finite time with initial data which is sufficiently large for positive solutions of parabolic and hyperbolic problems with reaction terms of local, nonlocal type involving a variable exponent for following problem:

$$\begin{cases} u_t = \Delta u + f(u),\ (x,t) \in \Omega \times [0,T)\,, \\ u(x,0) = u_0(x),\ x \in \Omega, \\ u(x,t) = 0, (x,t) \in \partial\Omega \times [0,T)\,, \end{cases} \tag{1.4}$$

where the source term is of the form

$$f(x,u) = a\,(x)\,u^{p(x)} \text{ or } f(u) = a\,(x) \int_\Omega u^{q(y)}\,(y,t)\,dy,$$

and functions $p, q : \Omega \to (1, +\infty)$ and the continuous function $a : \Omega \to \mathbb{R}$:

$$1 < p^- \leq p\,(x) \leq p^+ < +\infty, \tag{1.5}$$

$$1 < q^- \leq q\,(x) \leq q^+ < +\infty, \tag{1.6}$$

$$0 < c_a \leq a\,(x) \leq C_a < +\infty. \tag{1.7}$$

The author stated the following Theorem and proved the existence of initial data such that the corresponding solutions blow up at a finite time.

Theorem 1.1. (*Theorem 1.1 in* [13])*. Let* $\Omega \in \mathbb{R}^N$ *be a bounded smooth domain and let* u *be a positive solution of equation* (1.4)*, with* p, q *and* a *satisfying conditions* $(1.5) - (1.7)$*. Then, for a sufficiently large initial datum* $u_0\,(x)$*, there exists a finite time* $T_f > 0$ *such that*

$$\sup_{0 \leq t \leq T_f} \|u\,(.,t)\|_\infty = +\infty.$$

In this paper we study the blow up problem for positive solutions of parabolic problems with reaction terms of local and nonlocal type involving a variable sources. Based on a modified differential inequality technique, we establish some sufficient conditions to guarantee the existence of non-global solutions to the model and also derive the upper bounds for the blow-up time for any initial data of the problem (1.1).

We define the function

$$\eta\,(t) = \int_\Omega u\varphi_1 dx, \tag{1.8}$$

where $\varphi_1(x) > 0$ in Ω and $\lambda_1 > 0$, respectively, the first eigenfunction and the corresponding (smallest) eigenvalue of the problem

$$\Delta\varphi + \lambda\varphi = 0,\ x \in \Omega,\ \varphi|_{\partial\Omega} = 0, \tag{1.9}$$

and

$$\int_\Omega \varphi_1 dx = 1.$$

Definition 1.2. We say that the solution $u\,(x,t)$ blows up in a finite time if there exists an instant $T_f < +\infty$ such that

$$\|u(.,t)\|_\infty \to \infty \text{ as } t \to T_f.$$

It is easy to see that the finite time blow-up happens if, say, there exists a moment $T_f < +\infty$ such that $\eta(T_f) = +\infty$. Indeed:

$$\eta(t) = \int_\Omega u\varphi_1 dx \le \|u(.,t)\|_\infty \int_\Omega \varphi_1 dx = \|u(.,t)\|_\infty \to \infty \text{ as } t \to T_f.$$

This observation allows us to characterize blow-up of the solution $u(x,t)$ in terms of the function $\eta(t)$.

2. Main Results and Proofs

Next we will use Kaplan's method (see [10]) to investigate the upper bound for blow-up time of blow-up solution to problem (1.1).

Our main results are the following theorems:

Theorem 2.1. *Let Ω be a bounded smooth domain in $\mathbb{R}^N$ and let u be a positive solution of problem* (1.1)*, and function p satisfying condition* (1.2) *and*

$$f(x,u) = \beta u^{p(x)}.$$

If

$$0 < \lambda_1 < \frac{\eta^{p^-}(0)}{1+\eta^{p^-}(0)},$$

and

$$\beta > \max\left\{\left(\frac{1}{(1-\lambda_1)\eta^{p^-}(0) - \lambda_1}\right)^{\frac{p^+-p^-}{p^-}}, 1\right\},$$

then the problem (1.1) *has no global solutions in finite time $T_f > 0$ for any $\eta(0)$. We have*

$$\int_{\eta(0)}^{+\infty} \frac{ds}{(1-\lambda_1) s^{p^-} - \lambda_1 - \beta^{-\frac{p^-}{p^+-p^-}}} \ge T_f,$$

where

$$\eta(0) = \int_\Omega u_0 \varphi_1 dx.$$

Theorem 2.2. *Let Ω be a bounded smooth domain in $\mathbb{R}^N$ and let u be a positive solution of problem* (1.1)*, and function q satisfying condition* (1.3) *and*

$$f(u) = \beta \int_\Omega u^{q(y)}(y,t)\, dy.$$

If

$$0 < \lambda_1 < \frac{\eta^{q^-}(0)}{\|\varphi_1\|_\infty \left(1+\eta^{q^-}(0)\right)},$$

and

$$\beta > \max\left\{\left(\frac{1}{(1-\lambda_1\|\varphi_1\|_\infty)\eta^{q^-}(0) - \lambda_1\|\varphi_1\|_\infty}\right)^{\frac{q^+-q^-}{q^-}}, 1\right\},$$

then the problem (1.1) *has no global solutions in finite time* $T_f > 0$ *for any initial data* u_0. *We have*

$$\int_{\eta(0)}^{+\infty} \frac{\|\varphi_1\|_\infty d\xi}{(1-\lambda_1\|\varphi_1\|_\infty)\xi^{q^-} - \lambda_1\|\varphi_1\|_\infty - \beta^{-\frac{q^-}{q^+-q^-}}} \geq T_f.$$

We consider the problem (1.1) but now ask that p satisfying condition $0 < p^- \leq p^+ \leq 1$. In this case, we show that the solution remains bounded for all time when a restriction is imposed on the constant β.

Theorem 2.3. *Let* Ω *be a bounded smooth domain in* $\mathbb{R}^N$ *and let* u *be a positive solution of problem* (1.1) *and*

$$f(x,u) = \beta u^{p(x)}.$$

If function p *satisfying conditions* $0 < p^- \leq p^+ \leq 1$ *and* $0 < \beta \leq \frac{\lambda_1}{2}$, *then* u *is bounded for all time.*

Proof of Theorem 2.1. Taking the scalar product in $L^2(\Omega)$ with φ_1 of both parts of the equation (1.1) and integrating the resulting expression in t, we obtain the equality

$$(u,\varphi_1) - b_0 = -\lambda_1 \int_0^t \eta(s)\,ds + \beta \int_0^t \left(u^{p(x)},\varphi_1\right) ds, \tag{2.1}$$

where

$$b_0 = (u_0,\varphi_1) > 0.$$

Let remind the following elementary inequality (see [3]):

$$a\theta^l - b\theta^m \leq a\left(\frac{a}{b}\right)^{\frac{l}{m-l}}, \forall\theta > 0, \tag{2.2}$$

where $a, b > 0$ and $0 < l < m$. By using (1.2) and (2.2) , since $\beta > 1$ we derive

$$u^{p^-} - \beta u^{p(x)} \leq \left(\frac{1}{\beta}\right)^{\frac{p^-}{p(x)-p^-}} \leq \left(\frac{1}{\beta}\right)^{\frac{p^-}{p^+-p^-}}, \forall x \in \Omega, \tag{2.3}$$

and so we have

$$\beta u^{p(x)}\varphi_1 \geq u^{p^-}\varphi_1 - \beta^{-\frac{p^-}{p^+-p^-}}\varphi_1. \tag{2.4}$$

By (2.1) and (2.4), we get

$$\begin{aligned}&(u,\varphi_1) - b_0 \\ &\geq \int_0^t \left(u^{p^-},\varphi_1\right) ds - \lambda_1\int_0^t \eta(s)\,ds - \int_0^t\int_\Omega \beta^{-\frac{p^-}{p^+-p^-}}\varphi_1 dx ds.\end{aligned} \tag{2.5}$$

Furthermore, taking into account the fact that $p^- > 1$, by using Hölder's inequality, in (1.8), we obtain

$$\begin{aligned}\eta(t) &= \int_\Omega u\varphi_1^{\frac{1}{p^-}}\varphi_1^{1-\frac{1}{p^-}}dx \leq \left(\int_\Omega u^{p^-}\varphi_1 dx\right)^{\frac{1}{p^-}}\left(\int_\Omega \varphi_1 dx\right)^{\frac{p^--1}{p^-}} \\ &= \left(\int_\Omega u^{p^-}\varphi_1 dx\right)^{\frac{1}{p^-}}.\end{aligned} \tag{2.6}$$

By (2.1), (2.5) and (2.6), we can write

$$\eta(t) - b_0 \geq \int_0^t \left(\eta^{p^-}(s) - \beta^{-\frac{p^-}{p^+ - p^-}} \right) ds - \lambda_1 \int_0^t \eta(s)\, ds. \tag{2.7}$$

From (2.7), we obtain

$$\eta'(t) \geq \eta^{p^-}(t) - \lambda_1 \eta(t) - \beta^{-\frac{p^-}{p^+ - p^-}}, t > 0. \tag{2.8}$$

Since

$$\eta(t) \leq \max\left\{ \eta^{p^-}(t), 1 \right\},$$

from (2.8), we get

$$\eta'(t) \geq (1 - \lambda_1)\, \eta^{p^-}(t) - \lambda_1 - \beta^{-\frac{p^-}{p^+ - p^-}} \equiv f(\eta(t)), t > 0. \tag{2.9}$$

Obviously, since $p^- > 1$ and

$$\beta > \max\left\{ \left(\frac{1}{(1 - \lambda_1)\, \eta^{p^-}(0) - \lambda_1} \right)^{\frac{p^+ - p^-}{p^-}}, 1 \right\},$$

and

$$\lambda_1 < \min\left\{ \frac{\eta^{p^-}(0)}{1 + \eta^{p^-}(0)}, 1 \right\} = \frac{\eta^{p^-}(0)}{1 + \eta^{p^-}(0)},$$

we can get that the function η^{p^-} is monotone increasing for all $t \geq 0$, then we can know that the solution of problem (1.1) blows up in finite time. Therefore the solution of the boundary value problem is unbounded. Moreover, dividing the both parts of (2.9) by $f(s)$ and integrating, we have

$$I(\eta) = \int_{\eta(0)}^{\eta(t)} \frac{ds}{f(s)} \geq t.$$

Since the integral $I(s)$ is convergent at $s = +\infty$, this inequality is possible only if there exists T_f such that $\overline{\lim_{t \to T_f}}\, \eta(t) \to \infty$. Therefore u cannot exist globally. The proof of Theorem 2.1 is completed. □

Proof of Theorem 2.2. Let us now consider the case

$$f(u) = \beta \int_\Omega u^{q(y)}(y, t)\, dy.$$

We obtain in much the same way

$$\begin{aligned} & \eta(t) - b_0 \\ = \quad & -\lambda_1 \int_0^t \eta(s)\, ds + \int_0^t \int_\Omega \left(\int_\Omega \beta u^{q(y)}(y, s)\, dy \right) \varphi_1(x)\, dx ds, \end{aligned}$$

where

$$b_0 = (u_0, \varphi_1) > 0.$$

Similarly relation (2.3), we have

$$u^{q^-} - \beta u^{q(y)} \leq \beta^{-\frac{q^-}{q^+ - q^-}}, \forall y \in \Omega,$$

with $\beta > 1$, then

$$\begin{aligned}
&\eta(t) - b_0 \\
\geq\ & -\lambda_1 \int_0^t \eta(s)\,ds + \int_0^t \int_\Omega \left(u^{q^-}(y,s) - \beta^{-\frac{q^-}{q^+-q^-}} \right) dy \int_\Omega \varphi_1(x)\,dxds \\
\geq\ & -\lambda_1 \int_0^t \eta(s)\,ds + \frac{1}{\|\varphi_1\|_\infty} \int_0^t \int_\Omega \left(u^{q^-}(y,s) - \beta^{-\frac{q^-}{q^+-q^-}} \right) \varphi_1(y)\,dyds \\
\geq\ & -\lambda_1 \int_0^t \eta(s)\,ds + \frac{1}{\|\varphi_1\|_\infty} \int_0^t \left(\eta^{q^-}(s) - \beta^{-\frac{q^-}{q^+-q^-}} \right) ds,
\end{aligned}$$

since

$$\eta^{q^-}(t) \leq \int_\Omega u^{q^-} \varphi_1 dy.$$

Then we have

$$\eta(t) - b_0 \geq -\lambda_1 \int_0^t \eta(s)\,ds + \frac{1}{\|\varphi_1\|_\infty} \int_0^t \left(\eta^{q^-}(s) - \beta^{-\frac{q^-}{q^+-q^-}} \right) ds.$$

Similarly (2.9), we have

$$\begin{aligned}
\eta'(t) \ &\geq\ \frac{1}{\|\varphi_1\|_\infty} \eta^{q^-}(t) - \lambda_1 \eta(t) - \frac{1}{\beta^{\frac{q^-}{q^+-q^-}} \|\varphi_1\|_\infty} \\
&\geq\ \left(\frac{1}{\|\varphi_1\|_\infty} - \lambda_1 \right) \eta^{q^-}(t) - \lambda_1 - \frac{1}{\beta^{\frac{q^-}{q^+-q^-}} \|\varphi_1\|_\infty} \equiv g(\eta(t)), t > 0.
\end{aligned}$$

Since $q^- > 1$ and

$$\beta > \max \left\{ \left(\frac{1}{(1 - \lambda_1 \|\varphi_1\|_\infty) \eta^{q^-}(0) - \lambda_1 \|\varphi_1\|_\infty} \right)^{\frac{q^+-q^-}{q^-}}, 1 \right\},$$

with

$$\begin{aligned}
0 \ &<\ \lambda_1 < \min \left\{ \frac{\eta^{q^-}(0)}{\|\varphi_1\|_\infty \left(1 + \eta^{q^-}(0)\right)}, \frac{1}{\|\varphi_1\|_\infty} \right\} \\
&=\ \frac{\eta^{q^-}(0)}{\|\varphi_1\|_\infty \left(1 + \eta^{q^-}(0)\right)}
\end{aligned}$$

we can know that the solution to problem (1.1) blows up in finite time and

$$\int_{\eta(0)}^{+\infty} \frac{d\xi}{g(\xi)} \geq T_f.$$

The proof of Theorem 2.2 is completed. □

Proof of Theorem 2.3. We define the function

$$\mu(t) = \int_\Omega u^2 dx,$$

and compute

$$\begin{aligned}
&\mu'(t) \\
= \;& 2\int_\Omega u u_t dx \\
= \;& 2\int_\Omega u\left(\Delta u + \beta u^{p(x)}\right)dx = -2\int_\Omega |\nabla u|^2 dx + 2\beta \int_\Omega u^{p(x)+1} dx \\
\leq \;& -2\int_\Omega |\nabla u|^2 dx + 2\beta\left(\int_\Omega u^{p^-+1}dx + \int_\Omega u^{p^++1}dx\right) \\
\leq \;& -2\|\nabla u\|_2^2 + 2\beta\left(\int_\Omega u^{p^-+1}dx + \int_\Omega u^{p^++1}dx\right). \qquad (2.10)
\end{aligned}$$

Let us suppose that u becomes unbounded at some time T. Make use of the Rayleigh principle

$$\lambda_1 \int_\Omega \upsilon^2 dx \leq \int_\Omega |\nabla \upsilon|^2 dx,$$

where λ_1 is the first positive eigenvalue of the fixed membrane problem (1.9). From (2.10) and the assumption $0 < p^- \leq p^+ \leq 1$, we have

$$\mu'(t) \leq 2(2\beta - \lambda_1)\int_\Omega u^2 dx.$$

If we restrict β such that $\beta \leq \frac{\lambda_1}{2}$, where λ_1 the first positive eigenvalue of (1.9), we easily get $\mu'(t) \leq 0$.

Moreover it must be noticed that the blow-up time is T (supposes to exist), but $\mu'(t) \leq 0$ holds for every time t, which implies that u is bounded. A contradiction occurs. We can obtain that there is no time T such that u is unbounded. This is to say u is bounded for every time t. Thus, Theorem 2.3 is proved. □

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NOTES ON RANK PROPERTIES OF BLUPS IN LMM AND ITS TRANSFORMED MODEL

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Abstract. Consider a linear mixed model (LMM) and its transformed model without making any restrictions on the correlation of random effects and any full rank assumptions. LMMs include both fixed and random effects and supply helpful tools to account for the variability of model parameters that affect response variables. This study concerns rank relations of covariance matrices of predictors under the original LMM and its transformed model. Our aim is to establish the rank of covariance matrices between the best linear unbiased predictors (BLUPs) of unknown vectors under considered two LMMs by using various rank formulas. We also give some results for special cases by applying the results obtained for general cases.

1. Introduction

Consider a linear mixed model (LMM)

$$\mathcal{M} : \mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{Z}\mathbf{u} + \boldsymbol{\varepsilon} \tag{1.1}$$

and its transformed model

$$\mathcal{T} : \mathbf{T}\mathbf{y} = \mathbf{T}\mathbf{X}\boldsymbol{\beta} + \mathbf{T}\mathbf{Z}\mathbf{u} + \mathbf{T}\boldsymbol{\varepsilon}, \tag{1.2}$$

where $\mathbf{y} \in \mathbb{R}^{n\times 1}$ is a vector of observable response variables, $\mathbf{X} \in \mathbb{R}^{n\times k}$, $\mathbf{Z} \in \mathbb{R}^{n\times p}$, and $\mathbf{T} \in \mathbb{R}^{m\times n}$ are known matrices of arbitrary rank, $\boldsymbol{\beta} \in \mathbb{R}^{k\times 1}$ is a vector of fixed but unknown parameters, $\mathbf{u} \in \mathbb{R}^{p\times 1}$ is a vector of unobservable random effects, and $\boldsymbol{\varepsilon} \in \mathbb{R}^{n\times 1}$ is an unobservable vector of random errors. To establish some results on predictors of all unknown vectors under the models $\mathcal{M}$ and $\mathcal{T}$, we can consider the following vector

$$\boldsymbol{\phi} = \mathbf{K}\boldsymbol{\beta} + \mathbf{G}\mathbf{u} + \mathbf{H}\boldsymbol{\varepsilon} = \mathbf{K}\boldsymbol{\beta} + \begin{bmatrix}\mathbf{G}, & \mathbf{H}\end{bmatrix}\begin{bmatrix}\mathbf{u} \\ \boldsymbol{\varepsilon}\end{bmatrix} \tag{1.3}$$

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for given matrices $\mathbf{K} \in \mathbb{R}^{s\times k}$, $\mathbf{G} \in \mathbb{R}^{s\times p}$, and $\mathbf{H} \in \mathbb{R}^{s\times n}$. We assume the following general assumptions for considered models:

$$\mathrm{E}\begin{bmatrix}\mathbf{u}\\ \boldsymbol{\varepsilon}\end{bmatrix} = \mathbf{0} \text{ and } \mathrm{D}\begin{bmatrix}\mathbf{u}\\ \boldsymbol{\varepsilon}\end{bmatrix} = \mathrm{cov}\left\{\begin{bmatrix}\mathbf{u}\\ \boldsymbol{\varepsilon}\end{bmatrix}, \begin{bmatrix}\mathbf{u}\\ \boldsymbol{\varepsilon}\end{bmatrix}\right\} = \begin{bmatrix}\boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12}\\ \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22}\end{bmatrix} := \boldsymbol{\Sigma}, \tag{1.4}$$

where $\boldsymbol{\Sigma} \in \mathbb{R}^{(n+p)\times(n+p)}$ is a positive semi-definite matrix of arbitrary rank and all the entries of $\boldsymbol{\Sigma}$ are known. Let $\mathbf{B} = \begin{bmatrix}\mathbf{Z}, & \mathbf{I}_n\end{bmatrix}$ and $\mathbf{J} = \begin{bmatrix}\mathbf{G}, & \mathbf{H}\end{bmatrix}$. Then we obtain

$$\mathrm{E}(\mathbf{y}) = \mathbf{X}\boldsymbol{\beta},\ \ \mathrm{D}(\mathbf{y}) = \begin{bmatrix}\mathbf{Z}, & \mathbf{I}_n\end{bmatrix}\boldsymbol{\Sigma}\begin{bmatrix}\mathbf{Z}, & \mathbf{I}_n\end{bmatrix}' = \mathbf{B}\boldsymbol{\Sigma}\mathbf{B}' := \mathbf{R}, \tag{1.5}$$

$$\mathrm{E}(\boldsymbol{\phi}) = \mathbf{K}\boldsymbol{\beta},\ \ \mathrm{D}(\boldsymbol{\phi}) = \begin{bmatrix}\mathbf{G}, & \mathbf{H}\end{bmatrix}\boldsymbol{\Sigma}\begin{bmatrix}\mathbf{G}, & \mathbf{H}\end{bmatrix}' = \mathbf{J}\boldsymbol{\Sigma}\mathbf{J}' := \mathbf{S}, \tag{1.6}$$

$$\mathrm{cov}(\boldsymbol{\phi}, \mathbf{y}) = \begin{bmatrix}\mathbf{G}, & \mathbf{H}\end{bmatrix}\boldsymbol{\Sigma}\begin{bmatrix}\mathbf{Z}, & \mathbf{I}_n\end{bmatrix}' = \mathbf{J}\boldsymbol{\Sigma}\mathbf{B}' := \mathbf{C}. \tag{1.7}$$

Further, we assume that $\mathcal{M}$ is consistent, i.e., $\mathbf{y} \in \mathscr{C}\begin{bmatrix}\mathbf{X}, & \mathbf{R}\end{bmatrix}$ holds with probability 1, see, e.g., Rao (1973). The consistency of $\mathcal{T}$ is provided with the condition $\mathbf{Ty} \in \mathscr{C}\begin{bmatrix}\mathbf{TX}, & \mathbf{TRT}'\end{bmatrix}$ with probability 1. We note that $\mathcal{T}$ is consistent under the assumption of consistency of $\mathcal{M}$.

Investigating the relationships between two different linear models is one of the classical research problems in linear regression analysis. In this study, we consider a LMM and its transformed model. We establish a rank relation between the best linear unbiased predictors (BLUPs) of unknown vectors under these models through various rank formulas. We also give some results for different choices of the matrices in general vector of unknown variables in the models. For studies on transformation approach to linear models in the literature, see, e.g., Baksalary & Kala (1981), Dong, Guo, & Tian (2014), Güler (2020), Kala & Pordzik (2009), Morrell, Pearson & Brant (1997), Shao & Zhang (2015), Tian (2017b), Tian & Liu (2010), and Tian & Puntanen (2009). For studies on BLUPs and LMMs in the literature, see, e.g., Brown & Prescott (2006), Demidenko (2004), Güler & Büyükkaya (2019), Haslett & Puntanen (2011), Haslett, Puntanen & Arendacká (2015), Jiang (2007), Liu, Rong & Liu (2008), Liu & Wang (2013), Searle (1997), and Tian (2015). For more results on the Löwner partial ordering of real symmetric matrices and applications in statistical analysis and on rank of matrices, see, e.g., Puntanen, Styan & Isotalo (2011), Tian (2010), Tian (2017a), and Tian & Jiang (2016).

In the present paper, we use the following formulas for ranks of block matrices to establish the results. They are collected in the following lemma; see Marsaglia & Styan (1974).

Lemma 1.1. *Let* $\mathbf{A} \in \mathbb{R}^{m\times n}$, $\mathbf{B} \in \mathbb{R}^{m\times k}$, $\mathbf{C} \in \mathbb{R}^{l\times n}$, *and* $\mathbf{D} \in \mathbb{R}^{l\times k}$. *Then,*

$$\boldsymbol{r}\begin{bmatrix}\mathbf{A}, & \mathbf{B}\end{bmatrix} = \boldsymbol{r}(\mathbf{A}) + \boldsymbol{r}(\mathbf{E_A B}) = \boldsymbol{r}(\mathbf{B}) + \boldsymbol{r}(\mathbf{E_B A}), \tag{1.8}$$

$$\boldsymbol{r}\begin{bmatrix}\mathbf{A}\\ \mathbf{C}\end{bmatrix} = \boldsymbol{r}(\mathbf{A}) + \boldsymbol{r}(\mathbf{C F_A}) = \boldsymbol{r}(\mathbf{C}) + \boldsymbol{r}(\mathbf{A F_C}), \tag{1.9}$$

$$\boldsymbol{r}\begin{bmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{C} & \mathbf{D}\end{bmatrix} = \boldsymbol{r}(\mathbf{A}) + \boldsymbol{r}(\mathbf{D} - \mathbf{C}\mathbf{A}^{+}\mathbf{B}) \ \ \textit{if} \ \ \mathscr{C}(\mathbf{B}) \subseteq \mathscr{C}(\mathbf{A}), \mathscr{C}(\mathbf{C}') \subseteq \mathscr{C}(\mathbf{A}'). \tag{1.10}$$

We introduce the notation used in the paper. Let $\mathbb{R}^{m\times n}$ stand for the set of all $m\times n$ real matrices. $\mathbf{A}'$, $\boldsymbol{r}(\mathbf{A})$, $\mathscr{C}(\mathbf{A})$, and $\mathbf{A}^{+}$ denote the transpose, the rank, the column space, and the Moore–Penrose generalized inverse of $\mathbf{A}\in\mathbb{R}^{m\times n}$, respectively. $\mathbf{I}_m$ denotes the identity matrix of order m. $\mathbf{P}_{\mathbf{A}}=\mathbf{A}\mathbf{A}^{+}$, $\mathbf{E}_{\mathbf{A}}=\mathbf{A}^{\perp}=\mathbf{I}_m-\mathbf{A}\mathbf{A}^{+}$, $\mathbf{F}_{\mathbf{A}}=\mathbf{I}_n-\mathbf{A}^{+}\mathbf{A}$ stand for the orthogonal projectors.

2. BLUPs in LMMs

In this section, we review the predictability conditions of general linear function of all unknown vectors under the models $\mathcal{M}$ and $\mathcal{T}$, and also review the definition of BLUP. Then we give the fundamental BLUP equations and related properties under the considered models.

The predictability requirement of vector $\boldsymbol{\phi}$ under $\mathcal{M}$ is described as holding the following inclusion

$$\mathscr{C}(\mathbf{K}')\subseteq\mathscr{C}(\mathbf{X}'). \tag{2.1}$$

This requirement also corresponds to the estimability of vector $\mathbf{K}\boldsymbol{\beta}$ under $\mathcal{M}$; see, e.g., Alalouf & Styan (1979). For transformed model $\mathcal{T}$, the predictability requirement of vector $\boldsymbol{\phi}$ is

$$\mathscr{C}(\mathbf{K}')\subseteq\mathscr{C}(\mathbf{X}'\mathbf{T}'). \tag{2.2}$$

It is obvious that $\boldsymbol{\phi}$ is predictable under $\mathcal{M}$ if it is predictable under $\mathcal{T}$; see, Tian (2017b). Further, note that $\mathbf{X}\boldsymbol{\beta}$ is always estimable under $\mathcal{M}$ and the condition for estimability of vector $\mathbf{X}\boldsymbol{\beta}$ under two LMMs $\mathcal{M}$ and $\mathcal{T}$ is holding the equality $\boldsymbol{r}(\mathbf{X})=\boldsymbol{r}(\mathbf{T}\mathbf{X})$.

Let $\boldsymbol{\phi}$ predictable under $\mathcal{M}$. If there exists $\mathbf{L}\mathbf{y}$ such that

$$\mathrm{D}(\mathbf{L}\mathbf{y}-\boldsymbol{\phi})=\ \min\text{ subject to }\ \mathrm{E}(\mathbf{L}\mathbf{y}-\boldsymbol{\phi})=\mathbf{0} \tag{2.3}$$

holds in the Löwner partial ordering, the linear statistic $\mathbf{L}\mathbf{y}$ is defined to be the BLUP of $\boldsymbol{\phi}$ and is denoted by $\mathbf{L}\mathbf{y}=\mathrm{BLUP}_{\mathcal{M}}(\boldsymbol{\phi})=\mathrm{BLUP}_{\mathcal{M}}(\mathbf{K}\boldsymbol{\beta}+\mathbf{G}\mathbf{u}+\mathbf{H}\boldsymbol{\varepsilon})$, originated from Goldberger (1962). If $\mathbf{G}=\mathbf{0}$ and $\mathbf{H}=\mathbf{0}$ in $\boldsymbol{\phi}$, $\mathbf{L}\mathbf{y}$ corresponds the best linear unbiased estimator (BLUE) of $\mathbf{K}\boldsymbol{\beta}$, denoted by $\mathrm{BLUE}_{\mathcal{M}}(\mathbf{K}\boldsymbol{\beta})$, under $\mathcal{M}$.

To obtain some results of the BLUPs under the models $\mathcal{M}$ and $\mathcal{T}$, we need some fundamental facts on BLUPs under LMM. Concerning the matrix equations and the exact algebraic expressions of the BLUPs of $\boldsymbol{\phi}$, as well as properties of the BLUPs, we have the following comprehensive result; see Tian (2017b).

Lemma 2.1. *(Fundamental BLUP Equation) Let $\mathcal{T}$ be as given in* (1.2) *and let $\boldsymbol{\phi}$ in* (1.3) *be predictable under $\mathcal{T}$. In this case,*

$$\begin{aligned}&\mathrm{E}(\mathbf{L}_t\mathbf{y}-\boldsymbol{\phi})=\mathbf{0}\ \textit{and}\ \mathrm{cov}(\mathbf{L}_t\mathbf{y}-\boldsymbol{\phi})=\textit{min}\\&\quad\Leftrightarrow\mathbf{L}_t\left[\mathbf{T}\mathbf{X},\quad\mathbf{T}\mathbf{R}\mathbf{T}'(\mathbf{T}\mathbf{X})^{\perp}\right]=\left[\mathbf{K},\quad\mathbf{C}\mathbf{T}'(\mathbf{T}\mathbf{X})^{\perp}\right].\end{aligned} \tag{2.4}$$

The equation in (2.4)*, called the fundamental* BLUP *equation, is consistent and the general solution $\mathbf{L}_t$ of this equation and* $\mathrm{BLUP}_{\mathcal{T}}(\boldsymbol{\phi})$ *are given by*

$$\mathrm{BLUP}_{\mathcal{T}}(\boldsymbol{\phi})=\mathbf{L}_t\mathbf{T}\mathbf{y}=\left(\left[\mathbf{K},\quad\mathbf{C}\mathbf{T}'(\mathbf{T}\mathbf{X})^{\perp}\right]\mathbf{W}_t^{+}\mathbf{T}+\mathbf{U}_t\mathbf{W}_t^{\perp}\mathbf{T}\right)\mathbf{y}, \tag{2.5}$$

where $\mathbf{U}_t\in\mathbb{R}^{s\times m}$ is an arbitrary matrix and $\mathbf{W}_t=\left[\mathbf{T}\mathbf{X},\quad\mathbf{T}\mathbf{R}\mathbf{T}'(\mathbf{T}\mathbf{X})^{\perp}\right]$. In particular,

$$\mathbf{L}_t\ \textit{is unique}\ \Leftrightarrow\boldsymbol{r}\left[\mathbf{T}\mathbf{X},\quad\mathbf{T}\mathbf{R}\mathbf{T}'(\mathbf{T}\mathbf{X})^{\perp}\right]=m,$$

$$\mathrm{BLUP}_{\mathcal{T}}(\boldsymbol{\phi})\ \textit{is unique with probability 1}\ \Leftrightarrow\mathcal{T}\ \textit{is consistent},$$

$$\boldsymbol{r}\left[\mathbf{TX}, \quad \mathbf{TRT}'(\mathbf{TX})^{\perp}\right] = \boldsymbol{r}\left[\mathbf{TX}, \quad \mathbf{TRT}'\right] = \boldsymbol{r}\left[\mathbf{TX}, \quad (\mathbf{TX})^{\perp}\mathbf{TRT}'\right],$$
$$\mathscr{C}\left[\mathbf{TX}, \quad \mathbf{TRT}'(\mathbf{TX})^{\perp}\right] = \mathscr{C}\left[\mathbf{TX}, \quad \mathbf{TRT}'\right] = \mathscr{C}\left[\mathbf{TX}, \quad (\mathbf{TX})^{\perp}\mathbf{TRT}'\right].$$

The dispersion matrices of $\mathrm{BLUP}_{\mathcal{T}}(\boldsymbol{\phi})$ *and* $\boldsymbol{\phi} - \mathrm{BLUP}_{\mathcal{T}}(\boldsymbol{\phi})$ *are given as*

$$\mathrm{D}[\mathrm{BLUP}_{\mathcal{T}}(\boldsymbol{\phi})] = \left[\mathbf{K}, \quad \mathbf{CT}'(\mathbf{TX})^{\perp}\right]\mathbf{W}_t^{+}\mathbf{TRT}'\left(\left[\mathbf{K}, \quad \mathbf{CT}'(\mathbf{TX})^{\perp}\right]\mathbf{W}_t^{+}\right)', \tag{2.6}$$

$$\begin{aligned}&\mathrm{D}[\boldsymbol{\phi} - \mathrm{BLUP}_{\mathcal{T}}(\boldsymbol{\phi})]\\ &= \left(\left[\mathbf{K}, \quad \mathbf{CT}'(\mathbf{TX})^{\perp}\right]\mathbf{W}_t^{+}\mathbf{TB} - \mathbf{J}\right)\boldsymbol{\Sigma}\left(\left[\mathbf{K}, \quad \mathbf{CT}'(\mathbf{TX})^{\perp}\right]\mathbf{W}_t^{+}\mathbf{TB} - \mathbf{J}\right)'.\end{aligned} \tag{2.7}$$

Let $\boldsymbol{\phi}$ be predictable under $\mathcal{M}$. By setting $\mathbf{T} = \mathbf{I}_n$ in Lemma 2.1, we obtain the following well-known results on BLUP of $\boldsymbol{\phi}$ under $\mathcal{M}$.

$$\mathrm{BLUP}_{\mathcal{M}}(\boldsymbol{\phi}) = \mathbf{Ly} = \left(\left[\mathbf{K}, \quad \mathbf{CX}^{\perp}\right]\left[\mathbf{X}, \quad \mathbf{RX}^{\perp}\right]^{+} + \mathbf{U}\left[\mathbf{X}, \quad \mathbf{RX}^{\perp}\right]^{\perp}\right)\mathbf{y}, \tag{2.8}$$

$$\mathrm{D}[\mathrm{BLUP}_{\mathcal{M}}(\boldsymbol{\phi})] = \left[\mathbf{K}, \quad \mathbf{CX}^{\perp}\right]\mathbf{W}^{+}\mathbf{R}\left(\left[\mathbf{K}, \quad \mathbf{CX}^{\perp}\right]\mathbf{W}^{+}\right)', \tag{2.9}$$

$$\begin{aligned}&\mathrm{D}[\boldsymbol{\phi} - \mathrm{BLUP}_{\mathcal{M}}(\boldsymbol{\phi})]\\ &= \left(\left[\mathbf{K}, \quad \mathbf{CX}^{\perp}\right]\mathbf{W}^{+}\mathbf{B} - \mathbf{J}\right)\boldsymbol{\Sigma}\left(\left[\mathbf{K}, \quad \mathbf{CX}^{\perp}\right]\mathbf{W}^{+}\mathbf{B} - \mathbf{J}\right)',\end{aligned} \tag{2.10}$$

where $\mathbf{U} \in \mathbb{R}^{s \times n}$ is an arbitrary matrix and $\mathbf{W} = \left[\mathbf{X}, \quad \mathbf{RX}^{\perp}\right]$. In particular,

$$\mathbf{L} \text{ is unique } \Leftrightarrow \boldsymbol{r}\left[\mathbf{X}, \quad \mathbf{RX}^{\perp}\right] = n,$$

$$\mathrm{BLUP}_{\mathcal{M}}(\boldsymbol{\phi}) \text{ is unique with probability 1 } \Leftrightarrow \mathcal{M} \text{ is consistent },$$

$$\boldsymbol{r}\left[\mathbf{X}, \quad \mathbf{RX}^{\perp}\right] = \boldsymbol{r}\left[\mathbf{X}, \quad \mathbf{R}\right] = \boldsymbol{r}\left[\mathbf{X}, \quad \mathbf{X}^{\perp}\mathbf{R}\right],$$
$$\mathscr{C}\left[\mathbf{X}, \quad \mathbf{RX}^{\perp}\right] = \mathscr{C}\left[\mathbf{X}, \quad \mathbf{R}\right] = \mathscr{C}\left[\mathbf{X}, \quad \mathbf{X}^{\perp}\mathbf{R}\right].$$

We also note that the covariance between $\boldsymbol{\phi} - \mathrm{BLUP}_{\mathcal{M}}(\boldsymbol{\phi})$ and $\boldsymbol{\phi} - \mathrm{BLUP}_{\mathcal{T}}(\boldsymbol{\phi})$ is written as

$$\begin{aligned}&\mathrm{cov}[\boldsymbol{\phi} - \mathrm{BLUP}_{\mathcal{M}}(\boldsymbol{\phi}), \boldsymbol{\phi} - \mathrm{BLUP}_{\mathcal{T}}(\boldsymbol{\phi})]\\ &= \left(\left[\mathbf{K}, \quad \mathbf{CX}^{\perp}\right]\mathbf{W}^{+}\mathbf{B} - \mathbf{J}\right)\boldsymbol{\Sigma}\left(\left[\mathbf{K}, \quad \mathbf{CT}'(\mathbf{TX})^{\perp}\right]\mathbf{W}_t^{+}\mathbf{TB} - \mathbf{J}\right)'.\end{aligned} \tag{2.11}$$

3. Rank of covariance matrices between BLUPs in LMMs

In this section, we give the rank of covariance matrix between BLUPs of $\boldsymbol{\phi}$ under the models $\mathcal{M}$ and $\mathcal{T}$ by using block matrices' rank formulas and elementary matrix operations. Also we give some consequences for special cases.

Theorem 3.1. *Let consider the models* $\mathcal{M}$ *and* $\mathcal{T}$ *in* (1.1) *and* (1.2)*, respectively. Assume that the* $\boldsymbol{\phi}$ *in* (1.3) *is predictable under these models. Denote*

$$\mathbf{M} = \begin{bmatrix} \mathbf{RT}' & \mathbf{R} & \mathbf{X} & \mathbf{0} & \mathbf{C}' \\ \mathbf{TRT}' & \mathbf{0} & \mathbf{0} & \mathbf{TX} & \mathbf{TC}' \\ \mathbf{X}'\mathbf{T}' & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{K}' \\ \mathbf{0} & \mathbf{X}' & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{CT}' & \mathbf{C} & \mathbf{K} & \mathbf{0} & \mathbf{S} \end{bmatrix}. \tag{3.1}$$

Then,

$$\begin{aligned}&\boldsymbol{r}\left(\mathrm{cov}\left\{\boldsymbol{\phi} - \mathrm{BLUP}_{\mathcal{M}}(\boldsymbol{\phi}), \boldsymbol{\phi} - \mathrm{BLUP}_{\mathcal{T}}(\boldsymbol{\phi})\right\}\right)\\ &= \boldsymbol{r}(\mathbf{M}) - \boldsymbol{r}(\mathbf{X}) - \boldsymbol{r}(\mathbf{TX}) - \boldsymbol{r}\left[\mathbf{X}, \quad \mathbf{R}\right] - \boldsymbol{r}\left[\mathbf{TX}, \quad \mathbf{TRT}'\right].\end{aligned} \tag{3.2}$$

Proof. Applying (1.10) to (2.11), we obtain

$$
\begin{aligned}
&\boldsymbol{r}(\mathrm{cov}[\boldsymbol{\phi}-\mathrm{BLUP}_{\mathcal{M}}(\boldsymbol{\phi}), \boldsymbol{\phi}-\mathrm{BLUP}_{\mathcal{T}}(\boldsymbol{\phi})])\\
&= \boldsymbol{r}\left(\left(\begin{bmatrix}\mathbf{K}, & \mathbf{C}\mathbf{X}^{\perp}\end{bmatrix}\mathbf{W}^{+}\mathbf{B}-\mathbf{J}\right)\boldsymbol{\Sigma}\boldsymbol{\Sigma}^{+}\boldsymbol{\Sigma}\left(\begin{bmatrix}\mathbf{K}, & \mathbf{C}\mathbf{T}'(\mathbf{T}\mathbf{X})^{\perp}\end{bmatrix}\mathbf{W}_t^{+}\mathbf{T}\mathbf{B}-\mathbf{J}\right)'\right)\\
&= \boldsymbol{r}\begin{bmatrix}\boldsymbol{\Sigma} & \boldsymbol{\Sigma}\left(\begin{bmatrix}\mathbf{K}, & \mathbf{C}\mathbf{T}'(\mathbf{T}\mathbf{X})^{\perp}\end{bmatrix}\mathbf{W}_t^{+}\mathbf{T}\mathbf{B}-\mathbf{J}\right)'\\ \left(\begin{bmatrix}\mathbf{K}, & \mathbf{C}\mathbf{X}^{\perp}\end{bmatrix}\mathbf{W}^{+}\mathbf{B}-\mathbf{J}\right)\boldsymbol{\Sigma} & \mathbf{0}\end{bmatrix}\\
&\quad - \boldsymbol{r}(\boldsymbol{\Sigma})\\
&= \boldsymbol{r}\left(\begin{bmatrix}\boldsymbol{\Sigma} & -\boldsymbol{\Sigma}\mathbf{J}'\\ -\mathbf{J}\boldsymbol{\Sigma} & \mathbf{0}\end{bmatrix}+\begin{bmatrix}\boldsymbol{\Sigma}\mathbf{B}'\mathbf{T}' & \mathbf{0}\\ \mathbf{0} & \begin{bmatrix}\mathbf{K}, & \mathbf{C}\mathbf{X}^{\perp}\end{bmatrix}\end{bmatrix}\begin{bmatrix}\mathbf{0} & \mathbf{W}\\ \mathbf{W}_t' & \mathbf{0}\end{bmatrix}^{+}\right.\\
&\qquad\left.\times\begin{bmatrix}\mathbf{B}\boldsymbol{\Sigma} & \mathbf{0}\\ \mathbf{0} & \begin{bmatrix}\mathbf{K}, & \mathbf{C}\mathbf{T}'(\mathbf{T}\mathbf{X})^{\perp}\end{bmatrix}'\end{bmatrix}\right)-\boldsymbol{r}(\boldsymbol{\Sigma})
\end{aligned}
\tag{3.3}
$$

where $\mathbf{W}=\begin{bmatrix}\mathbf{X}, & \mathbf{R}\mathbf{X}^{\perp}\end{bmatrix}$ and $\mathbf{W}_t=\begin{bmatrix}\mathbf{T}\mathbf{X}, & \mathbf{T}\mathbf{R}\mathbf{T}'(\mathbf{T}\mathbf{X})^{\perp}\end{bmatrix}$. We can reapply (1.10) to last equality in (3.3) since $\mathscr{C}(\mathbf{T}\mathbf{B}\boldsymbol{\Sigma})=\mathscr{C}(\mathbf{T}\mathbf{R}\mathbf{T}')\subseteq\mathscr{C}(\mathbf{W}_t)$, $\mathscr{C}(\mathbf{B}\boldsymbol{\Sigma})=\mathscr{C}(\mathbf{R})\subseteq\mathscr{C}(\mathbf{W})$, $\mathscr{C}\left(\begin{bmatrix}\mathbf{K}, & \mathbf{C}\mathbf{T}'(\mathbf{T}\mathbf{X})^{\perp}\end{bmatrix}'\right)\subseteq\mathscr{C}(\mathbf{W}_t')$, and $\mathscr{C}\left(\begin{bmatrix}\mathbf{K}, & \mathbf{C}\mathbf{X}^{\perp}\end{bmatrix}'\right)\subseteq\mathscr{C}(\mathbf{W}')$ hold. Then, by simplifying Lemma 1.1, (3.3) is written as

$$
\begin{aligned}
&\boldsymbol{r}\begin{bmatrix}\mathbf{0} & -\mathbf{X} & -\mathbf{R}\mathbf{X}^{\perp} & \mathbf{B}\boldsymbol{\Sigma} & \mathbf{0}\\ -\mathbf{X}'\mathbf{T}' & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{K}'\\ -(\mathbf{T}\mathbf{X})^{\perp}\mathbf{T}\mathbf{R}\mathbf{T}' & \mathbf{0} & \mathbf{0} & \mathbf{0} & (\mathbf{T}\mathbf{X})^{\perp}\mathbf{T}\mathbf{C}'\\ \boldsymbol{\Sigma}\mathbf{B}'\mathbf{T}' & \mathbf{0} & \mathbf{0} & \boldsymbol{\Sigma} & -\boldsymbol{\Sigma}\mathbf{J}'\\ \mathbf{0} & \mathbf{K} & \mathbf{C}\mathbf{X}^{\perp} & -\mathbf{J}\boldsymbol{\Sigma} & \mathbf{0}\end{bmatrix}-\boldsymbol{r}\begin{bmatrix}\mathbf{T}\mathbf{X}, & \mathbf{T}\mathbf{R}\mathbf{T}'(\mathbf{T}\mathbf{X})^{\perp}\end{bmatrix}\\
&\quad-\boldsymbol{r}\begin{bmatrix}\mathbf{X}, & \mathbf{R}\mathbf{X}^{\perp}\end{bmatrix}-\boldsymbol{r}(\boldsymbol{\Sigma})\\
&=\boldsymbol{r}\begin{bmatrix}-\mathbf{R}\mathbf{T}' & -\mathbf{X} & -\mathbf{R}\mathbf{X}^{\perp} & \mathbf{C}'\\ -\mathbf{X}'\mathbf{T}' & \mathbf{0} & \mathbf{0} & \mathbf{K}'\\ -(\mathbf{T}\mathbf{X})^{\perp}\mathbf{T}\mathbf{R}\mathbf{T}' & \mathbf{0} & \mathbf{0} & (\mathbf{T}\mathbf{X})^{\perp}\mathbf{T}\mathbf{C}'\\ \mathbf{C}\mathbf{T}' & \mathbf{K} & \mathbf{C}\mathbf{X}^{\perp} & -\mathbf{S}\end{bmatrix}-\boldsymbol{r}\begin{bmatrix}\mathbf{T}\mathbf{X}, & \mathbf{T}\mathbf{R}\mathbf{T}'\end{bmatrix}\\
&\quad-\boldsymbol{r}\begin{bmatrix}\mathbf{X}, & \mathbf{R}\end{bmatrix}\\
&=\boldsymbol{r}\begin{bmatrix}-\mathbf{R}\mathbf{T}' & -\mathbf{X} & -\mathbf{R} & \mathbf{C}' & \mathbf{0}\\ -\mathbf{X}'\mathbf{T}' & \mathbf{0} & \mathbf{0} & \mathbf{K}' & \mathbf{0}\\ -\mathbf{T}\mathbf{R}\mathbf{T}' & \mathbf{0} & \mathbf{0} & \mathbf{T}\mathbf{C}' & \mathbf{T}\mathbf{X}\\ \mathbf{C}\mathbf{T}' & \mathbf{K} & \mathbf{C} & -\mathbf{S} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{X}' & \mathbf{0} & \mathbf{0}\end{bmatrix}-\boldsymbol{r}\begin{bmatrix}\mathbf{T}\mathbf{X}, & \mathbf{T}\mathbf{R}\mathbf{T}'\end{bmatrix}-\boldsymbol{r}\begin{bmatrix}\mathbf{X}, & \mathbf{R}\end{bmatrix}\\
&\quad-\boldsymbol{r}(\mathbf{T}\mathbf{X})-\boldsymbol{r}(\mathbf{X})\\
&=\boldsymbol{r}\begin{bmatrix}\mathbf{R}\mathbf{T}' & \mathbf{R} & \mathbf{X} & \mathbf{0} & \mathbf{C}'\\ \mathbf{T}\mathbf{R}\mathbf{T}' & \mathbf{0} & \mathbf{0} & \mathbf{T}\mathbf{X} & \mathbf{T}\mathbf{C}'\\ \mathbf{X}'\mathbf{T}' & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{K}'\\ \mathbf{0} & \mathbf{X}' & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{C}\mathbf{T}' & \mathbf{C} & \mathbf{K} & \mathbf{0} & \mathbf{S}\end{bmatrix}-\boldsymbol{r}\begin{bmatrix}\mathbf{T}\mathbf{X}, & \mathbf{T}\mathbf{R}\mathbf{T}'\end{bmatrix}-\boldsymbol{r}\begin{bmatrix}\mathbf{X}, & \mathbf{R}\end{bmatrix}\\
&\quad-\boldsymbol{r}(\mathbf{T}\mathbf{X})-\boldsymbol{r}(\mathbf{X}).
\end{aligned}
\tag{3.4}
$$

The required result is seen from (3.4). □

Corollary 3.1. *Let consider the models $\mathcal{M}$ and $\mathcal{T}$ in* (1.1) *and* (1.2)*, respectively. Assume that $\mathbf{K}\boldsymbol{\beta}$ is estimable under these models. Then*

$$\begin{aligned}&\operatorname{cov}[\,\mathrm{BLUE}_{\mathcal{M}}(\mathbf{K}\boldsymbol{\beta}),\mathrm{BLUE}_{\mathcal{T}}(\mathbf{K}\boldsymbol{\beta})]\\ &= \boldsymbol{r}\begin{bmatrix}\mathbf{RT}' & \mathbf{R} & \mathbf{X} & \mathbf{0} & \mathbf{0}\\ \mathbf{TRT}' & \mathbf{0} & \mathbf{0} & \mathbf{TX} & \mathbf{0}\\ \mathbf{X}'\mathbf{T}' & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{K}'\\ \mathbf{0} & \mathbf{X}' & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{K} & \mathbf{0} & \mathbf{0}\end{bmatrix} - \boldsymbol{r}\begin{bmatrix}\mathbf{TX}, & \mathbf{TRT}'\end{bmatrix} - \boldsymbol{r}\begin{bmatrix}\mathbf{X}, & \mathbf{R}\end{bmatrix}\\ &\quad - \boldsymbol{r}(\mathbf{TX}) - \boldsymbol{r}(\mathbf{X}).\end{aligned} \tag{3.5}$$

If $\mathbf{X}\boldsymbol{\beta}$ is estimable under the models $\mathcal{M}$ and $\mathcal{T}$, then

$$\begin{aligned}&\boldsymbol{r}(\operatorname{cov}[\mathrm{BLUE}_{\mathcal{M}}(\mathbf{X}\boldsymbol{\beta}),\mathrm{BLUE}_{\mathcal{T}}(\mathbf{X}\boldsymbol{\beta})])\\ &= \boldsymbol{r}\begin{bmatrix}\mathbf{RT}' & \mathbf{R} & \mathbf{0}\\ \mathbf{TRT}' & \mathbf{0} & \mathbf{TX}\\ \mathbf{0} & \mathbf{X}' & \mathbf{0}\end{bmatrix} - \boldsymbol{r}\begin{bmatrix}\mathbf{TX}, & \mathbf{TRT}'\end{bmatrix} - \boldsymbol{r}\begin{bmatrix}\mathbf{X}, & \mathbf{R}\end{bmatrix}.\end{aligned} \tag{3.6}$$

4. Conclusion

In this study, we consider comparison problems of predictors under a LMM $\mathcal{M}$ and its transformed model $\mathcal{T}$. We present rank relations between BLUPs of unknown vectors under considered models by using various rank formulas of block matrices and elementary matrix operations. In order to establish the general results on the predictors, we consider the general linear function of all unknown vectors. Besides, results for special cases are also given.

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Γ-SEMIGROUPS REGARDED AS SEMIGROUPS UNDER Γ

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Abstract. There is a striking similarity between Γ-semigroups on the one hand, and semigroups on the other one. In this paper we express this similarity using the language of the category theory. To this end we consider two categories. The category Γ-**Sgrp** of Γ-semigroups and Γ-semigroup morphisms, and the category $\Gamma \downarrow$ **Sgrp** of semigroups under a given semigroup $(\Gamma, \bullet)$, and prove that there are functors Ψ : Γ-**Sgrp** $\to \Gamma \downarrow$ **Sgrp** and Ψ' : $\Gamma \downarrow$ **Sgrp** $\to$ Γ-**Sgrp** such that Ψ is a left adjoint of Ψ'.

1. Introduction and Preliminaries

Let S and Γ be two non empty sets. Every map from $S\times\Gamma\times S$ to S will be called a Γ-multiplication in S and is denoted by $(\cdot)_\Gamma$. The result of this multiplication for $a, b \in S$ and $\gamma \in \Gamma$ is denoted by $a\gamma b$. According to Sen and Saha [7], a Γ-semigroup S is an ordered pair $(S, (\cdot)_\Gamma)$ where S and Γ are non empty sets and $(\cdot)_\Gamma$ is a Γ-multiplication on S which satisfies the following property

$$\forall(a, b, c, \alpha, \beta) \in S^3 \times \Gamma^2, (a\alpha b)\beta c = a\alpha(b\beta c).$$

Let $(\Gamma, \bullet)$ be any semigroup. The category $\Gamma \downarrow$ **Sgrp** of semigroups under Γ has objects all pairs (j, T) with T a semigroup and $j : \Gamma \to T$ a homomorphism of semigroups. If (j_1, T_1) and (j_2, T_2) are two such objects, a morphism $h : (j_1, T_1) \to (j_2, T_2)$ is a homomorphism $h : T_1 \to T_2$ of semigroups such that $hj_1 = j_2$.

Recall that the category of Γ-semigroups Γ-**Sgrp** has objects all Γ-semigroups and morphisms all homomorphisms between them. In this paper we will define two functors Ψ : Γ-**Sgrp** $\to \Gamma \downarrow$ **Sgrp** dhe Ψ' : $\Gamma \downarrow$ **Sgrp** $\to$ Γ-**Sgrp** for which we will prove that are adjoints of each other.

2. The enveloping semigroup $\Lambda(S, \gamma_0)$

For any nonempty set Γ we let Γ^+ be the free semigroup on Γ. Every congruence on Γ^+ which has the set Γ as a cross-section defines a multiplication $\bullet$ on Γ in such a way that $(\Gamma, \bullet)$ becomes a semigroup. We mention here by passing that such congruences really exist. For example the congruence generated by all pairs

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$((\alpha, \beta), \alpha)$ where α and β vary in Γ. In this case a congruence class contains all words beginning with the same letter α.

Let now σ be a congruence having Γ as a cross-section. Define in Γ the multiplication

$$\alpha \bullet \beta = \gamma$$

where $\gamma \in \Gamma$ is (the unique letter) such that $\overline{\gamma}^{\sigma} = \overline{\alpha\beta}^{\sigma}$. Define now the onto map

$$\nu : \Gamma^{+} \to \Gamma$$

such that

$$u \mapsto \gamma$$

with $\overline{\gamma}^{\sigma} = \overline{u}^{\sigma}$. This map is a homomorphis since if $u, v \in \Gamma^{+}$ are such that $\overline{\alpha}^{\sigma} = \overline{u}^{\sigma}$ and $\overline{\beta}^{\sigma} = \overline{v}^{\sigma}$, then

$$\begin{aligned} \nu(uv) &= \nu(\alpha\beta) && (\text{since } \overline{\alpha\beta}^{\sigma} = \overline{uv}^{\sigma}) \\ &= \gamma && (\text{if } \overline{\gamma}^{\sigma} = \overline{\alpha\beta}^{\sigma}) \\ &= \alpha \bullet \beta && (\text{from the definition of } \bullet) \\ &= \nu(u) \bullet \nu(v). \end{aligned}$$

It follows that $(\Gamma, \bullet)$ is a semigroup.

Let (S, Γ) be a Γ-semigroup and $\gamma_0 \in \Gamma$ a fixed element. As we explained above, there is an associative multiplication $\bullet$ in Γ and so the pair $(S, \bullet)$ is a semigroup. In a similar way with [5] we can define a semigroup $\Lambda(S, \gamma_0)$ out of (S, Γ). We quotient the free semigroup $(S \cup \Gamma)^{+}$ on $S \cup \Gamma$ by the congruence generated by all pairs

$$((x, \alpha, y), x\alpha y),$$

$$((x, y), x\gamma_0 y)$$

and

$$((\alpha, \beta), \alpha \bullet \beta).$$

The result will be a semigroup which is denoted here by $\Lambda(S, \gamma_0)$. We prove the following which is the analogue of Lemma 2.4 of [5].

Lemma 2.1. *Every element of $\Lambda(S, \gamma_0)$ is represented by an irreducible word which has one of the following five forms (γ, x, γ'), (γ, x), (x, γ), γ or x with $x \in S$ and $\gamma, \gamma' \in \Gamma$.*

Proof. To prove the lemma, we must prove first that the reduction system arsing from the presentation of $\Lambda(S, \gamma_0)$ is Noetherian and confluent, which would imply that any element of Σ_{γ_0} is given by an irreducible word from $(S \cup \Gamma)^{+}$. Secondly, we must prove that the irreducible words have the required forms.
If a word w of $(S \cup \Gamma)^{+}$ has the form $(u, \gamma_1, \gamma_2, v)$ with $\gamma_1, \gamma_2 \in \Gamma$, and u, v are words from $(S \cup \Gamma)^{+}$, then w reduces to $w' = (u, \gamma_1 \bullet \gamma_2, v)$. If for some $x, y \in S$ and $\gamma \in \Gamma$, the word w contains a subword of the form (x, γ, y), which means that $w = (u, x, \gamma, y, v)$ where u, v are words from $(S \cup \Gamma)^{+}$ or are empty words, then it reduces to $w' = (u, x\gamma y, v)$. Lastly, if the word w contains two adjacent letters from S, which means that $w = (u, x, y, v)$ where u and v are as before and $x, y \in S$, then

it reduces to $w' = (u, x\gamma_0 y, v)$. In this way we have obtained a reduction system consisting in three reductions:

$$\begin{array}{rcl} (u,\gamma_1,\gamma_2,v) & \to & (u,\gamma_1 \bullet \gamma_2,v) \\ (u,x,\gamma,y,v) & \to & (u,x\gamma y,v) \\ (u,x,y,v) & \to & (u,x\gamma_0 y,v) \end{array}$$

which is obviously Noetherian since it is length reducing. To prove that it is confluent, from the Newman lemma, it is sufficient to prove that it is locally confluent. As this system does not contain inclusion ambiguities, it is enough to check all overlapping pairs. There are only five such pairs:
1- $(x,y,\gamma,z) \to (x\gamma_0 y,\gamma,z)$ and $(x,y,\gamma,z) \to (x,y\gamma z)$. Both resolve to $(x\gamma_0 y\gamma z)$.
2- $(x,\gamma,y,z) \to (x,\gamma,y\gamma_0 z)$ and $(x,\gamma,y,z) \to (x\gamma y,z)$ which resolve to $(x\gamma y\gamma_0 z)$.
3- $(x,\gamma,y,\gamma',z) \to (x,\gamma,y\gamma' z)$ and $(x,\gamma,y,\gamma',z) \to (x\gamma y,\gamma',z)$ which resolve to $(x\gamma y\gamma' z)$.
4- $(x,y,z) \to (x\gamma_0 y,z)$ and $(x,y,z) \to (x,y\gamma_0 z)$, which resolve to $(x\gamma_0 y\gamma_0 z)$.
5- $(\gamma_1,\gamma_2,\gamma_3) \to (\gamma_1 \bullet \gamma_2,\gamma_3)$ and $(\gamma_1,\gamma_2,\gamma_3) \to (\gamma_1,\gamma_2 \bullet \gamma_3)$ which resolve to $(\gamma_1 \bullet \gamma_2 \bullet \gamma_3)$.

To conclude the proof we must show that the irreducible word that represents an element of Σ_{γ_0} has the required form. First observe that any word which has neither a prefix nor a suffix made of letters from Γ, reduces to an element of S by performing the three type of the above reductions. Otherwise, if the word is (η, U, η') where η, η' are words from the free monoid on Γ and U has neither a prefix nor a suffix made of letters from Γ, then we reduce η and η' in a single letter from Γ by performing reductions of the first type, and then reduce U into a single letter from S. □

Lemma 2.2. *The semigroup* $(\Gamma, \bullet)$ *embeds into* $\Lambda(S, \gamma_0)$.

Proof. The map $\chi : \Gamma \to \Lambda(S,\gamma_0)$ defined by $\gamma \mapsto \mu(\gamma)$ where $\mu : (S \cup \Gamma)^+ \to \Lambda(S,\gamma_0)$ is the canonical epimorphism, is a homomorphism, and injective as well. □

3. The adjunction

From now and on we will assume that the set Γ which appears in Γ-**Sgrp** i equipped with a multiplication $\bullet$ such that $(\Gamma, \bullet)$ is a semigroup. Along with Γ-**Sgrp** we consider the category $\Gamma \downarrow$ **Sgrp** which we defined in the previous paragraph. The following lemma gives a recipe to associate each object of $\Gamma \downarrow$ **Sgrp** a certain Γ-semigroup. This will be used in the proof of our main theorem.

Lemma 3.1. *For every* $(j, T) \in \Gamma \downarrow$ **Sgrp** *we can give the semigroup* T *the structure of a* Γ*-semigroup* (T, Γ).

Proof. Define the map

$$\cdot : T \times \Gamma \times T \to T$$

such that

$$(x, \gamma, y) \mapsto xj(\gamma)y$$

where the multiplication on the right hand side is that of T. For every $x, y, z \in T$ and $\alpha, \beta \in \Gamma$ we see that

$$\begin{aligned} x\alpha(y\beta z) &= xj(\alpha)(yj(\beta)z) \\ &= (xj(\alpha)y)j(\beta)z \\ &= (x\alpha y)\beta z, \end{aligned}$$

which shows that the pair (T, Γ) is a Γ-semigroup. □

Theorem 3.2. *There is an adjunction between the two categories* Γ-**Sgrp** *and* $\Gamma \downarrow$ **Sgrp**.

Proof. First we define a functor $\Psi : \Gamma\text{-}\mathbf{Sgrp} \to \Gamma \downarrow \mathbf{Sgrp}$. On objects, Ψ is defined by assigning to each Γ-semigroup (S, Γ) the pair $(j, \Lambda(S, \gamma_0))$ where $\Lambda(S, \gamma_0)$ is the enveloping semigroup of (S, Γ) and $j : \Gamma \to \Lambda(S, \gamma_0)$ is the monomorphism that $\gamma \mapsto \mu(\gamma)$. To define Ψ on morphisms, it is enough to show that if (S, Γ) and (S', Γ) are two Γ-semigroups, then for every morphism of Γ-semigroups $\varphi : S \to S'$, there is a unique homomorphism of semigroups $\phi : \Lambda(S, \gamma_0) \to \Lambda(S', \gamma_0)$ such that $\phi j_1 = j_2$ which has the additional property that $\phi\mu = \mu'\varphi$. Here μ and μ' are the canonical epimorphisms $\mu : (S \cup \Gamma)^+ \to \Lambda(S, \gamma_0)$ and $\mu : (S' \cup \Gamma)^+ \to \Lambda(S', \gamma_0)$. Indeed, let us write by $f : (S \cup \Gamma)^+ \to (S' \cup \Gamma)^+$ the homomorphism of free semigroups induced by the extension of φ on $S \cup \Gamma$ which fixes the elements of Γ. We prove that f induces a homomorphism $\phi : \Lambda(S, \gamma_0) \to \Lambda(S', \gamma_0)$, which amounts to saying that the defining relations of $\Lambda(S, \gamma_0)$ belong to the kernel of $\mu' f$. Indeed, for the relations of the first kind $((\gamma_1, \gamma_2), \gamma_1 \bullet \gamma_2)$ we have that

$$\begin{aligned} \mu' f(\gamma_1, \gamma_2) &= \mu'(\varphi(\gamma_1), \varphi(\gamma_2)) \\ &= \gamma_1 \bullet \gamma_2 \\ &= \mu' f(\gamma_1 \bullet \gamma_2). \end{aligned}$$

For the relations of the second kind $((x, \gamma, y), x\gamma y)$ we have

$$\begin{aligned} \mu' f(x, \gamma, y) &= \mu'(\varphi(x), \gamma, \varphi(y)) \\ &= \varphi(x)\gamma\varphi(y) \\ &= \varphi(x\gamma y) \\ &= \mu' f(x\gamma y), \end{aligned}$$

and for those of the third kind $((x, y), x1y)$ we have that

$$\begin{aligned} \mu' f(x, y) &= \mu'(\varphi(x), \varphi(y)) \\ &= \varphi(x)1\varphi(y) \\ &= \varphi(x1y) \\ &= \mu' f(x1y). \end{aligned}$$

Thus $\mu' f$ induces a $\phi : \Lambda(S, \gamma_0) \to \Lambda(S', \gamma_0)$ with the property that $\phi\mu = \mu' f$. Since φ is the restriction of f on $S \cup \Gamma$, then we get that $\phi\mu = \mu'\varphi$. The uniqueness of ϕ with the given property follows from the fact that any other homomorphism $\hat{\phi} : \Lambda(S, \gamma_0) \to \Lambda(S', \gamma_0)$ which satisfies the equality $\hat{\phi}\mu = \mu'\varphi$ coincides with ϕ on the generators of $\Lambda(S, \gamma_0)$ and as a result coincides with ϕ. Also ϕ satisfies the condition $\phi j = j'$ since for every $\gamma \in \Gamma$ we have that

$$(\phi j)(\gamma) = \phi(cls(\gamma)) = cls(\gamma) = j'(\gamma).$$

So we have obtained a morphism $\phi : (j, \Lambda(S, \gamma_0)) \to (j', \Lambda(S', \gamma_0))$ in the category $\Gamma \downarrow \mathbf{Sgrp}$, which is more convenient to write as $\Psi(f)$.

Now we prove the functorial properties of Ψ. It is straightforward that $\Psi(id_{(S,\Gamma)}) = id_{\Lambda(S,\gamma_0)}$. To check the covariance we see that if

$$\varphi : (S, \Gamma) \to (S', \Gamma) \text{ and } \varphi' : (S', \Gamma) \to (S'', \Gamma)$$

are morphisms in Γ-**Sgrp**, then

$$\Psi(\varphi'\varphi) : \Lambda(S, \gamma_0) \to \Lambda(S'', \gamma_0)$$

coincides with $\Psi(\varphi') \circ \Psi(\varphi)$ since both agree on generators. Indeed, if f and f' are morphisms induced by φ and φ' respectively, then for every $x \in S$ we have that,

$$\begin{aligned}\Psi(\varphi'\varphi)(\mu(x)) &= (\mu''(f'f))(x)\\ &= (\Psi(\varphi')\mu')(f(x))\\ &= (\Psi(\varphi')\Psi(\varphi))(\mu(x)).\end{aligned}$$

In a similar way we have that for all $\gamma \in \Gamma$,

$$\Psi(\varphi'\varphi)(\mu(\gamma)) = (\Psi(\varphi')\Psi(\varphi))(\mu(\gamma)).$$

Next we define another functor $\Psi' : \Gamma \downarrow \mathbf{Sgrp} \to \Gamma\text{-}\mathbf{Sgrp}$ on objects by assigning to each object (j, S) of $\Gamma \downarrow \mathbf{Sgrp}$ the Γ-semigroup (S, Γ) of lemma 3.1, and to each morphism $f : (j, S) \to (j', S')$, the map

$$\Psi'(f) : (S, \Gamma) \to (S', \Gamma)$$

such that for every $x \in S$,

$$\Psi'(f)(x) = f(x).$$

This map is a homomorphism of Γ-semigroups since for every $x, y \in S$ and $\gamma \in \Gamma$ we have that

$$\begin{aligned}\Psi'(f)(x\gamma y) &= f(x\gamma y)\\ &= f(x)f(\gamma)f(y)\\ &= f(x)\gamma f(y)\\ &= \Psi'(f)(x)\gamma\Psi'(f)(y).\end{aligned}$$

The functorial properties of Ψ' are easy to prove and are omitted here.

In the second part of the proof we show that Ψ is a left adjoint of Ψ'. Let $(S, \Gamma) \in \Gamma$-**Sgrp** and $(j, M) \in \Gamma \downarrow \mathbf{Sgrp}$ be arbitrary objects. Define

$$\xi_{(S,\Gamma),M} : \Gamma\text{-}\mathbf{Sgrp}((S, \Gamma), \Psi'(M)) \to \Gamma \downarrow \mathbf{Sgrp}(\Psi(S, \Gamma), M),$$

such that

$$h \mapsto h_*$$

where for the two types of generators, $x \in S$ and $\gamma \in \Gamma$, h_* is defined by

$$h_*(x) = h(x) \text{ and } h_*(\gamma) = j(\gamma).$$

To see that h_* is a homomorphism, we need to check that it is compatible with the defining relations of $\Psi(S, \Gamma) = \Lambda(S, \gamma_0)$. Indeed, for the relations $(x, \gamma, y) \sim (x\gamma y)$

we see that

$$\begin{aligned} h_*(x\gamma y) &= h(x\gamma y) \\ &= h(x)\gamma h(y) \\ &= h(x)j(\gamma)h(y) \\ &= h_*(x)h_*(\gamma)h_*(y). \end{aligned}$$

For the relations $(x, y) \sim (x\gamma_0 y)$ we have that

$$\begin{aligned} h_*(xy) &= h(x\gamma_0 y) \\ &= h(x)\gamma_0 h(y) \\ &= h(x)j(\gamma_0)h(y) \\ &= h_*(x)h_*(\gamma_0)h_*(y). \end{aligned}$$

For the relations of the last type $(\alpha, \beta) \sim (\alpha \bullet \beta)$ we have

$$\begin{aligned} h_*(\alpha, \beta) &= j(\alpha)j(\beta) \\ &= j(\alpha \bullet \beta) \\ &= h_*(\alpha \bullet \beta). \end{aligned}$$

Also we observe that the map $\xi_{(S,\Gamma),M}$ is a bijection. It is an injection since if $h, g : (S, \Gamma) \to (M, \Gamma)$ are such that $h_* = g_*$, then for every $x \in S$, we have

$$h(x) = h_*(x) = g_*(x) = g(x).$$

The map is a surjection as well since for every homomorphism of semigroups $g : \Lambda(S, \gamma_0) \to M$ such that $g(\mu(\gamma)) = j(\gamma)$ for every $\gamma \in \Gamma$, we can define $h : S \to M$ such that $x \mapsto g(x)$. This map is a Γ-homomorphism since for every $x, y \in S$ and $\gamma \in \Gamma$ we have that

$$\begin{aligned} h(x\gamma y) &= g(x\gamma y) \\ &= g(x)g(\gamma)g(y) \\ &= g(x)j(\gamma)g(y) \\ &= h(x)j(\gamma)h(y) \\ &= h(x)\gamma h(y). \end{aligned}$$

We prove now that $\xi_{(S,\Gamma),M}$ is natural in both variables. To prove the naturality in the first variable, we must prove that for every Γ-homomorphism $f : (S, \Gamma) \to (S', \Gamma)$, the following diagram commutes.

$$\begin{array}{ccc} \Gamma\text{-}\mathbf{Sgrp}((S,\Gamma), \Psi'(M)) & \xrightarrow{\xi_{(S,\Gamma),M}} & \Gamma \downarrow \mathbf{Sgrp}(\Lambda(S,\gamma_0), M) \\ \Big\uparrow {\scriptstyle \Gamma\text{-}\mathbf{Sgrp}(f,\Psi'(M))} & & \Big\uparrow {\scriptstyle \Gamma\text{-}\mathbf{Sgrp}(\Psi(f),M)} \\ \Gamma\text{-}\mathbf{Sgrp}((S',\Gamma), \Psi'(M)) & \xrightarrow[\xi_{(S',\Gamma),M}]{} & \Gamma \downarrow \mathbf{Sgrp}(\Lambda(S',\gamma_0), M) \end{array}$$

This means that for every $h' : (S', \Gamma) \to (M, \Gamma)$ we have to show that $(h'f)_* = h'_*\Psi(f)$. It is sufficient to see that $(h'f)_*$ and $h'_*\Psi(f)$ agree on generators. This is true since for every $x \in S$ we have that

$$(h'f)_*(x) = (h'f)(x) = h'_*(f(x)) = h'_*(\Psi(f)(x)) = (h'_*\Psi(f))(x).$$

Also for all $\gamma \in \Gamma$ we have that

$$(h'f)_*(\gamma) = j(\gamma) = h'_*(\gamma) = h'_*(\Psi(f)(\gamma)) = (h'_*\Psi(f))(\gamma).$$

Finally, we check the naturality in the second variable. Let $\chi : M \to N$ be a homomorphism such that $\chi j_M = j_N$ where $j_M : \Gamma \to M$ and $j_N : \Gamma \to N$. We have to prove that the following diagram commutes too

$$\begin{array}{ccc}
\Gamma\text{-}\mathbf{Sgrp}((S,\Gamma), \Psi'(M)) & \xrightarrow{\xi_{(S,\Gamma),M}} & \Gamma \downarrow \mathbf{Sgrp}(\Lambda(S,\gamma_0), M) \\
\Big\downarrow {\scriptstyle \Gamma\text{-}\mathbf{Sgrp}((S,\Gamma),\Psi'(\chi))} & & \Big\downarrow {\scriptstyle \Gamma\text{-}\mathbf{Sgrp}(\Lambda(S,\gamma_0),\chi)} \\
\Gamma\text{-}\mathbf{Sgrp}((S,\Gamma), \Psi'(N)) & \xrightarrow{\xi_{(S,\Gamma),N}} & \Gamma \downarrow \mathbf{Sgrp}(\Lambda(S,\gamma_0), N)
\end{array}$$

which means that $\chi h_* = (\Psi'(\chi)h)_*$. As before, it is enough to prove that both morphisms agree on the generators of $\Lambda(S, \gamma_0)$. For every $x \in S$ we see that

$$\begin{aligned}
(\Psi'(\chi)h)_*(x) &= (\Psi'(\chi)h)(x) \\
&= (\chi h)(x) \\
&= (\chi h_*)(x).
\end{aligned}$$

Also for every $\gamma \in \Gamma$ we have that

$$\begin{aligned}
(\Psi'(\chi)h)_*(\gamma) &= \Psi'(\chi)(h_*(\gamma)) \\
&= \Psi'(\chi)(\gamma) \\
&= \gamma \\
&= (\chi h_*)(\gamma).
\end{aligned}$$

This conclude the proof. □

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CATEGORICAL ASPECTS OF Γ-SEMIGROUPS

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Abstract. It is well known that any nonempty set Γ can be equipped with a multiplication $\bullet$ such that $(\Gamma, \bullet)$ is a group. Related to $(\Gamma, \bullet)$, we consider two categories. The first one is the category Γ-**Sgrp** of Γ-semigroups and their homomorphisms, and the second one is the category $\mathbf{Mon}(\Gamma)$ of monoids having the same group of units $(\Gamma, \bullet)$ and with morphisms those monoid homomorphisms which fix Γ. Then we define a functor $\Psi : \Gamma\text{-}\mathbf{Sgrp} \to \mathbf{Mon}(\Gamma)$ which maps each (S, Γ) to its enveloping monoid $\Omega_1(S, \Gamma)$, and another functor $\Psi' : \mathbf{Mon}(\Gamma) \to \Gamma\text{-}\mathbf{Sgrp}$ which maps each monoid M to its Γ semigroup of units (M, Γ), and prove that Ψ is a left adjoint of Ψ'.

1. Introduction and preliminaries

Let S and Γ be two non empty sets. Every map from $S \times \Gamma \times S$ to S will be called a Γ-multiplication in S and is denoted by $(\cdot)_\Gamma$. The result of this multiplication for $a, b \in S$ and $\gamma \in \Gamma$ is denoted by $a\gamma b$. According to Sen and Saha [8], a Γ-semigroup S is an ordered pair $(S, (\cdot)_\Gamma)$ where S and Γ are non empty sets and $(\cdot)_\Gamma$ is a Γ-multiplication on S which satisfies the following property

$$\forall (a, b, c, \alpha, \beta) \in S^3 \times \Gamma^2, (a\alpha b)\beta c = a\alpha(b\beta c).$$

Given a Γ-semigroup S, we have defined in [2] a monoid $\Omega_{\gamma_0}(S, \Gamma)$ where γ_0 is a fixed element of Γ. The construction is very similar to that of [1] except for the fact that here we have a unit element. The definition of $\Omega_{\gamma_0}(S, \Gamma)$ is based on a result from [3] which states that we can always define a multiplication $\bullet$ on any non empty set Γ in such a way that $(\Gamma, \bullet)$ becomes a group. Letting the unit element of $(\Gamma, \bullet)$ be γ_0 we define $\Omega_{\gamma_0}(S, \Gamma)$ in the following way. We first let $(F, \cdot)$ be the free semigroup on S whose elements are finite strings $(x_1, ..., x_n)$ where each $x_i \in S$ and the product $\cdot$ is the concatenation of words. Then we define $\Omega_{\gamma_0}(S, \Gamma)$ as the quotient semigroup of the free product $F * \Gamma$ of two semigroups $(F, \cdot)$ with $(\Gamma, \bullet)$ by the congruence generated from the set of relations

$$((x, y), x\gamma_0 y), ((x, \gamma, y), x\gamma y), (\gamma_0 x, x), (x\gamma_0, x),$$

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for all $x, y \in S, \gamma \in \Gamma$ and with $\gamma_0 \in \Gamma$ being the unit element of $(\Gamma, \bullet)$. We can also regard the group $(\Gamma, \bullet)$ as given by a presentation with generators the elements of Γ, and relations arising from the multiplication table of the group. So a presentation of $\Omega_{\gamma_0}(S, \Gamma)$ has now as a generating set $S \cup \Gamma$, and relations those mentioned above together with those arising from the multiplication table of $(\Gamma, \bullet)$. It is obvious that $\Omega_{\gamma_0}(S, \Gamma)$ becomes thus a monoid with unit element the class of γ_0 modulo the defining relations. We recall from [2] the following useful results.

Lemma 1.1. *Every element of $\Omega_{\gamma_0}(S, \Gamma)$ can be represented by an irreducible word which has the form $(\alpha, x, \beta), (\alpha, x), (x, \beta), \gamma$ or x where $x \in S$ and $\alpha, \beta \in \Gamma \setminus \{\gamma_0\}$ and $\gamma \in \Gamma$.*

The map
$$\iota : S \to \Omega_{\gamma_0}(S, \Gamma) \text{ such that } x \mapsto cls(x)$$
where $cls(x)$ is the class of $x \in F$ modulo the defining relations of $\Omega_{\gamma_0}(S, \Gamma)$ is proved to be an injection and this allows us to regard S as a subset of $\Omega_{\gamma_0}(S, \Gamma)$. There is also a monomorphism of monoids
$$j : \Gamma \to \Omega_{\gamma_0}(S, \Gamma)$$
defined by setting
$$\gamma \mapsto cls(\gamma).$$
This allows us to regard the group $(\Gamma, \bullet)$ as a submonoid of $\Omega_{\gamma_0}(S, \Gamma)$ via j.

If S is any Γ-semigroup, and $(\Gamma, \bullet)$ is a group on Γ with unit $1 \in \Gamma$, then, as we exhibited above, we have defined $\Omega_1(S, \Gamma)$ which we call *the enveloping monoid of* (S, Γ) *relative to the group* $(\Gamma, \bullet)$. The following is also proved in [2].

Lemma 1.2. *The group of units of $\Omega_1(S, \Gamma)$ is exactly the group $(\Gamma, \bullet)$ regarded as a subgroup of $\Omega_1(S, \Gamma)$ via j.*

We have observed in [2] that the reverse of this process can be performed to any given monoid M with group of units Γ to define a Γ-semigroup (M, Γ) where the Γ-multiplication is given by
$$\cdot : M \times \Gamma \times M \to M \text{ such that } (x, \gamma, y) \mapsto x\gamma y,$$
and the multiplication on the right hand side is the one defined in M. We have called (M, Γ) the Γ-*semigroup of units of* M. We note here by passing that if a monoid $M \in \mathbf{Mon}(\Gamma)$ is von Neumann regular, then its Γ-semigroup of units (M, Γ) is a regular Γ-semigroup, and conversely. Indeed, for every $x \in M$, there is some $x' \in M$ such that $x = xx'x$. This can be rewritten as $x = x \cdot 1 \cdot x' \cdot 1 \cdot x$ which means that x' is a $(1, 1)$-inverse of x in (M, Γ). For the converse, assume that for every $x \in M$, there are $\alpha, \beta \in \Gamma$ and $x' \in M$ such that $x = x\alpha x'\beta x$. This means that in M the element $\alpha x'\beta$ is an inverse of x, hence x is regular and M is von Neumman regular.

The main result of [2] states that (S, Γ) embeds into the Γ-semigroup of units $(\Omega_1(S, \Gamma), \Gamma)$ of the enveloping monoid $\Omega_1(S, \Gamma)$ of (S, Γ). This result says roughly that every Γ-semigroup arises from a certain monoid and its group of units giving thus hope that one can entirely describe Γ-semigroups by means of monoids and groups. Finally, we mention briefly that the pair of Γ-semigroups, (S, Γ) and $(\Omega_1(S, \Gamma), \Gamma)$ are simultaneously regular. Indeed, if (S, Γ) is regular, then from Proposition 2.3 of [1] $\Omega_1(S, \Gamma)$ is von Neumann regular, and then from the above we derive that its Γ-semigroup of units $(\Omega_1(S,\Gamma),\Gamma)$ is a regular Γ-semigroup. Conversely, if $(\Omega_1(S,\Gamma),\Gamma)$ is a regular Γ-semigroup, then every $x \in S$, has an inverse which has to be of one of the following five forms: $x' \in S$, $\alpha x'\beta \in \Gamma S\Gamma$, $\alpha x' \in \Gamma S$, $x'\beta \in S\Gamma$, $\alpha \in \Gamma$. In either case it follows that x has an inverse in (S,Γ). We give the proof for convenience in the last case. In this case $x = x\alpha x$, hence $x = x\alpha x\alpha x$ and this means that x is an (α,α)-inverse of x.

2. The adjunction

This paper is a continuation of [2] and aims to express in a mathematical way the similarity between Γ-semigroups and monoids. This is done by considering two categories. The first one is the category of Γ-semigroups, and the second one is the subcategory of the category of monoids whose objects are all those monoids having the same group of units Γ, and whose morphisms are those monoid homomorphisms that fix Γ. We then define two functors from one category to the other, and prove that they form an adjoint pair.

In what follows $(\Gamma,\bullet)$ is a group with unit element 1.

Definition 2.1. We let Γ-**Sgrp** be the category of Γ-semigroups and their homomorphisms, and **Mon**(Γ) the category of monoids having the same group of units $(\Gamma,\bullet)$ and with morphisms those monoid homomorphisms which fix $(\Gamma,\bullet)$. An alternative way of defining the objects of **Mon**(Γ) is to regard them as monoid extensions of the same group $(\Gamma,\bullet)$.

Theorem 2.2. *There is an adjunction between the two categories* Γ-**Sgrp** *and* **Mon**(Γ).

Proof. First we define a functor $\Psi : \Gamma\text{-}\mathbf{Sgrp} \to \mathbf{Mon}(\Gamma)$. In objects, Ψ will be defined by mapping each Γ-semigroup (S,Γ) to the enveloping monoid $\Omega_1(S,\Gamma)$. To define Ψ on morphisms, we proceed as follows. Let (S,Γ) and (S',Γ) be both Γ-semigroups. For every homomorphism of Γ-semigroups $\varphi : S \to S'$, there is a unique homomorphism of monoids $\phi : \Omega_1(S,\Gamma) \to \Omega_1(S',\Gamma)$ identical on $(\Gamma,\bullet)$ such that $\phi\mu = \mu'\varphi$. Indeed, let $f : F(S\cup\Gamma) \to F(S'\cup\Gamma)$ be the homomorphism of free semigroups induced from the extension of φ on $S\cup\Gamma$ that fixes Γ. We prove that f induces a homomorphism $\phi : \Omega_1(S,\Gamma) \to \Omega_1(S',\Gamma)$. To do this we need to show that every relation that defines $\Omega_1(S,\Gamma)$ lies in the kernel of $\mu' f$ where $\mu' : F(S'\cup\Gamma) \to \Omega_1(S',\Gamma)$ is the canonical homomorphism. Indeed, for the first type of relations $((\gamma_1,\gamma_2),\gamma_1\bullet\gamma_2)$ we have

$$\begin{aligned}\mu' f(\gamma_1,\gamma_2) &= \mu'(\varphi(\gamma_1),\varphi(\gamma_2))\\ &= \gamma_1\bullet\gamma_2\\ &= \mu' f(\gamma_1\bullet\gamma_2).\end{aligned}$$

For the second type $((x,\gamma,y),x\gamma y)$ we have

$$\begin{aligned}\mu' f(x,\gamma,y) &= \mu'(\varphi(x),\gamma,\varphi(y))\\ &= \varphi(x)\gamma\varphi(y)\\ &= \varphi(x\gamma y)\\ &= \mu' f(x\gamma y),\end{aligned}$$

and for the last type $((x, y), x1y)$ we have

$$\begin{aligned}\mu' f(x, y) &= \mu'(\varphi(x), \varphi(y)) \\ &= \varphi(x)1\varphi(y) \\ &= \varphi(x1y) \\ &= \mu' f(x1y).\end{aligned}$$

Therefore $\mu' f$ induces $\phi : \Omega_1(S, \Gamma) \to \Omega_1(S', \Gamma)$ such that $\phi\mu = \mu' f$. Since φ is the restriction of f in $S \cup \Gamma$, then we derive that $\phi\mu = \mu'\varphi$. The uniqueness of ϕ with the given property follows easily from the fact any other homomorphism $\hat{\phi} : \Omega_1(S, \Gamma) \to \Omega_1(S', \Gamma)$ satisfying $\hat{\phi}\mu = \mu'\varphi$ coincides with ϕ on the generators of $\Omega_1(S, \Gamma)$ and therefore equals with ϕ. Also ϕ is identical on $(\Gamma, \bullet)$ since it is induced by f and f fixes Γ. Now we prove the functorial properties of Ψ. It is straightforward from the definition that $\Psi(id_{(S,\Gamma)}) = id_{\Omega_1(S,\Gamma)}$. If now

$$\varphi : (S, \Gamma) \to (S', \Gamma) \text{ and } \varphi' : (S', \Gamma) \to (S'', \Gamma)$$

are morphisms in Γ-**Sgrp**, then

$$\Psi(\varphi'\varphi) : \Omega_1(S, \Gamma) \to \Omega_1(S'', \Gamma)$$

coincides with $\Psi(\varphi') \circ \Psi(\varphi)$ since they coincide on generators. Indeed, if we let f and f' be the morphisms induced by φ and φ' respectively, then for every $x \in S$ we have,

$$\begin{aligned}\Psi(\varphi'\varphi)(\mu(x)) &= (\mu''(f'f))(x) \\ &= (\Psi(\varphi')\mu')(f(x)) \\ &= (\Psi(\varphi')\Psi(\varphi))(\mu(x)).\end{aligned}$$

In a similar way we get that for every $\gamma \in \Gamma$,

$$\Psi(\varphi'\varphi)(\mu(\gamma)) = (\Psi(\varphi')\Psi(\varphi))(\mu(\gamma)).$$

Next we define a functor $\Psi' : \mathbf{Mon}(\Gamma) \to \Gamma$-**Sgrp** on objects by assigning to each monoid S its Γ-semigroup of units (S, Γ), and to each homomorphism of monoids $f : S \to S'$ which fixes Γ, the map

$$\Psi'(f) : (S, \Gamma) \to (S', \Gamma)$$

such that for every $x \in S$,

$$\Psi'(f)(x) = f(x).$$

This map is a homomorphism of Γ semigroups since for every $x, y \in S$ and $\gamma \in \Gamma$ we have that

$$\begin{aligned}\Psi'(f)(x\gamma y) &= f(x\gamma y) \\ &= f(x)f(\gamma)f(y) \\ &= f(x)\gamma f(y) \\ &= \Psi'(f)(x)\gamma\Psi'(f)(y).\end{aligned}$$

The functorial properties of Ψ' are easy to prove. Next we will prove that Ψ is a left adjoint of Ψ'. Let $(S, \Gamma) \in \Gamma$-**Sgrp** and $M \in \mathbf{Mon}(\Gamma)$ be two arbitrary objects. We define

$$\xi_{(S,\Gamma),M} : \Gamma\text{-}\mathbf{Sgrp}((S, \Gamma), \Psi'(M)) \to \mathbf{Mon}(\Gamma)(\Psi(S, \Gamma), M),$$

by

$$h \mapsto h_*$$

where for the two types of generators, $x \in S$ and $\gamma \in \Gamma$, h_* is defined by

$$h_*(x) = h(x) \text{ and } h_*(\gamma) = \gamma.$$

To see that h_* is a homomorphism we need to check that it is compatible with the defining relations of $\Psi(S,\Gamma) = \Omega_1(S,\Gamma)$. Indeed, for the relations of the type $(x,\gamma,y) \sim (x\gamma y)$ we have that

$$\begin{aligned} h_*(x\gamma y) &= h(x\gamma y) \\ &= h(x)\gamma h(y) \\ &= h_*(x)h_*(\gamma)h_*(y). \end{aligned}$$

For relations of the type $(x,y) \sim (x1y)$ we have that

$$\begin{aligned} h_*(xy) &= h(x1y) \\ &= h(x)h(1)h(y) \\ &= h_*(x)h_*(1)h_*(y). \end{aligned}$$

And for the last type of relations $(\alpha,\beta) \sim (\alpha \bullet \beta)$ we have

$$\begin{aligned} h_*(\alpha,\beta) &= \alpha \bullet \beta \\ &= h_*(\alpha \bullet \beta). \end{aligned}$$

The map $\xi_{(S,\Gamma),M}$ is indeed a bijection. It is an injection since if $h, g : (S,\Gamma) \to (M,\Gamma)$ are such that $h_* = g_*$, then for every $x \in S$,

$$h(x) = h_*(x) = g_*(x) = g(x).$$

It is also surjective since every homomorphism of monoids $g : \Omega_1(S,\Gamma) \to M$ that fixes Γ is induced by the restriction $h = g|_S$ and so $g = h_*$. Next we prove that $\xi_{(S,\Gamma),M}$ is natural in both variables. To see the naturality in the first variable, we must prove that for every Γ-homomorphism $f : (S,\Gamma) \to (S',\Gamma)$, the following diagram commutes.

$$\begin{array}{ccc} \Gamma\text{-}\mathbf{Sgrp}((S,\Gamma),\Psi'(M)) & \xrightarrow{\xi_{(S,\Gamma),M}} & \mathbf{Mon}(\Gamma)(\Omega_1(S,\Gamma),M) \\ \uparrow{\scriptstyle \Gamma\text{-}\mathbf{Sgrp}(f,\Psi'(M))} & & \uparrow{\scriptstyle \Gamma\text{-}\mathbf{Sgrp}(\Psi(f),M)} \\ \Gamma\text{-}\mathbf{Sgrp}((S',\Gamma),\Psi'(M)) & \xrightarrow[\xi_{(S',\Gamma),M}]{} & \mathbf{Mon}(\Gamma)(\Omega_1(S',\Gamma),M) \end{array}$$

This means that for every $h' : (S',\Gamma) \to (M,\Gamma)$ we must have that $(h'f)_* = h'_*\Psi(f)$. It is enough that $(h'f)_*$ and $h'_*\Psi(f)$ coincide on generators. This is true since for every $x \in S$ we have that

$$(h'f)_*(x) = (h'f)(x) = h'_*(f(x)) = h'_*(\Psi(f)(x)) = (h'_*\Psi(f))(x).$$

For for every $\gamma \in \Gamma$ we have that

$$(h'f)_*(\gamma) = \gamma = h'_*(\gamma) = h'_*(\Psi(f)(\gamma)) = (h'_*\Psi(f))(\gamma).$$

Now we see the naturality in the second variable. Let $\chi : M \to N$ be a monoid morphism that fixes the common group of units Γ. We must prove that the following diagram commutes.

$$\begin{array}{ccc}
\Gamma\text{-}\mathbf{Sgrp}((S,\Gamma),\Psi'(M)) & \xrightarrow{\xi_{(S,\Gamma),M}} & \mathbf{Mon}(\Gamma)(\Omega_1(S,\Gamma),M) \\
\Big\downarrow{\scriptstyle \Gamma\text{-}\mathbf{Sgrp}((S,\Gamma),\Psi'(\chi))} & & \Big\downarrow{\scriptstyle \Gamma\text{-}\mathbf{Sgrp}(\Omega_1(S,\Gamma),\chi)} \\
\Gamma\text{-}\mathbf{Sgrp}((S,\Gamma),\Psi'(N)) & \xrightarrow{\xi_{(S,\Gamma),N}} & \mathbf{Mon}(\Gamma)(\Omega_1(S,\Gamma),N)
\end{array}$$

which amounts to saying that $\chi h_* = (\Psi'(\chi)h)_*$. As before, we need to prove that the above maps coincide in the generators of $\Omega_1(S,\Gamma)$. For every $x \in S$ we see that

$$\begin{aligned}
(\Psi'(\chi)h)_*(x) &= (\Psi'(\chi)h)(x) \\
&= (\chi h)(x) \\
&= (\chi h_*)(x).
\end{aligned}$$

Also for every $\gamma \in \Gamma$ we have

$$\begin{aligned}
(\Psi'(\chi)h)_*(\gamma) &= \Psi'(\chi)(h_*(\gamma)) \\
&= \Psi'(\chi)(\gamma) \\
&= \gamma \\
&= (\chi h_*)(\gamma).
\end{aligned}$$

This concludes the proof. □

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BLOW-UP OF SOLUTIONS TO A PARABOLIC SYSTEM WITH VARIABLE SOURCES

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Abstract. In this paper, we establish some sufficient conditions on variable sources and parametris to guarantee the existence blow-up of solutions parabolic system with variable sources.

1. Introduction

This paper is concerned with the properties of solutions of a parabolic system involving two semilinear equations associated with nonlinear heat diffusion with variable sources:

$$\begin{cases} u_t = \Delta u^{\gamma+1} + \delta \upsilon^{p(x)},\ x \in \Omega,\ t > 0, \\ \upsilon_t = \Delta \upsilon^{\mu+1} + \delta u^{q(x)},\ x \in \Omega,\ t > 0, \\ u(x,t) = \upsilon(x,t) = 0,\ x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x),\ \upsilon(x,0) = \upsilon_0(x),\ x \in \Omega, \end{cases} \tag{1.1}$$

where $\gamma, \mu > 0$ such that

$$1 + \mu \leq p^- \leq p(x) < +\infty,\ x \in \Omega,$$
$$1 + \gamma \leq q^- \leq q(x) < +\infty,\ x \in \Omega,$$

and

$$1 < p^- \leq p(x) < +\infty,\ x \in \Omega, \tag{1.2}$$

$$1 < q^- \leq q(x) < +\infty,\ x \in \Omega. \tag{1.3}$$

Parabolic systems such as (1.1) appear in population dynamics with nonlocal growth terms, heat propagation in a two component combustible mixture, porous medium, electro-rheological fluids and chemical processes (see [8, 9] and references therein). There is a rich literature devoted to the existence of global solution, unique solvablity and blow-up rates for parabolic equations (see [1, 2, 3, 4, 7]). Also, Souplet [10] considered the nonlocal problems with homogeneous Dirichlet boundary condition and determined some sharp critical exponents for blow-up or global existence.

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In [5, 6], Galaktionov et al. considered the system

$$u_t = \Delta u^{\gamma+1} + \delta \upsilon^p, \quad \upsilon_t = \Delta \upsilon^{\mu+1} + \delta u^q, \ (x,t) \in \Omega \times (0,T)$$

with homogeneous Dirichlet boundary conditions. In the proof of the propositions, the authors assumed that the local solution of the boundary value problem satisfies the natural inclusions $u^{\frac{\gamma+1}{2}}, \upsilon^{\frac{\mu+1}{2}} \in L^2\left(0,T; L^2(\Omega)\right)$ and $u^{\gamma+1}$, $\upsilon^{\mu+1} \in L^\infty\left(0,T; H_0^1(\Omega)\right)$, $T < T_0$. Under the assumption of boundedness of u, υ these was derived by Galerkin approximations and also the authors established a weak Maximum Principle, so that if u, υ a bounded in $(0,T) \times \Omega$, then $u, \upsilon \geq 0$ a.e. in Ω, $0 < t < T$.

In this paper, we establish some sufficient conditions on variable sources and parametris to guarantee the existence blow-up of solutions parabolic system with variable sources for any initial data of the problem (1.1).

2. Main Results and Proofs

Let $\lambda_1 > 0$ be the first eigenvalue and φ_1 be the corresponding eigenfunction of the following fixed membrane problem

$$\Delta\varphi + \lambda\varphi = 0, \varphi > 0 \text{ in } x \in \Omega,$$
$$\varphi = 0, \text{ on } \partial\Omega,$$

with

$$\int_\Omega \varphi_1 dx = 1. \tag{2.1}$$

We define the auxiliary function of the form

$$\eta(t) = \int_\Omega u\varphi_1 dx, \ t > 0$$

with

$$a_0 = \int_\Omega u_0 \varphi_1 dx > 0,$$

and

$$b_0 = \int_\Omega \upsilon_0 \varphi_1 dx > 0.$$

Our main result is the following theorem:

Theorem 2.1. *Let Ω be a bounded smooth domain in $\mathbb{R}^N$ and let u, υ be a positive solution of problem* (1.1).

$i)_a$ Let $p^- = 1 + \mu$, $q^- = 1 + \gamma$ and $\mu = \gamma > 0$. If $\lambda_1 < \delta$ with

$$\delta > \frac{\lambda_1 (a_0 + b_0)^{\gamma+1}}{(a_0 + b_0)^{\gamma+1} - 2^{\gamma+1}},$$

and $a_0 + b_0 > 2$, then we can derive that the blow-up time T satisfies

$$\int_{a_0+b_0}^{+\infty} \frac{2^\gamma d\zeta}{(\delta - \lambda_1)\zeta^{\gamma+1} - 2^{\gamma+1}\delta} \geq T.$$

$i)_b$ Let $p^- = 1 + \mu$, $q^- = 1 + \gamma$, $\mu \neq \gamma$ such that $0 < \gamma < \mu$. If $\lambda_1 < \delta$ with

$$\delta > \frac{\lambda_1 \left[(\gamma+1)(a_0 + b_0)^{\gamma+1} - 2^\gamma(\mu - \gamma)\right]}{(\gamma+1)\left[(a_0 + b_0)^{\gamma+1} - 2^{\gamma+1}\right] - 2^\gamma(\mu - \gamma)},$$

then we can derive that the blow-up time T satisfies

$$\int_{a_0+b_0}^{+\infty} \frac{2^{\gamma} d\zeta}{(\delta-\lambda_1)\zeta^{\gamma+1} - \frac{2^{\gamma}(\delta-\lambda_1)(\mu-\gamma)+2\delta(\gamma+1)}{\gamma+1}} \geq T,$$

for a_0+b_0 satisfying

$$a_0+b_0 > \left(\frac{(\delta-\lambda_1)(\mu-\gamma)}{2^{-\gamma}(\delta-\lambda_1)(\gamma+1)}\right)^{\frac{1}{\gamma+1}}.$$

$ii)_a$ *Let $p^- > 1+\mu$, $q^- > 1+\gamma$ and $\mu = \gamma > 0$. If $\lambda_1 > 0$ and*

$$\delta > \frac{2^{\gamma} C_1}{(a_0+b_0)^{\gamma+1}},$$

then we can derive that the blow-up time T satisfies

$$\int_{a_0+b_0}^{+\infty} \frac{2^{\gamma} d\zeta}{\delta\zeta^{\gamma+1} - 2^{\gamma} C_1} \geq T,$$

for a_0+b_0 satisfying

$$a_0+b_0 > (2^{\gamma} C_1)^{\frac{1}{1+\gamma}},$$

where C_1 is a positive constant which will be determined later.

$ii)_b$ *Let $p^- > 1+\mu$, $q^- > 1+\gamma$ and $\mu > \gamma > 0$. If $\lambda_1 > 0$ and*

$$\delta > \frac{2^{\gamma}(C_1+D_1)}{(a_0+b_0)^{\gamma+1}},$$

then we can derive that the blow-up time T satisfies

$$\int_{a_0+b_0}^{+\infty} \frac{2^{\gamma} d\zeta}{\delta\zeta^{\gamma+1} - 2^{\gamma}(C_1+D_1)} \geq T,$$

for a_0+b_0 satisfying

$$a_0+b_0 > (2^{\gamma}(C_1+D_1))^{\frac{1}{1+\gamma}},$$

where C_1, D_1 are positive constants which will be determined later.

Proof. By integrating in t both parts of the equation (1.1), we obtain the system of equalities

$$\eta(t) - a_0 = -\lambda_1 \int_0^t \int_{\Omega} u^{\gamma+1}\varphi_1 dxds + \delta \int_0^t \int_{\Omega} \upsilon^{p(x)}\varphi_1 dxds, \tag{2.2}$$

and

$$\eta(t) - b_0 = -\lambda_1 \int_0^t \int_{\Omega} \upsilon^{\mu+1}\varphi_1 dxds + \delta \int_0^t \int_{\Omega} u^{q(x)}\varphi_1 dxds. \tag{2.3}$$

Let define the functions

$$f(t) = \left(\int_{\Omega} u^{\gamma+1}\varphi_1 dx\right)^{\frac{1}{\gamma+1}}, \tag{2.4}$$

and

$$g(t) = \left(\int_{\Omega} \upsilon^{\mu+1}\varphi_1 dx\right)^{\frac{1}{\eta+1}}. \tag{2.5}$$

From the Hölder inequality, (2.4) and (2.5) it follows that

$$\eta(t) \le \left(\int_\Omega u^{\gamma+1}\varphi_1 dx\right)^{\frac{1}{\gamma+1}} = f(t), \tag{2.6}$$

and

$$\eta(t) \le \left(\int_\Omega \upsilon^{\mu+1}\varphi_1 dx\right)^{\frac{1}{\mu+1}} = g(t). \tag{2.7}$$

If p and q satisfies the condition (1.2), (1.3), by using (2.1) we have

$$\begin{aligned}
\int_\Omega \upsilon^{p(x)}\varphi_1 dx &= \int_{\Omega\cap\{u\ge1\}} \upsilon^{p(x)}\varphi_1 dx + \int_{\Omega\cap\{u<1\}} \upsilon^{p(x)}\varphi_1 dx \\
&\ge \int_{\Omega\cap\{u\ge1\}} \upsilon^{p^-}\varphi_1 dx + \int_{\Omega\cap\{u<1\}} \upsilon^{p^+}\varphi_1 dx \\
&\ge \int_{\Omega\cap\{u\ge1\}} \upsilon^{p^-}\varphi_1 dx \\
&\ge \int_{\Omega\cap\{x:u<1\}} \upsilon^{p^-}\varphi_1 dx + \int_{\Omega\cap\{u\ge1\}} \upsilon^{p^-}\varphi_1 dx - \int_{\Omega\cap\{x:u<1\}} \upsilon^{p^-}\varphi_1 dx \\
&\ge \int_\Omega \upsilon^{p^-}\varphi_1 dx - \int_\Omega \varphi_1 dx \\
&= \int_\Omega \upsilon^{p^-}\varphi_1 dx - 1, \qquad (2.8)
\end{aligned}$$

and

$$\int_\Omega u^{q(x)}\varphi_1 dx \ge \int_\Omega u^{q^-}\varphi_1 dx - 1. \tag{2.9}$$

Then from $(2.2), (2.3), (2.6), (2.7), (2.8)$ and (2.9) we obtain

$$\eta(t) - a_0 \ge -\lambda_1 \int_0^t f^{\gamma+1}(s)\, ds + \int_0^t \left(\delta \int_\Omega \upsilon^{p^-}\varphi_1 dx - \delta\right) ds,$$

and

$$\eta(t) - b_0 \ge -\lambda_1 \int_0^t g^{\mu+1}(s)\, ds + \int_0^t \left(\delta \int_\Omega u^{q^-}\varphi_1 dx - \delta\right) ds.$$

Furthermore, taking into account the fact that $p^- \ge 1+\eta$, $q^- \ge 1+\gamma$, we obtain

$$\begin{aligned}
f(t) &= \left(\int_\Omega u^{\gamma+1}\varphi_1^{\frac{1+\gamma}{p^-}}\varphi_1^{1-\frac{1+\gamma}{p^-}} dx\right)^{\frac{1}{\gamma+1}} \\
&\le \left(\int_\Omega \upsilon^{p^-}\varphi_1 dx\right)^{\frac{1}{p^-}} \left(\int_\Omega \varphi_1 dx\right)^{\frac{p^- -\gamma-1}{p^-}} = \left(\int_\Omega \upsilon^{p^-}\varphi_1 dx\right)^{\frac{1}{p^-}},
\end{aligned}$$

we get

$$\int_\Omega \upsilon^{p^-}\varphi_1 dx \ge f^{p^-}(t),$$

and similarly

$$\int_\Omega u^{q^-}\varphi_1 dx \ge g^{q^-}(t).$$

Then, we have

$$f(t) - a_0 \geq -\lambda_1 \int_0^t f^{\gamma+1}(s)\,ds + \delta \int_0^t \left(g^{p^-}(t) - 1\right) ds,\ t > 0, \tag{2.10}$$

and

$$g(t) - b_0 \geq -\lambda_1 \int_0^t g^{\mu+1}(s)\,ds + \delta \int_0^t \left(f^{q^-}(t) - 1\right) ds,\ t > 0. \tag{2.11}$$

In conjunction with (2.10) and (2.11) let us consider the following of ordinary differential equations:

$$\widetilde{f}'(t) = -\lambda_1 \widetilde{f}^{\gamma+1}(t) + \delta \widetilde{g}^{p^-}(t) - \delta,\ t > 0, \tag{2.12}$$

and

$$\widetilde{g}'(t) = -\lambda_1 \widetilde{g}^{\mu+1}(t) + \delta \widetilde{f}^{q^-}(t) - \delta,\ t > 0. \tag{2.13}$$

Let the functions $\widetilde{f}$, $\widetilde{g}$ satisfy the conditions

$$\widetilde{f}(0) = a_0 > 0,\ \widetilde{g}(0) = b_0 > 0. \tag{2.14}$$

A direct comparison of (2.10), (2.11) with the equations (2.12), (2.13) and (2.14) shows that for all admissible t we have the inequalities

$$f(t) \geq \widetilde{f}(t),\ g(t) \geq \widetilde{g}(t). \tag{2.15}$$

Therefore the system of equations (2.12), (2.13) allows us, in view of (2.15), to define the conditions under which the functions $f(t)$, $g(t)$ cannot both be bounded for all $t > 0$, that is

$$\overline{\lim} \max\{f(t), g(t)\} = \infty,\ t \to T_0^- < \infty. \tag{2.16}$$

In view of the inequalities

$$f(t) \leq \left(\left\|u^{\gamma+1}\right\|_2^2 \|\varphi_1\|_2^2\right)^{\frac{1}{2(\gamma+1)}},$$

and

$$g(t) \leq \left(\left\|\upsilon^{\mu+1}\right\|_2^2 \|\varphi_1\|_2^2\right)^{\frac{1}{2(\mu+1)}},$$

this ensures that (2.16) holds.

$i)_a$ For $p^- = 1 + \mu$, $q^- = 1 + \gamma$ and $\mu = \gamma$.

We define the functions $\widetilde{f}$ and $\widetilde{g}$. Adding up equations (2.12), (2.13), we have

$$\widetilde{f}'(t) = -\lambda_1 \widetilde{f}^{\gamma+1}(t) + \delta \widetilde{g}^{\gamma+1}(t) - \delta,\ t > 0, \tag{2.17}$$

and

$$\widetilde{g}'(t) = -\lambda_1 \widetilde{g}^{\gamma+1}(t) + \delta \widetilde{f}^{\gamma+1}(t) - \delta,\ t > 0. \tag{2.18}$$

Let as before $E(t) := \widetilde{f}(t) + \widetilde{g}(t)$ with $E(0) = a_0 + b_0 > 0$, we get

$$\begin{aligned} E'(t) &\geq -\lambda_1 \left(\widetilde{f}^{\gamma+1}(t) + \widetilde{g}^{\gamma+1}(t)\right) + \delta \left(\widetilde{f}^{\gamma+1}(t) + \widetilde{g}^{\gamma+1}(t)\right) - 2\delta \\ &= (\delta - \lambda_1) \left(\widetilde{f}^{\gamma+1}(t) + \widetilde{g}^{\gamma+1}(t)\right) - 2\delta. \end{aligned}$$

From (2.17), (2.18) and inequality

$$c^\sigma + d^\sigma \geq 2^{1-\sigma}(c+d)^\sigma, \text{ where } c, d > 0,\ \sigma \geq 1,$$

it follows that

$$\begin{aligned} E'(t) &\geq (\delta-\lambda_1)\,2^{-\gamma}\left(\widetilde{f}^{\gamma+1}(t)+\widetilde{g}^{\gamma+1}(t)\right)-2\delta \\ &= (\delta-\lambda_1)\,2^{-\gamma}E^{\gamma+1}(t)-2\delta,\ t>0. \end{aligned} \tag{2.19}$$

Obviously, since $\lambda_1<\delta$, $\gamma>0$ we can get that the function $E^{\gamma+1}$ is monotone increasing and with $E(0)>2$ and

$$\delta>\max\left\{\lambda_1,\frac{\lambda_1E^{\gamma+1}(0)}{E^{\gamma+1}(0)-2^{\gamma+1}}\right\}=\frac{\lambda_1E^{\gamma+1}(0)}{E^{\gamma+1}(0)-2^{\gamma+1}}.$$

Then we can know that the solution to problem (1.1) blows up in finite time, by virtue of (2.19), we can derive that the blow-up time T satisfies

$$\int_{E(0)}^{E(t)}\frac{2^{\gamma}d\zeta}{(\delta-\lambda_1)\,\zeta^{\gamma+1}-2^{\gamma+1}\delta}\geq T,$$

such that $E(t)\to\infty$ as $t\to T_0^-$.

$i)_b$ For $p^-=1+\mu$, $q^-=1+\gamma$, $\mu\neq\gamma$ with $0<\gamma<\mu$. Then from (2.17), (2.18), using Young's inequality

$$\widetilde{g}^{\mu+1}(t)\geq\frac{\mu+1}{\gamma+1}\widetilde{g}^{\gamma+1}(t)-\frac{\mu-\gamma}{\gamma+1}\geq\widetilde{g}^{\gamma+1}(t)-\frac{\mu-\gamma}{\gamma+1}.$$

By (2.19), we obtain

$$\begin{aligned} E'(t) &\geq (\delta-\lambda_1)\left(\widetilde{f}^{\gamma+1}(t)+\widetilde{g}^{\mu+1}(t)\right)-2\delta \\ &\geq (\delta-\lambda_1)\left(\widetilde{f}^{\gamma+1}(t)+\widetilde{g}^{\gamma+1}(t)\right)-\left(\frac{(\delta-\lambda_1)(\mu-\gamma)}{\gamma+1}+2\delta\right) \\ &\geq (\delta-\lambda_1)\,2^{-\gamma}E^{\gamma+1}(t)-\frac{(\delta-\lambda_1)(\mu-\gamma)+2\delta(\gamma+1)}{\gamma+1}. \end{aligned}$$

Therefore, since $\lambda_1<\delta$, $\mu>\gamma>0$ we can get that the function $E^{\gamma+1}$ is monotone increasing with

$$E(0)>\left(\frac{(\delta-\lambda_1)(\mu-\gamma)}{2^{-\gamma}(\delta-\lambda_1)(\gamma+1)}\right)^{\frac{1}{\gamma+1}},$$

and

$$\delta>\frac{\lambda_1\left[(\gamma+1)E^{\gamma+1}(0)-2^{\gamma}(\mu-\gamma)\right]}{(\gamma+1)\left[E^{\gamma+1}(0)-2^{\gamma+1}\right]-2^{\gamma}(\mu-\gamma)}>0.$$

We can derive that the blow-up time T satisfies

$$\int_{E(0)}^{+\infty}\frac{2^{\gamma}d\zeta}{(\delta-\lambda_1)\,\zeta^{\gamma+1}-\frac{2^{\gamma}(\delta-\lambda_1)(\mu-\gamma)+2\delta(\gamma+1)}{\gamma+1}}\geq T.$$

Let us estimate $\widetilde{f}(t)$, $\widetilde{g}(t)$ using Young's inequality:

$$\widetilde{g}^{q^-}(t)\geq(\delta+\lambda_1)\,\widetilde{g}^{\gamma+1}(t)-A_0,$$

and

$$\widetilde{f}^{p^-}(t)\geq(\delta+\lambda_1)\,\widetilde{f}^{\mu+1}(t)-B_0,$$

where A_0 and B_0 are constants:

$$A_0=\frac{q^--\gamma-1}{\gamma+1}\left[\frac{(\delta+\lambda_1)(\gamma+1)}{q^-}\right]^{\frac{q^-}{q^--\gamma-1}},$$

and

$$B_0 = \frac{p^- - \mu - 1}{\mu + 1} \left[\frac{(\delta + \lambda_1)(\mu + 1)}{p^-} \right]^{\frac{p^-}{p^- - \mu - 1}}.$$

Then we obtain from (2.19) that

$$E'(t) \geq \delta \left(\widetilde{f}^{\gamma+1}(t) + \widetilde{g}^{\mu+1}(t) \right) - C_0, \ t > 0, \tag{2.20}$$

where $C_0 = A_0 + B_0 - 2\delta > 0$ for small enough $\delta > 0$.

$ii)_a$ For $p^- > 1 + \mu$, $q^- > 1 + \gamma$. If $\gamma = \mu$, using the inequality

$$1 + \omega^{\gamma+1} \geq 2^{-\gamma} (1 + \omega)^{\gamma+1}, \ \omega \geq 0,$$

we derive from (2.20) the inequality

$$E'(t) \geq 2^{-\gamma} \delta E^{\gamma+1}(t) - C_1, \ t > 0,$$

where $C_1 = C_0 - 2 > 0$ and

$$\delta > \frac{C_1 2^\gamma}{E^{\gamma+1}(0)} = \frac{C_1 2^\gamma}{(a_0 + b_0)^{\gamma+1}}.$$

$ii)_b$ For $p^- > 1 + \mu$, $q^- > 1 + \gamma$. If, on the other hand, $\gamma \neq \mu$, then, setting for definiteness $\mu > \gamma$, and using the estimate

$$\widetilde{g}^{\mu+1}(t) \geq \widetilde{g}^{\gamma+1}(t) - D_1, \ t > 0,$$

where $D_1 = \frac{\mu-\gamma}{\mu+1} \left(\frac{\gamma+1}{\mu+1} \right)^{\frac{\gamma+1}{\mu-\gamma}}$. Then we obtain from (2.20)

$$E'(t) \geq 2^{-\gamma} \delta E^{\gamma+1}(t) - (C_1 + D_1), \ t > 0, \tag{2.21}$$

where

$$\delta > \frac{2^\gamma (C_1 + D_1)}{E^{\gamma+1}(0)} = \frac{2^\gamma (C_1 + D_1)}{(a_0 + b_0)^{\gamma+1}}.$$

Hence, if $E(0)$ satisfying

$$E(0) = \lambda_1 \int_\Omega u_0 dx + \lambda_1 \int_\Omega \upsilon_0 dx > 2^{\frac{\gamma}{\gamma+1}} (C_1 + D_1)^{\frac{1}{\gamma+1}},$$

since $\gamma > 0$, we can get that the function $E^{\gamma+1}$ is monotone increasing with $E(0)$ such that

$$E(0) > (2^\gamma (C_1 + D_1))^{\frac{1}{\gamma+1}}.$$

Then we can know that the solution to problem (1.1) blows up in finite time. It is easy to see that the right-hand side of this inequality admits passing to the $\lim_{\mu \to \gamma^+}$ that is, for $\gamma = \mu$ it is the same as (2.21). Thus, the proof of Theorem 2.1 is completed. □

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A MULTI-STRAIN SEIR OUTBREAK MODEL WITH GENERAL INCIDENCE RATES: APPLICATION OF THE NEW CORONAVIRUS DISEASE

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Abstract. The global usability analysis of the multi-epidemic model with an overall incidence rate on this page is being investigated. The problem, exposure, is modeled by a system of 4 untranslated ordinary differential equations that describe the treated individuals. The creation model is well defined, apart from its solutions, except for its positivity and speech. Generally speaking, 3 equilibrium points; disease-free equilibrium point, endemic equilibrium point according to Type i and the last endemic equilibrium point according to the species. Appropriate Lyapunov news, global applications of disease-free equilibria points are proved depending on the basic reproduction number R_0. In addition, the global practical results of the other suitable Lyapunov annotated endemic equilibrium, species with -1 reproduction number R_0^1, Type-2 reproduction number R_0^2 and species reproduction number R_0^i. Simulations are made to verify the different theoretical results. An important broad view on the application of equilibrium is presented that the generalized incidence function model covers multiple models with classical incidence rates. Comparisons were made between model results and numerical results of the new coronavirus. It is pointed out that this realized model fits well with the actual results. It is an undeniable fact that some strategies such as quarantine, isolation, wearing a mask, and disinfection have an undeniable importance in controlling the spread of the epidemic during this period of the disease.

1. Introduction

Today, many infectious diseases still grow large populations targeting. It is the leading cause of deaths, especially in many developing countries. They are counted among the years. Accordingly, mathematical modeling has become an increasingly dominant place in epidemiology. These studies do indeed contribute to a good understanding of the epidemiological phenomenon under study and the different factors that could lead to a serious worldwide epidemic and even a dangerous

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pandemic. Therefore, multiple SEIR epidemic patterns include a long incubation period, ie the time from the entry of an infectious agent into the body to the emergence of disease symptoms; It offers an important tool for studying incubation time as well as infectious diseases involving various types of infections. The importance of studying multifarious models is to find different conditions that allow all their influential types to coexist. Global dynamics of a single SEIR model, with known or nonlinear incidence functions; By including the two incidence functions in the global stability of the two types of SEIR models, the first is bilineer, the second is not monotonous, and recently they have been investigating non-monotonous cases of two incidence functions of similar problems. Thus, the overall incidence function has the purpose to represent the incidence rates of a major outbreak. The aim of this research, then, is to generalize the previous models, taking into account a multifaceted SEIR model with an overall incidence rate of n. So, our research will be done on n types of expanded epidemic model as follows:

$$
\begin{aligned}
\frac{dS}{dt} &= \Lambda - \sum_{i=1}^{n} f_i\left(S, I_i\right) I_i - \delta S \\
\frac{dE_i}{dt} &= f_i\left(S, I_i\right) I_i - \left(\gamma_i + \delta\right) E_i, \quad i = 1, 2, \ldots, n \\
\frac{dI_i}{dt} &= \gamma_i E_i - \left(\mu_i + \delta\right) I_i, \quad i = 1, 2, \ldots, n \\
\frac{dR}{dt} &= \sum_{i=1}^{n} \mu_i I_i - \delta R
\end{aligned}
\tag{1.1}
$$

with $S(0) \geq 0, E_i(0) \geq 0, I_i(0) \geq 0, R(0) \geq 0$,
$\forall i \in \{1, 2, \ldots, n\}$.

(S) is the number of susceptible individuals, $\left(E_1\right), \left(E_2\right),$ $\ldots, \left(E_n\right)$ each latent (exposed) are the numbers of the individual class, $\left(I_1\right), \left(I_2\right), \ldots, \left(I_n\right)$ respectively for each contagious are the numbers of the individual class and (R) is the number of people who have recovered.

Table 1. Explanation of the Parameters of the Model

Parametre	Explanation
Λ	Population birth rate
δ	Mortality rate of the population
γ_i	ith hatching rate of the species
μ_i	ith infection rate of the species

Finally, the general incidence function $f_i\left(S, I_i\right)$, ith represents the infection transmission rates for the species and confirms the following conditions:

$$f_i\left(0, I_i\right) = 0 \quad, \forall I_i \geq 0 \quad, \quad i = 1, 2, \ldots, n \tag{1.2}$$

$$\frac{\partial f_i\left(0, I_i\right)}{\partial S} > 0 \quad, \forall S > 0, \forall I_i \geq 0 \quad, i = 1, 2, \ldots, n \tag{1.3}$$

$$\frac{\partial f_i\left(0, I_i\right)}{\partial I_i} \leq 0 \quad, \forall S \geq 0, \forall I_i \geq 0 \quad, i = 1, 2, \ldots, n \tag{1.4}$$

The properties (1.2),(1.3) and (1.4) for f_i functions
bilineer incidence functions βS [1, 2, 3, 4, 5],
saturated incidence function $\frac{\beta S}{1+\alpha_1 S}$ or $\frac{\beta S}{1+\alpha_2 I}$ [6, 7, 8],
Beddington-DeAngelis incidence function $\frac{\beta S}{1+\alpha_1 S+\alpha_2 I}$ [9],
Crowley-Martin incidence function $\frac{\beta S}{1+\alpha_1 S+\alpha_2 I+\alpha_1\alpha_2 I}$ [10],
specific nonlinear incidence function $\frac{\beta S}{1+\alpha_1 S+\alpha_2 I+\alpha_3 IS}$ [11, 12, 13, 14, 15],
and non-monotonous incidence function $\frac{\beta S}{1+\alpha I^2}$ [16, 17, 18, 19, 20]

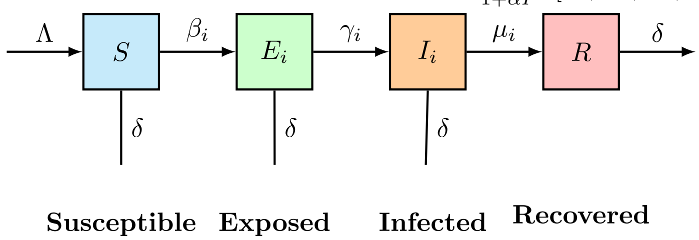

Figure: Flowchart of multi-strain SEIR model

The flow chart of the multifarious epidemiological SEIR model is shown in figure. Our focus revolves around the overall incidence rates and the global stability of the multifarious Seir epidemic model.

The following parts of our study are summarized as follows. In the section 1 the model is examined by proving the existence, positivity and limitation of the solutions. In section 3, we make the global stability analysis of the equilibrium points of the model, calculate and prove the basic reproduction number of our epidemic model. In the section 4 numerical simulations were obtained using different incidence functions and comments were made about the progress of the epidemic in the light of these numerical results.

2. Positivity and Limitedness of Model Solutions

For problems with population dynamics, the variables should be positive and limited. First, let's assume that the model parameters are positive.

Definition of the total population is

$$N(t) = S(t) + \sum_{i=1}^{n} E_i(t) + \sum_{i=1}^{n} I_i(t) + R(t).$$

Proposition 1. From every non-negative initial conditions, solutions (1.1) are limited and non-negative.

$$\text{Otherwise,} \quad N(t) \leq \frac{\Lambda}{\delta} + N(0)$$

.

Proof. With the basic differential equation theory, it is verified that there is only one solution for [21, 22, 23, 24, 25, 26] (1.1) in this framework.

We'll show it stays there forever at
$\mathbb{R}_+^{2n+2} = \{(S, E_1, E_2, \ldots, E_n, I_1, I_2, \ldots I_n, R) \in \mathbb{R}^{2n+2} : S \geq 0,$
$E_1 \geq 0, E_2 \geq 0, \ldots, E_n \geq 0, I_1 \geq 0, I_2 \geq 0, \ldots I_n \geq 0, R \geq 0\}$ to prove its positivity.

Primarily, let

$$\begin{aligned} T = &\sup\{\tau \geq 0 \mid \forall t \in [0,\tau] \text{ such that} \\ &S(t) \geq 0, E_1(t) \geq 0, E_2(t) \geq 0, \ldots, E_n(t) \geq 0, \\ &I_1(t) \geq 0, I_2(t) \geq 0, \ldots I_n(t) \geq 0, R(t) \geq 0\} \end{aligned} \tag{2.1}$$

Now let's prove it's $T = +\infty$.

Suppose that T is finite; by continuity of solutions, we have

$S(T) = 0$ either $E_1(T) = 0$ either $E_2(T) = 0$ either $\cdots$ either $E_n(T) = 0$ either $I_1(T) = 0$ either $I_2(T) = 0$ either $\cdots$ either $I_n(T) = 0$ either $R(T) = 0$.

If $S(T) = 0$ before the other variables $E_1, E_2, \ldots, E_n, I_1, I_2, \ldots I_n, R$ become zero then

$$\frac{dS(T)}{dt} = \lim_{t \to T^-} \frac{S(T) - S(t)}{T - t} = \lim_{t \to T^-} \frac{-S(t)}{T - t} \leq 0 \tag{2.2}$$

From the 1 st equation of system (1.1), we have

$$\dot{S}(T) = \Lambda - \sum_{i=1}^{n} f_i\left(S(T), I_i(T)\right) I_i(T) - \delta S(T) \tag{2.3}$$

then,

$$\dot{S}(T) = \Lambda - \sum_{i=1}^{n} f_i\left(0, I_i(T)\right) I_i(T) \tag{2.4}$$

However, from (1.2) we have

$$\dot{S}(T) = \Lambda > 0 \tag{2.5}$$

which presents a contradiction.

If $E_1(T) = 0$, before the other parameters $S, E_2, \ldots,$ $E_n, I_1, I_2, \ldots I_n, R$ become zero then,

$$\frac{dE_1(T)}{dt} = \lim_{t \to T^-} \frac{E_1(T) - E_1(t)}{T - t} = \lim_{t \to T^-} \frac{-E_1(t)}{T - t} \leq 0 \tag{2.6}$$

From the 2 nd equation of system (1.1), we have

$$\frac{dE_1(T)}{dt} = f_1\left(S, I_1\right) I_1 \tag{2.7}$$

However, from (1.2) and (1.3), $f_1\left(S, I_1\right) I_1$ is positive, then we will have

$$\frac{dE_1(T)}{dt} > 0 \tag{2.8}$$

Similar calculations are the same for $E_2(T) = 0, E_3(T) = 0, \ldots, E_n(T) = 0$.

If $I_1(T) = 0$, before the other variables $S, E_1, E_2, \ldots, E_n,$ $I_2, I_3, \ldots I_n, R$ become zero then

$$\frac{dI_1(T)}{dt} = \lim_{t\to T^-} \frac{I_1(T) - I_1(t)}{T-t} = \lim_{t\to T^-} \frac{-I_1(t)}{T-t} \leq 0 \tag{2.9}$$

From the third equation of system (1.1), we will have

$$\frac{dI_1(T)}{dt} = \gamma_1 E_1 \tag{2.10}$$

Since $\gamma_1 > 0$, we have

$$\frac{dI_1(T)}{dt} = \gamma_1 E_1 > 0 \tag{2.11}$$

This leads to contradiction.

Similar calculations are the same for $I_2(T) = 0, I_3(T) = 0, \ldots, I_n(T) = 0$.

If $R(T) = 0$, before the other variables $S, E_1, E_2, \ldots, E_n, I_1, I_2, \ldots I_n$, become zero then

$$\frac{dR(T)}{dt} = \lim_{t\to T^-} \frac{R(T) - R(t)}{T-t} = \lim_{t\to T^-} \frac{-R(t)}{T-t} \leq 0 \tag{2.12}$$

But from the fourth equation of system (1.1), we will have

$$\frac{dR(T)}{dt} = \sum_{i=1}^{n} \mu_i I_i(T) \tag{2.13}$$

Since $\mu_i > 0$, we have

$$\frac{dR(T)}{dt} > 0 \tag{2.14}$$

This leads to contradiction.

T is not finite. Hence, $S(t) \geq 0, E_1(t) \geq 0, E_2(t) \geq 0, \ldots, E_n(t) \geq 0, I_1(t) \geq 0, I_2(t) \geq 0, \ldots I_n(t) \geq 0, R(t) \geq 0$ for all $t > 0$. This proves to be positive.

Let's examine the case of limitedness. The total population becomes

$$N(t) = S(t) + \sum_{i=1}^{n} E_i(t) + \sum_{i=1}^{n} I_i(t) + R(t) \tag{2.15}$$

Using system (1.1),

$$\frac{dN(t)}{dt} = \Lambda - \delta N(t) \tag{2.16}$$

therefore,

$$N(t) = \frac{\Lambda}{\delta} + Ke^{-\delta t} \tag{2.17}$$

At $t = 0$, we have

$$N(0) = \frac{\Lambda}{\delta} + K \tag{2.18}$$

then

$$N(t) = \frac{\Lambda}{\delta} + \left(N(0) - \frac{\Lambda}{\delta}\right) e^{-\delta t} \tag{2.19}$$

As a result,

$$\lim_{t \to +\infty} N(t) = \frac{\Lambda}{\delta} \tag{2.20}$$

in that case

$$N(t) \leq \frac{\Lambda}{\delta} + N(0) \tag{2.21}$$

So it means that $N(t)$ is limited and hence $S(t), E_1(t), E_2(t), \ldots, E_n(t), I_1(t), I_2(t), \ldots I_n(t)$ and $R(t)$ are limited. The solution then means it exists globally for any $t > 0$.

□

3. Stability Analysis of the Model

In this subsection, we show that there is a disease free equilibrium point and $(2^n + 1)$ endemic equilibrium point. We can omit the fourth equation of the (1.1) system, since we know that the total population confirms (2.17) and is independent of R. So, the equation is written as:

$$\begin{aligned} \frac{dS}{dt} &= \Lambda - \sum_{i=1}^{n} f_i\left(S, I_i\right) I_i - \delta S \\ \frac{dE_i}{dt} &= f_i\left(S, I_i\right) I_i - \left(\gamma_i + \delta\right) E_i, \quad i = 1, 2, \ldots, n \\ \frac{dI_i}{dt} &= \gamma_i E_i - \left(\mu_i + \delta\right) I_i, \quad i = 1, 2, \ldots, n \end{aligned} \tag{3.1}$$

with $R = N - S - E_1 - E_2 - \ldots - E_n - I_1 - I_2 - \ldots - I_n$.

As always, the basic reproduction number can be written as the mean value of new cases of contamination caused by the contamination when the entire population is susceptible. We will use the definition of the next generation matrix to calculate the basic reproduction number.

The formula for the basic reproduction number:

$$R_0 = \rho\left(FV^{-1}\right),$$

The spectral radius ρ is the matrix of new non-negative infection cases and the matrix of the passage of infections associated with the V model (3.1).

$$F = \begin{bmatrix} 0 & f_i\left(\frac{\Lambda}{\delta}, 0\right) \\ 0 & 0 \end{bmatrix}, \quad V = \begin{bmatrix} \gamma_i + \delta & 0 \\ -\gamma_i & \mu_i + \delta \end{bmatrix}$$

$$FV^{-1} = \frac{1}{(\gamma_i+\delta)+(\mu_i+\delta)} \begin{bmatrix} 0 & f_i\left(\frac{\Lambda}{\delta},0\right) \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \mu_i+\delta & 0 \\ \gamma_i & \gamma_i+\delta \end{bmatrix}$$
$$= \frac{1}{(\gamma_i+\delta)+(\mu_i+\delta)} \begin{bmatrix} f_i\left(\frac{\Lambda}{\delta},0\right)\gamma_i & f_i\left(\frac{\Lambda}{\delta},0\right)(\gamma_i+\delta) \\ 0 & 0 \end{bmatrix}$$

We can write the basic reproduction number as follows:

$$R_0 = \max_{i\in\{1,2,\ldots,n\}} \left\{ \frac{f_i\left(\frac{\Lambda}{\delta},0\right)\gamma_i}{(\gamma_i+\delta)+(\mu_i+\delta)} \right\} \tag{3.2}$$

and ith basic reproduction number of species

$$R_0^i = \frac{f_i\left(\frac{\Lambda}{\delta},0\right)\gamma_i}{(\gamma_i+\delta)+(\mu_i+\delta)}, \quad i=1,2,\ldots,n \tag{3.3}$$

.

Theorem 3.1. *The problem (3.1) has the disease-free equilibrium E_f, endemic equilibrium ε_{S_t} and ε_{S_i}, $i=1,2,...,n$ with respect to the i th strain. Moreover, we have*
-When $R_0^i > 1$, ith strain endemic equilibrium point ε_{S_i} exists,
-When $R_0^1, R_0^2, \ldots, R_0^n > 1$, endemic equilibrium point ε_{S_t} exists.

Proof. In order to examine the stability states of the (3.1) system, we must analyze the following equations.

$$\Lambda - \sum_{i=1}^{i=n} f_i(S,I_i)\,I_i - \delta S = 0 \tag{3.4}$$
$$f_i(S,I_i)\,I_i - (\gamma_i+\delta)\,E_i = 0, \qquad i=1,2,\ldots,n \tag{3.5}$$
$$\gamma_i E_i - (\mu_i+\delta)\,I_i = 0, \qquad i=1,2,\ldots,n \tag{3.6}$$

-We find the disease-free equilibrium point when $I_i = 0, i=1,2,\ldots,n$.

$$\varepsilon_f = \left(\frac{\Lambda}{\delta}, 0, \ldots, 0\right)$$

-When $I_i \neq 0, I_1 = 0, I_2 = 0 \ldots, I_n = 0$, we find the ith type endemic equilibrium point.

$$\varepsilon_{S_i} = \Bigg(S_i^*, 0, 0, \ldots, \frac{1}{(\gamma_i+\delta)}(\Lambda - \delta S_i^*), 0, 0, \ldots, 0, \frac{\gamma_i}{(\gamma_i+\delta)(\mu_i+\delta)}(\Lambda - \delta S_i^*) \Bigg)$$

where $S_i^* \in \left[0, \frac{\Lambda}{\delta}\right]$.
Define also a function Ω on $[0,+\infty[$ as follows

$$\Omega(S) = f_i\left(S, \frac{\gamma_i}{(\gamma_i+\delta)(\mu_i+\delta)}(\Lambda - \delta S)\right) - \frac{(\gamma_i+\delta)(\mu_i+\delta)}{\gamma_i} \tag{3.7}$$

$$\frac{\partial \Omega(S)}{\partial S} = \frac{\partial f_i(S, I_i)}{\partial S} + \frac{\partial f_i(S, I_i)}{\partial I_i}\left(-\frac{\delta\gamma_i}{(\gamma_i+\delta)(\mu_i+\delta)}\right) \tag{3.8}$$

Using the conditions (1.3) and (1.4) we conclude that

$$\frac{\partial \Omega(S)}{\partial S} \geq 0 \tag{3.9}$$

However,

$$\Omega(0) = f_i\left(0, I^*_{i,S_i}\right) - \frac{(\gamma_i+\delta)(\mu_i+\delta)}{\gamma_i} = -\frac{(\gamma_i+\delta)(\mu_i+\delta)}{\gamma_i} < 0$$

So, for $R_0^i > 1$, we have

$$\Omega\left(\frac{\Lambda}{\delta}\right) = f_i\left(\frac{\Lambda}{\delta}, 0\right) - \frac{(\gamma_i+\delta)(\mu_i+\delta)}{\gamma_i} = -\frac{(\gamma_i+\delta)(\mu_i+\delta)}{\gamma_i}\left(R_0^i - 1\right) > 0 \tag{3.10}$$

Thus, there exists a unique i th species endemic equilibrium point.

$$\varepsilon_{S_i} = \left(S_i^*, E^*_{1,S_i}, E^*_{2,S_i}, \ldots, E^*_{n,S_i}, I^*_{1,S_i}, I^*_{2,S_i}, \ldots, I^*_{n,S_i}\right) \tag{3.11}$$

with $S_i^* \in \left]0, \frac{\Lambda}{\delta}\right[, E^*_{i,S_i} > 0, I^*_{i,S_i} > 0$ ve $E^*_{1,S_i} = \ldots = E^*_{n,S_i} = I^*_{1,S_i} = \ldots = I^*_{n,S_i} = 0$.

-When $I_1 \neq 0, I_2 \neq 0 \ldots, I_n \neq 0$, We find the 3 rd endemic equilibrium point.

$$\varepsilon_{S_t} = \left(S_t^*, E^*_{1,t}, E^*_{2,t}, \ldots,, E^*_{n,t}, I^*_{1,t}, I^*_{2,t}, \ldots,, I^*_{n,t}\right) \tag{3.12}$$

where

$$E^*_{1,t} = \frac{\mu_1+\delta}{\gamma_1} I^*_{1,t}, \quad E^*_{2,t} = \frac{\mu_2+\delta}{\gamma_2} I^*_{2,t}, \quad \ldots \quad , \quad E^*_{n,t} = \frac{\mu_n+\delta}{\gamma_n} I^*_{n,t} \tag{3.13}$$

$$S_t^* = \frac{1}{\delta}\left[\Lambda - \frac{f_i\left(\frac{\Lambda}{\delta}, 0\right)}{R_0^i} I^*_{i,t}\right], \quad i = 1, 2, \ldots, n \tag{3.14}$$

with $\Lambda \geq \frac{f_i\left(\frac{\Lambda}{\delta}, 0\right)}{R_0^i} I^*_{i,t}, R_0^1, R_0^2, \ldots, R_0^n > 1$.

□

4. Global Stability of Equilibrium Points

4.1. **Global Stability of Disease Free Equilibrium Points.**

Theorem 4.1. *When $R_0 \leq 1$, then the disease-free equilibrium ε_f is globally asymptotically stable.*

Proof. First, we consider the following Lyapunov function in $\mathbb{R}_+^{2n-1}$:

$$\begin{aligned}
&\mathcal{L}_f\left(S, E_1, E_2, \ldots, E_n, I_1, I_2, \ldots, I_n\right)\\
&\quad = S - S_0^* - \int_{S_0^*}^{S} \frac{f\left(S_0^*, 0\right)}{f(X,0)} dX + E_1 + E_2 + \ldots +\\
&\quad E_n + \frac{\gamma_1+\delta}{\gamma_1} I_1 + \frac{\gamma_2+\delta}{\gamma_2} I_2 + \ldots + \frac{\gamma_n+\delta}{\gamma_n} I_n
\end{aligned} \tag{4.1}$$

If we take the derivative of both sides of (4.1) depending on t (time):

$$\begin{aligned}
&\dot{\mathcal{L}}_f\left(S, E_1, E_2, \ldots, E_n, I_1, I_2, \ldots, I_n\right)\\
&\quad = \dot{S} - \frac{f\left(S_0^*, 0\right)}{f(X,0)} \dot{S} + \dot{E}_1 + \dot{E}_2 + \ldots + \dot{E}_n\\
&\quad + \frac{\gamma_1+\delta}{\gamma_1} \dot{I}_1 + \frac{\gamma_2+\delta}{\gamma_2} \dot{I}_2 + \ldots + \frac{\gamma_n+\delta}{\gamma_n} I_n\\
&\quad = \delta S_0^* \left(1 - \frac{S}{S_0^*}\right)\left(1 - \frac{f\left(S_0^*, 0\right)}{f(X,0)}\right)\\
&\quad + \frac{\left(\gamma_1+\delta\right)\left(\mu_1+\delta\right)}{\gamma_1} I_1 \left(\frac{f\left(S, I_1\right)}{f(S,0)} R_0^1 - 1\right) &(4.2)\\
&\quad + \frac{\left(\gamma_2+\delta\right)\left(\mu_2+\delta\right)}{\gamma_2} I_2 \left(\frac{f\left(S_0^*, 0\right)}{f(S,0)} \frac{\tilde{f}\left(S, I_2\right)}{\tilde{f}\left(S_0^*, 0\right)} R_0^2 - 1\right) &(4.3)\\
&\quad \vdots\\
&\quad + \frac{\left(\gamma_n+\delta\right)\left(\mu_n+\delta\right)}{\gamma_n} I_n \left(\frac{f\left(S_0^*, 0\right)}{f(S,0)} \frac{f\left(S, I_n\right)}{f\left(S_0^*, 0\right)} R_0^n - 1\right) &(4.4)\\
&\quad \leq \delta S_0^* \left(1 - \frac{S}{S_0^*}\right)\left(1 - \frac{f\left(S_0^*, 0\right)}{f(X,0)}\right)\\
&\quad + \frac{\left(\gamma_1+\delta\right)\left(\mu_1+\delta\right)}{\gamma_1} I_1 \left(R_0^1 - 1\right)\\
&\quad + \frac{\left(\gamma_2+\delta\right)\left(\mu_2+\delta\right)}{\gamma_2} I_2 \left(\frac{f\left(S_0^*, 0\right)}{f(S,0)} \frac{\tilde{f}\left(S, I_2\right)}{\tilde{f}\left(S_0^*, 0\right)} R_0^2 - 1\right) &(4.5)\\
&\quad \vdots\\
&\quad + \frac{\left(\gamma_n+\delta\right)\left(\mu_n+\delta\right)}{\gamma_n} I_n \left(\frac{f\left(S_0^*, 0\right)}{f(S,0)} \cdots \frac{\tilde{f}\left(S, I_n\right)}{\tilde{f}\left(S_0^*, 0\right)} R_0^n - 1\right) &(4.6)
\end{aligned}$$

We will discuss 2 cases:

- If $S \leq S_0^*$ using the condition (1.3), we will have $\frac{\tilde{f}(S,I_i)}{\tilde{f}(S_0^*,0)} \leq 1$, $i = 1, 2, ..., n$, then

$$\begin{aligned}
&\dot{\mathcal{L}}_f\left(S, E_1, E_2, \ldots, E_n, I_1, I_2, \ldots, I_n\right) \\
&\quad \leq \delta S_0^*\left(1 - \frac{S}{S_0^*}\right)\left(1 - \frac{f\left(S_0^*, 0\right)}{f(X, 0)}\right) \\
&\quad + \frac{\left(\gamma_1+\delta\right)\left(\mu_1+\delta\right)}{\gamma_1} I_1\left(R_0^1 - 1\right) \\
&\quad + \frac{\left(\gamma_2+\delta\right)\left(\mu_2+\delta\right)}{\gamma_2} I_2\left(\frac{f\left(S_0^*, 0\right)}{f(S, 0)} R_0^2 - 1\right) \qquad (4.7)\\
&\vdots \\
&\quad + \frac{\left(\gamma_n+\delta\right)\left(\mu_n+\delta\right)}{\gamma_n} I_n\left(\frac{f\left(S_0^*, 0\right)}{f(S, 0)} R_0^n - 1\right) \qquad (4.8)
\end{aligned}$$

Since $R_0^i \leq \frac{f(S,0)}{f(S_0^*,0)} \leq 1$, we obtain

$$\frac{f\left(S_0^*, 0\right)}{f(S, 0)} R_0^i - 1 \leq 0 \qquad (4.9)$$

Otherwise,
$1 - \frac{f(S_0^*,0)}{f(S,0)} \leq 0$, therefore

$$\delta S_0^*\left(1 - \frac{S}{S_0^*}\right)\left(1 - \frac{f\left(S_0^*, 0\right)}{f(S, 0)}\right) \leq 0. \qquad (4.10)$$

- If $S_0^* < S$, using the condition (1.3), we will have $\frac{f(S,I_i)}{f\left(S_0^*,0\right)} > 1, i = 1, 2, \ldots, n$ and $\frac{f(S_0^*,0)}{f(S,0)} < 1$ then.

$$\begin{aligned}
&\dot{\mathcal{L}}_f\left(S, E_1, E_2, \ldots, E_n, I_1, I_2, \ldots, I_n\right) \\
&\quad \leq \delta S_0^*\left(1 - \frac{S}{S_0^*}\right)\left(1 - \frac{f\left(S_0^*, 0\right)}{f(X, 0)}\right) \\
&\quad + \frac{\left(\gamma_1+\delta\right)\left(\mu_1+\delta\right)}{\gamma_1} I_1\left(R_0^1 - 1\right) \\
&\quad + \frac{\left(\gamma_2+\delta\right)\left(\mu_2+\delta\right)}{\gamma_2} I_2\left(\frac{\tilde{f}\left(S_0^*, 0\right)}{\tilde{f}(S, 0)} R_0^2 - 1\right) \qquad (4.11)\\
&\vdots \\
&\quad + \frac{\left(\gamma_n+\delta\right)\left(\mu_n+\delta\right)}{\gamma_n} I_n\left(\frac{\tilde{f}\left(S_0^*, 0\right)}{f(S, 0)} R_0^n - 1\right) \qquad (4.12)
\end{aligned}$$

Since $R_0^i < \frac{\tilde{f}(S_0^*,0)}{f(S,I_i)} < 1$, we obtain

$$\frac{\tilde{f}\,(S, I_i)}{\tilde{f}\,(S_0^*, 0)} R_0^i - 1 < 0. \tag{4.13}$$

Otherwise,
$\frac{f(S_0^*,0)}{f(S,0)} < 1$, therefore

$$\delta S_0^* \left(1 - \frac{S}{S_0^*}\right)\left(1 - \frac{f\,(S_0^*, 0)}{f(S,0)}\right) \leq 0. \tag{4.14}$$

By the above discussion, we deduce that, if $R_0^i \leq 1, i = 1, 2, \ldots, n$((which means that $R_0 \leq 1$), then

$$\dot{\mathcal{L}}_f\,(S, E_1, E_2, \ldots, E_n, I_1, I_2, \ldots, I_n) \leq 0. \tag{4.15}$$

Thus, the disease free equilibrium point is globally asymptotically stable when $\varepsilon_f, R_0 \leq 1$.

□

4.2. **Global Stability of i th Species Equilibrium Points.** For the global stability of ε_{S_i}, we assume that the function $\tilde{f}_i$ satisfies the following condition:

$$(1 - \Gamma)\left(\frac{1}{\Gamma} - \frac{I_i}{I_{i,S_i}^*}\right) \leq 0, \quad \forall S, I_i > 0, \quad i = 1, 2, \ldots, n \tag{4.16}$$

with $\Gamma = \prod_{i=1}^{i=n} \frac{f_i\left(S, I_{i,S_i}^*\right)}{f_i\left(S_i^*, I_{i,S_i}^*\right)} \frac{f_j\left(S_j^*, I_{j,S_j}^*\right)}{f_j\left(S, I_{j,S_j}^*\right)}$ such that $I_{i,S_i}^* \neq 0, I_{j,S_j}^* \neq 0, i \in 1, 2, \ldots, n, j \in 1, 2, \ldots, n$ and $i \neq j$.

Theorem 4.2. *When $R_0^k \leq 1 < R_0^i$, $k \neq i$, the i. species endemic equilibra points ϵ_{S_i} are globally asymptotically stable.*

Proof. First, we consider the Lyapunov function $\mathcal{L}_i$ defined by:

$$\begin{aligned}
&\mathcal{L}_i\,(S, E_1, E_2, \ldots, E_n, I_1, I_2, \ldots, I_n) \\
&\quad = S - S_i^* - \int_{S_i^*}^{S} \frac{f\,(S_{S_i}^*, I_{i,S_i}^*)}{f\left(X, I_{i,S_i}^*\right)} dX \\
&\quad + \sum_{i=1}^{n} E_{i,S_i}^* \left(\frac{E_i}{E_{i,S_i}^*} - \ln\left(\frac{E_i}{E_{i,S_i}^*}\right) - 1\right) \\
&\quad + \sum_{i=1}^{n} \frac{\gamma_i + \delta}{\gamma_i} I_{i,S_i}^* \left(\frac{I_i}{I_{i,S_i}^*} - \ln\left(\frac{I_i}{I_{i,S_i}^*}\right) - 1\right)
\end{aligned} \tag{4.17}$$

If we take the derivative of both sides of equation (4.17) depending on t (time):

$$\begin{aligned}
&\dot{\mathcal{L}}_i\left(S, E_1, E_2, \ldots, E_n, I_1, I_2, \ldots, I_n\right)\\
&\quad=\left(\Lambda-f_i\left(S, I_i\right) I_i-\delta S\right)\left(1-\frac{f\left(S_i^*, I_{i, S_i}^*\right)}{f\left(S, I_{i, S_i}^*\right)}\right)\\
&\quad+\left(f_i\left(S, I_i\right) I_i-\left(\gamma_i+\delta\right) E_i\right)\left(1-\frac{E_{i, S_i}^*}{E_i}\right)\\
&\quad+\frac{\gamma_i+\delta}{\gamma_i}\left(\gamma_i E_i-\left(\mu_i+\delta\right) I_i\right)\left(1-\frac{I_i^*, S_i}{I_i}\right) \qquad (4.18)
\end{aligned}$$

We have

$$\begin{aligned}
&\Lambda=\delta S_i^*+f\left(S_i^*, I_{i, S_i}^*\right) I_{i, S_i}^*\\
&f\left(S_i^*, I_{i, S_i}^*\right) I_{i, S_i}^*=\frac{\left(\gamma_i+\delta\right)\left(\mu_i+\delta\right)}{\gamma_i} I_{i, S_i}^*=\left(\gamma_i+\delta\right) E_{i, S_i}^* \qquad (4.19)\\
&\frac{E_{i, S_i}^*}{I_{i, S_i}^*}=\frac{\mu_i+\delta}{\gamma_i}
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\dot{\mathcal{L}}_i\left(S, E_1, E_2, \ldots, E_n, I_1, I_2, \ldots, I_n\right)\\
&\quad=\delta S_i^*\left(1-\frac{f\left(S_i^*, I_{i, S_i}^*\right)}{f\left(S, I_{i, S_i}^*\right)}\right)\\
&\quad+f\left(S_i^*, I_{i, S_i}^*\right) I_{i, S_i}^*-f\left(S_i^*, I_{i, S_i}^*\right) I_{i, S_i}^* \frac{f\left(S_i^*, I_{i, S_i}^*\right)}{f\left(S, I_{i, S_i}^*\right)}\\
&\quad-f_i\left(S, I_i\right) I_i+f_i\left(S, I_i\right) I_i \frac{f\left(S_i^*, I_{i, S_i}^*\right)}{f\left(S, I_{i, S_i}^*\right)}\\
&\quad-\delta S\left(1-\frac{f\left(S_i^*, I_{i, S_i}^*\right)}{f\left(S, I_{i, S_i}^*\right)}\right)\\
&\quad+f_i\left(S, I_i\right) I_i-\left(\gamma_i+\delta\right) E_i-f_i\left(S, I_i\right) \frac{I_i E_{i, S_i}^*}{E_i}+\left(\gamma_i+\delta\right) E_i^*\\
&\quad+(\gamma_i+\delta)E_i-\frac{(\gamma_i+\delta)(\mu_i+\delta)}{\gamma_i} I_i-(\gamma_i+\delta)\frac{E_i I_{i, S_i}^*}{I_i}\\
&\quad+\frac{\left(\gamma_i+\delta\right)\left(\mu_i+\delta\right)}{\gamma_i} I_{i, S_i}^* \qquad (4.20)
\end{aligned}$$

Then,

$$
\begin{aligned}
&\dot{\mathcal{L}}_i \left(S, E_1, E_2, \ldots, E_n, I_1, I_2, \ldots, I_n\right)\\
&= \left(\gamma_i + \delta\right) E^*_{i,S_i} \left[4 - \frac{\left(\gamma_i + \delta\right) E^*_{i,S_i}}{\left[f\left(S, I^*_{i,S_i}\right) f\left(S, I^*_{j,S_j}\right)\right] I^*_{i,S_i}} \right.\\
&\quad \left. - \frac{\left[f\left(S, I_i\right) f\left(S, I_j\right)\right] I_i}{\left(\gamma_i + \delta\right) E_i} - \frac{I^*_{i,S_i} E_i}{I_i E^*_{i,S_i}} - \frac{f\left(S, I^*_{i,S_i}\right)}{f\left(S, I_i\right)} \frac{f\left(S, I^*_{j,S_j}\right)}{f\left(S, I_j\right)} \right]\\
&\quad + \left(\gamma_i + \delta\right) E^*_{i,S_i} \left[\frac{f\left(S, I^*_{i,S_i}\right)}{f\left(S, I_i\right)} \frac{f\left(S, I^*_{j,S_j}\right)}{f\left(S, I_j\right)} \right.\\
&\quad \left. + \frac{f\left(S, I_i\right)}{f\left(S, I^*_{i,S_i}\right)} \frac{f\left(S, I_j\right)}{f\left(S, I^*_{j,S_j}\right)} \frac{I_i}{I^*_{i,S_i}} - \frac{I_i}{I^*_{i,S_i}} - 1 \right]\\
&\quad + \frac{f\left(S^*_i, I^*_{i,S_i}\right)}{f\left(S, I^*_{i,S_i}\right)} \frac{f\left(S^*_j, I^*_{j,S_j}\right)}{f\left(S, I^*_{j,S_j}\right)} \left[\sum_{i=1}^{n} f_i\left(S, I_i\right) I_i \right]\\
&\quad - \sum_{i=1}^{n} \frac{\left(\gamma_i + \delta\right)\left(\mu_i + \delta\right)}{\gamma_i} I_i
\end{aligned}
\tag{4.21}
$$

$$
\begin{aligned}
&= \left(\gamma_i + \delta\right) E^*_{i,S_i} \left[4 - \frac{\left(\gamma_i + \delta\right) E^*_{i,S_i}}{\left[f\left(S, I^*_{i,S_i}\right) f\left(S, I^*_{j,S_j}\right)\right] I^*_{i,S_i}} \right.\\
&\quad \left. - \frac{\left[f\left(S, I_i\right) f\left(S, I_j\right)\right] I_i}{\left(\gamma_i + \delta\right) E_i} - \frac{I^*_{i,S_i} E_i}{I_i E^*_{i,S_i}} - \frac{f\left(S, I^*_{i,S_i}\right)}{f\left(S, I_i\right)} \frac{f\left(S, I^*_{j,S_j}\right)}{f\left(S, I_j\right)} \right]\\
&\quad + \left(\gamma_i + \delta\right) E^*_{i,S_i} \left[\frac{f\left(S, I^*_{i,S_i}\right)}{f\left(S, I_i\right)} \frac{f\left(S, I^*_{j,S_j}\right)}{f\left(S, I_j\right)} \right.\\
&\quad \left. + \frac{f\left(S, I_i\right)}{f\left(S, I^*_{i,S_i}\right)} \frac{f\left(S, I_j\right)}{f\left(S, I^*_{j,S_j}\right)} \frac{I_i}{I^*_{i,S_i}} - \frac{I_i}{I^*_{i,S_i}} - 1 \right]\\
&\quad + \frac{\left(\gamma_i + \delta\right)\left(\mu_i + \delta\right)}{\gamma_i} I_i\\
&\quad \left[\sum_{i=1}^{n} \frac{f\left(S^*_i, I^*_{i,S_i}\right)}{f\left(S, I^*_{i,S_i}\right)} \frac{f\left(S^*_j, I^*_{j,S_j}\right)}{f\left(S, I^*_{j,S_j}\right)} \frac{f_i\left(S, I_i\right) I_i}{f_i\left(S^*_0, 0\right)} R^i_0 - 1 \right]
\end{aligned}
\tag{4.22}
$$

Using the condition (1.3) and (1.4), we have $\frac{f_i(S,I_i)}{f\left(S^*_0,0\right)} \leq 1$, then,

$$
\begin{aligned}
&\dot{\mathcal{L}}_i\left(S, E_1, E_2, \ldots, E_n, I_1, I_2, \ldots, I_n\right) \leq \left(\gamma_i+\delta\right) E_{i,S_i}^* \\
&\left[4-\frac{\left(\gamma_i+\delta\right) E_{i,S_i}^*}{\left[f\left(S, I_{i,S_i}^*\right) f\left(S, I_{j,S_j}^*\right)\right] I_{i,S_i}^*}-\frac{\left[f\left(S, I_i\right) f\left(S, I_j\right)\right] I_i}{\left(\gamma_i+\delta\right) E_i}\right. \\
&\left.-\frac{I_{i,S_i}^*, E_i}{I_i E_{i,S_i}^*}-\frac{f\left(S, I_{i,S_i}^*\right)}{f\left(S, I_i\right)} \frac{f\left(S, I_{j,S_j}^*\right)}{f\left(S, I_j\right)}\right] \\
&+\left(\gamma_i+\delta\right) E_{i,S_i}^*\left[\frac{f\left(S, I_{i,S_i}^*\right)}{f\left(S, I_i\right)} \frac{f\left(S, I_{j,S_j}^*\right)}{f\left(S, I_j\right)}\right. \\
&\left.+\frac{f\left(S, I_i\right)}{f\left(S, I_{i,S_i}^*\right)} \frac{f\left(S, I_j\right)}{f\left(S, I_{j,S_j}^*\right)} \frac{I_i}{I_{i,S_i}^*}-\frac{I_i}{I_{i,S_i}^*}-1\right] \\
&+\sum_{i=1}^n\left[\frac{\left(\gamma_i+\delta\right)\left(\mu_i+\delta\right)}{\gamma_i} I_i\right. \\
&\left.\left(\frac{f\left(S_i^*, I_{i,S_i}^*\right)}{f\left(S, I_{i,S_i}^*\right)} \frac{f\left(S_j^*, I_{j,S_j}^*\right)}{f\left(S, I_{j,S_j}^*\right)} R_0^i-1\right)\right]
\end{aligned} \tag{4.23}
$$

from (4.16)

$$
\begin{aligned}
&\left[\frac{f\left(S, I_{i,S_i}^*\right)}{f\left(S, I_i\right)} \frac{f\left(S, I_{j,S_j}^*\right)}{f\left(S, I_j\right)}+\frac{f\left(S, I_i\right)}{f\left(S, I_{i,S_i}^*\right)} \frac{f\left(S, I_j\right)}{f\left(S, I_{j,S_j}^*\right)} \frac{I_i}{I_{i,S_i}^*}\right. \\
&\left.-\frac{I_i}{I_{i,S_i}^*}-1\right]=(1-\Gamma)\left(\frac{1}{\Gamma}-\frac{I_i}{I_{i,S_i}^*}\right) \leq 0
\end{aligned}
$$

By the relation between arithmetic and geometric means, we have

$$
\begin{aligned}
&4-\frac{\left(\gamma_i+\delta\right) E_{i,S_i}^*}{\left[f\left(S, I_{i,S_i}^*\right) f\left(S, I_{j,S_j}^*\right)\right] I_{i,S_i}^*}-\frac{\left[f\left(S, I_i\right) f\left(S, I_j\right)\right] I_i}{\left(\gamma_i+\delta\right) E_i} \\
&-\frac{I_{i,S_i}^* E_i}{I_i E_{i,S_i}^*}-\frac{f\left(S, I_{i,S_i}^*\right)}{f\left(S, I_i\right)} \frac{f\left(S, I_{j,S_j}^*\right)}{f\left(S, I_j\right)} \leq 0
\end{aligned} \tag{4.24}
$$

We discuss 2 cases:

- If $S_i^* \leq S$, from the condition (1.3), we will have $\frac{f\left(S_i^*, I_{i,S_i}^*\right)}{f\left(S, I_{i,S_i}^*\right)} \frac{f\left(S_j^*, I_{j,S_j}^*\right)}{f\left(S, I_{j,S_j}^*\right)} \leq 1$, we obtain, for $\left(\frac{f\left(S_i^*, I_{i,S_i}^*\right)}{f\left(S, I_{i,S_i}^*\right)} \frac{f\left(S_j^*, I_{j,S_j}^*\right)}{f\left(S, I_{jS_j}^*\right)} R_0^k-1\right) \leq 1$ $\Rightarrow R_0^k \leq 1$, the following $\dot{\mathcal{L}}_i \leq 0$.

- If $S \leq S_i^*$ from the condition (1.3), we will have
  $\frac{f\left(S_i^*,I_{i,S_i}^*\right)}{f\left(S,I'_{i,S_i}\right)} \frac{f\left(S_j^*,I_{j,S}^*\right)}{f\left(S,I_{j,S_j}^*\right)} \geq 1$, we obtain, for,
  $R_0^k \leq \frac{f\left(S,I_{i,S_i}^*\right)}{f\left(S_i^*,I_{i,S_i}^*\right)} \frac{f\left(S,I_{j,S_j}^*\right)}{f\left(S_j^*,I_{j,S_j}^*\right)} \leq 1 \Rightarrow \frac{f\left(S_i^*,I_{i,S_i}^*\right)}{f\left(S,I_{i,S_i}^*\right)} \frac{f\left(S_j^*,I_{j,S_j}^*\right)}{f\left(S,I_{j,S_j}^*\right)} R_0^k - 1 \leq 0$ the following $\dot{\mathcal{L}}_i \leq 0$.

  By the above discussion, we deduce that if $R_0^k \leq 1$, then $\dot{\mathcal{L}}_i \leq 0$.
  ith endemic equilibrium point ε_{S_i} is globally asymptotically stable when $R_0^k \leq 1 < R_0^i$, $k \neq i$.

□

4.3. **Global Stability of The Last Endemic Equilibrium Points.** For the global stability analysis of ε_{S_t} of the last endemic equilibrium point, suppose the functions f and $\tilde{f}$ satisfy the following condition:

$$\left(1 - \frac{\tilde{f}\left(S, I_2\right)}{\tilde{f}\left(S_1^*, I_{2,t}^*\right)} \frac{f\left(S_t^*, I_{1,t}^*\right)}{f\left(S, I_{1,t}^*\right)}\right) \left(\frac{\tilde{f}\left(S_1^*, I_{2,t}^*\right)}{\tilde{f}\left(S, I_2\right)} \frac{f\left(S, I_{1,t}^*\right)}{f\left(S_t^*, I_{1,t}^*\right)} - \frac{I_2}{I_{2,t}^*}\right) \leq 0 \tag{4.25}$$

$\forall S, I_1, I_2, \ldots, I_n > 0$

Theorem 4.3. *When $R_0^1, R_0^2, \ldots, R_0^n > 1$, the endemic equilibrium point ε_{S_t} is globally asymptotically stable.*

Proof. First, we consider the Lyapunov function $\mathcal{L}_3$ defined by:

$$
\begin{aligned}
&\mathcal{L}_3\left(S, E_1, E_2, \ldots, E_n, I_1, I_2, \ldots, I_n\right)\\
&= S - S_t^* - \int_{S_i^*}^{S} \frac{f\left(S_t^*, I_{1,t}^*\right)}{f\left(X, I_{1,t}^*\right)} dX\\
&+ E_{1,t}^*\left(\frac{E_1}{E_{1,t}^*} - \ln\left(\frac{E_1}{E_{1,t}^*}\right) - 1\right)\\
&+ E_{2,t}^*\left(\frac{E_2}{E_{2,t}^*} - \ln\left(\frac{E_2}{E_{2,t}^*}\right) - 1\right)\\
&+\\
&\vdots\\
&+ E_{n,t}^*\left(\frac{E_n}{E_{n,t}^*} - \ln\left(\frac{E_n}{E_{n,t}^*}\right) - 1\right)\\
&+ \frac{\gamma_1 + \delta}{\gamma_1} I_{1,t}^*\left(\frac{I_1}{I_{1,t}^*} - \ln\left(\frac{I_1}{I_{1,t}^*}\right) - 1\right)\\
&+ \frac{\gamma_2 + \delta}{\gamma_2} I_{2,t}^*\left(\frac{I_2}{I_{2,t}^*} - \ln\left(\frac{I_2}{I_{2,t}^*}\right) - 1\right)\\
&+\\
&\vdots\\
&+ \frac{\gamma_n + \delta}{\gamma_n} I_{n,t}^*\left(\frac{I_n}{I_{n,t}^*} - \ln\left(\frac{I_n}{I_{n,t}^*}\right) - 1\right)
\end{aligned}
\tag{4.26}
$$

If we take the derivative of both sides of the equation (4.26) depending on t (time):

$$
\begin{aligned}
&\mathcal{L}_3\left(S, E_1, E_2, \ldots, E_n, I_1, I_2, \ldots, I_n\right) \\
&\quad = \left(1 - \frac{f\left(S_t^*, I_{1,t}^*\right)}{f\left(S, I_{1,t}^*\right)}\right) \dot{S} \\
&\quad + \left(1 - \frac{E_{1,t}^*}{E_1}\right) \dot{E}_1 \\
&\quad + \\
&\quad \vdots \\
&\quad + \left(1 - \frac{E_{n,t}^*}{E_n}\right) \dot{E}_n \\
&\quad + \frac{\gamma_1 + \delta}{\gamma_1}\left(1 - \frac{I_{1,t}^*}{I_1}\right) \dot{I}_1 \\
&\quad + \\
&\quad \vdots \\
&\quad + \frac{\gamma_n + \delta}{\gamma_n}\left(1 - \frac{I_{n,t}^*}{I_n}\right) \dot{I}_n
\end{aligned} \tag{4.27}
$$

It is easy to verify that

$$
\begin{aligned}
&\Lambda = \delta S_t^* + f\left(S_t^*, I_{1,t}^*\right) I_{1,t}^* + \tilde{f}\left(S_t^*, I_{2,t}^*\right) I_{2,t}^* \\
&f\left(S_t^*, I_{1,t}^*\right) I_{1,t}^* = \left(\gamma_1 + \delta\right) E_{1,t}^* \\
&\tilde{f}\left(S_t^*, I_{2,t}^*\right) I_{2,t}^* = \left(\gamma_2 + \delta\right) E_{2,t}^* \\
&\frac{E_{1,t}^*}{I_{1,t}^*} = \frac{\mu_1 + \delta}{\gamma_1} \\
&\frac{E_{2,t}^*}{I_{2,t}^*} = \frac{\mu_2 + \delta}{\gamma_2}
\end{aligned} \tag{4.28}
$$

Consequently,

$$
\begin{aligned}
&\dot{\mathcal{L}}_3\left(S, E_1, E_2, \ldots, E_n, I_1, I_2, \ldots, I_n\right) \\
&= \delta S_t^*\left(1-\frac{S}{S_t^*}\right)\left(1-\frac{f\left(S_t^*, I_{1,t}^*\right)}{f\left(S, I_{1,t}^*\right)}\right) \\
&\left(\gamma_1+\delta\right) E_{1,t}^*\left[4-\frac{\left(\gamma_1+\delta\right) E_{1,t}^*}{f\left(S, I_{1,t}^*\right) I_{1,t}^*}\right. \\
&\left.-\frac{f\left(S, I_1\right) I_1}{\left(\gamma_1+\delta\right) E_1}-\frac{I_{1,t}^* E_1}{I_1 E_{1,t}^*}-\frac{f\left(S, I_{1,t}^*\right)}{f\left(S, I_1\right)}\right] \\
&\vdots \\
&+\left(\gamma_2+\delta\right) E_{2,t}^*\left[\frac{\tilde{f}\left(S_t^*, I_{2,t}^*\right)}{\tilde{f}\left(S, I_2\right)} \frac{f\left(S, I_{1,t}^*\right)}{f\left(S_t^*, I_{1,t}^*\right)}\right. \\
&\left.+\frac{f\left(S, I_2\right)}{\tilde{f}\left(S_t^*, I_{2,t}^*\right)} \frac{I_2}{I_{2,t}^*}-\frac{I_2}{I_{2,t}^*}-1\right] \\
&+\frac{\tilde{f}\left(S, I_2\right)}{\tilde{f}\left(S_t^*, I_{2,t}^*\right)} \frac{f\left(S_t^*, I_{1,t}^*\right)}{f\left(S, I_{1,t}^*\right)}\left[f_3\left(S, I_3\right) I_3+\ldots+f_n\left(S, I_n\right) I_n\right] \\
&-\left[\left(\gamma_3+\delta\right) E_3+\ldots+\left(\gamma_n+\delta\right) E_n\right]
\end{aligned}
\tag{4.29}
$$

With the help of the following inequality

$$
1-\frac{f\left(S_t^*, I_{1,t}^*\right)}{f\left(S, I_{1,t}^*\right)} \geq 0 \quad \text{for} \quad S \geq S_t^* \tag{4.30}
$$

$$
1-\frac{f\left(S_t^*, I_{1,t}^*\right)}{f\left(S, I_{1,t}^*\right)}<0 \quad \text{for} \quad S<S_t^* \tag{4.31}
$$

Using the inequalities (4.30) and (4.31) then

$$
\left(1-\frac{S}{S_t^*}\right)\left(1-\frac{f\left(S_t^*, I_{1,t}^*\right)}{f\left(S, I_{1,t}^*\right)}\right) \leq 0 \tag{4.32}
$$

by the relation between arithmetic and geometric means, we have

$$
\left[4-\frac{\left(\gamma_1+\delta\right) E_{1,t}^*}{f\left(S, I_{1,t}^*\right) I_{1,t}^*}-\frac{f\left(S, I_1\right) I_1}{\left(\gamma_1+\delta\right) E_1}-\frac{I_{1,t}^* E_1}{I_1 E_{1,t}^*}-\frac{f\left(S, I_{1,t}^*\right)}{f\left(S, I_1\right)}\right] \leq 0 \tag{4.33}
$$

$$
\begin{aligned}
&\left[4-\frac{f\left(S_t^*, I_{1,t}^*\right)}{f\left(S, I_{1,t}^*\right)}-\frac{\tilde{f}\left(S, I_2\right) I_2}{\left(\gamma_2+\delta\right) E_2}-\frac{I_{2,t}^* E_2}{I_2 E_{2,t}^*}\right. \\
&\left.-\frac{\left(\gamma_2+\delta\right) E_{2,t}^* f\left(S, I_{1, S_{1,t}}^*\right)}{\tilde{f}\left(S, I_2\right) f\left(S_t^*, I_{1,t}^*\right) I_{2,t}^*}\right] \leq 0
\end{aligned}
\tag{4.34}
$$

from (4.16) we have

$$\left[\frac{f\left(S, I_{1,t}^{*}\right)}{f\left(S, I_{1}\right)}+\frac{f\left(S, I_{1}\right)}{f\left(S, I_{1,t}^{*}\right)} \frac{I_{1}}{I_{1,t}^{*}}-\frac{I_{1}}{I_{1,t}^{*}}-1\right]$$

$$=\left(1-\frac{f\left(S, I_{1}\right)}{f\left(S, I_{1,t}^{*}\right)}\right)\left(\frac{f\left(S, I_{1,t}^{*}\right)}{f\left(S, I_{1}\right)}-\frac{I_{1}}{I_{1,t}^{*}}\right) \leq 0 \tag{4.35}$$

also from (4.25) we have

$$\left[\frac{\tilde{f}\left(S_{t}^{*}, I_{2,t}^{*}\right)}{\tilde{f}\left(S, I_{2}\right)} \frac{f\left(S, I_{1,t}^{*}\right)}{f\left(S_{t}^{*}, I_{1,t}^{*}\right)}+\frac{\tilde{f}\left(S, I_{2}\right)}{\tilde{f}\left(S_{t}^{*}, I_{2,t}^{*}\right)} \frac{I_{2}}{I_{2,t}^{*}}-\frac{I_{2}}{I_{2,t}^{*}}-1\right]$$

$$=(1-\Gamma)\left(\frac{1}{\Gamma}-\frac{I_{2}}{I_{2,t}^{*}}\right) \leq 0 \tag{4.36}$$

with $\Gamma=\frac{f(S,I_2)}{f\left(S_t^*,I_{2,t}^*\right.} \frac{f\left(S_i^*,r_{1,t}^r\right)}{f(S,I_{1,t})}$.

When $R_0^1, R_0^2, \ldots, R_0^n>1$, the endemic equilibrium point ε_t is globally asymptotically stable.

□

5. Application of New Coronavirus Disease

As we mentioned in the introduction, the latest epidemic New Coronavirus Disease is a very kind of infection. For this reason, the main area of interest of this subsection is the numerical simulations caused by our multi-species SEIR epidemic model, Coronavirus disease 2019 in short "COVID-19", Middle East Respiratory Syndrome in short "MERS-CoV" and Severe Acute. Respiratory Failure Syndrome (Severe Acute Respiratory Syndrome) is briefly compared with "SARS-CoV" clinical data. We used clinical data from Saudi Arabia during March-December 2020 for our comparison. [27, 28]

The behavior of the infection was examined for the $\Lambda=2.64$, $\beta_1=4.27$, $\beta_2=2.85$, $\beta_3=2.13$, $\gamma_1=7$, $\gamma_2=2.5$, $\gamma_3=5$, $\mu_1=0.001$, $\mu_2=0.005$, $\mu_3=0.087$, $\delta=0.2$, $w_1=1.2$, $w_2=0.05$, $w_3=1.2$, $w_4=0.05$, $x_1=0.4$, $x_2=0.05$, $x_3=0.4$, $x_4=0.05$, $\alpha_1=0.14$, $\alpha_2=0.145$, $\alpha_3=0.150$ parameter values. If we write down the parameter values in the mathematical model using different incidence functions represented with the help of COVID-19, MERS-CoV and SARS-CoV clinical data, we observe that there is a significant relationship in the numerical simulations resulting from this. Therefore, due to the mathematical model, a good agreement can be observed between the relation of infected cases with clinical data. Numerical analysis with various incidence functions only fits clinical data very well for a certain observation period. Therefore, multiple mathematical models with generalized incidence functions are suitable to represent the disease under study.

The behavior of the infection was examined for the $\Lambda=2.34$, $\beta_1=0.01$, $\beta_2=0.01$, $\beta_3=0.01$, $\gamma_1=0.14$, $\gamma_2=0.5$, $\gamma_3=0.2$, $\mu_1=0.01$, $\mu_2=0.08$, $\mu_3=0.087$, $\delta=0.2$, $w_1=0.4$, $w_2=0.5$, $w_3=1.5$, $w_4=2.8$, $x_1=1.5$, $x_2=2$, $x_3=2.5$, $x_4=3$, $\alpha_1=2$, $\alpha_2=2.5$, $\alpha_3=3$ parameter values. We observe that the solution of the model converges to the disease-free equilibrium point ϵ_f for different incidence functions. In this case, the disease disappears, susceptible individuals reach their maximum value and other parameters are lost. We can simply calculate the basic reproduction number within the given numerical values. Therefore, the basic reproduction number of the Bilineer incidence function is $R_0 = 0.29846$; The base reproduction number of the Beddington-DeAngelis incidence function is $R_0 = 0.05254$; Basic reproduction number of Crowley-Martin incidence function $R_0 = 0.20892$; The basic reproduction number of the Non-monotonous incidence function is calculated as $R_0 = 0.29846$.

The behavior of the infection was examined for the $\Lambda = 2.34$, $\beta_1 = 7.54$, $\beta_2 = 7.25$, $\beta_3 = 11.6$, $\gamma_1 = 0.5$, $\gamma_2 = 0.4$, $\gamma_3 = 0.1$, $\mu_1 = 2.8$, $\mu_2 = 2.5$, $\mu_3 = 2$, $\delta = 0.2$, $w_1 = 1.2$, $w_2 = 0.05$, $w_3 = 1.2$, $w_4 = 0.05$, $x_1 = 0.4$, $x_2 = 0.05$, $x_3 = 0.4$, $x_4 = 0.05$, $\alpha_1 = 0.14$, $\alpha_2 = 0.145$, $\alpha_3 = 0.150$ parameter values. We draw attention to the convergence of the solution to the endemic balance for all incidence functions taken. As a matter of fact, the basic reproduction number of the Bilineer incidence function is $R_0 = 2.963344$; The base reproduction number of the Beddington-DeAngelis incidence function is $R_0 = 1.03051$; Basic reproduction number of Crowley-Martin incidence function $R_0 = 1.904111$; The base reproduction number of the Non-monotonous incidence function is calculated as $R_0 =1.776946$ and is greater than 1. As a result of numerical simulations with clinical data of COVID-19, MERS-CoV and SARS-CoV, we can reach a conclusion in harmony. Our numerical simulations can evolve into two situations for this epidemic. In the first case, the extinction of the disease and the other event occurs when the basic reproduction number is greater than one; The disease will continue in this event. It will be important whether strict measures such as quarantine, isolation, wearing a mask, and disinfection will significantly reduce the spread of the epidemic during this period of the disease.

6. Results and Discussion

In this study, we examined the overall incidence function of n and the global stability of the multifarious epidemic model. The model consists of 4 parts: The category of susceptible (S), exposed (E), infected (I), and recovered individuals (R), this type of model is called SEIR abbreviation. We have established the presence, positivity and limitation of the solution results that guarantee that our SEIR model is well defined. Disease-free equilibrium point, endemic equilibrium point according to Type-i and endemic equilibrium point according to each species. Using an appropriate Lyapunov function, the global stability of the balance is determined depending on the basic reproduction number R_0 and the Type-i reproduction number R_0^i. Numerical simulations are made to verify our different theoretical results. Comparisons have been made between our model results and the numerical results of the new coronavirus disease. There is a good correlation between numerical simulations and the numerical results of the new coronavirus disease, suggesting that our multiple mathematical models can adapt and predict the evolution of the epidemic. We can suggest that strict measures such as quarantine, isolation, wearing a mask, and disinfection can significantly reduce the spread of the epidemic during this period of the disease.

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A STUDY ON PICTURE FUZZY SETS

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Abstract. Recently, the concept of picture fuzzy sets (pf-sets) has propounded as a generalization of intuitionistic fuzzy sets to overcome further uncertainties than intuitionistic fuzzy uncertainties. In this paper, this concept and some of its operations are modified to ensure its consistency. Afterwards, some of the basic properties of the modified pf-sets are investigated. Finally, the need for further research is discussed.

1. Introduction

Many problems in daily life contain various uncertainty. Since existing standard mathematical tools may not model such uncertainties, new ones are needed. Fuzzy sets [1], introduced to deal with uncertainty, are one of the well-known mathematical tools for the aforesaid purpose. The concept of fuzzy sets has been applied many different fields from algebra to computer science [2]. In the fuzzy sets, the element has a membership degree denoted by $\mu(x)$. We can easily calculate the non-membership degree by subtracting the membership degree from 1 since the sum of membership and non-membership degrees is equal to 1. However, this sum may be less than 1 when the problems containing different uncertainty comes into question. Shortly after the definition of fuzzy sets, intuitionistic fuzzy sets [3] have been introduced as a generalisation of fuzzy sets to model such an uncertainty. In the intuitionistic fuzzy sets, the membership degree is denoted by $\mu(x)$ similar to those in fuzzy sets and the non-membership degree is denoted by $\nu(x)$. Thus, intuitionistic fuzzy sets can model the problems where $\mu(x) + \nu(x) \leq 1$. Moreover, the indeterminacy degree is $1 - (\mu(x) + \nu(x))$ therein. Although fuzzy sets and intuitionistic fuzzy sets can model many problems [4], there is much more problems and uncertainties in the real-life. For instance, in voting for an election, decisions of the electorate may split into three types: Yes, no, and abstain. To model this problem and the problems similar to this, Cuong has put forward the concept of picture fuzzy sets (pf-sets) [5]. In the pf-sets, the membership, neutral membership, and non-membership degrees are denoted by $\mu(x)$, $\eta(x)$, and $\nu(x)$, respectively. Here, the indeterminacy results from the refusal of the voting or non-participating in the

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voting, and the indeterminacy degree is denoted by $1-(\mu(x)+\eta(x)+\nu(x))$. Despite of the modelling success of *pf*-sets, Cuong's definitions and operations have theoretical inconsistencies. Therefore, this paper aims to redefine the concept of *pf*-sets and their operations to ensure their consistency.

In Section 2 of the present study, concepts of fuzzy sets, intuitionistic fuzzy sets, *pf*-sets, and basic operations of *pf*-sets are presented. In Section 3, the counter-examples concerning Cuong's definitions and operations are provided. In Section 4, the concept of *pf*-sets and some of its basic operations are redefined, and their properties are investigated. Finally, the need for further research related to the new definitions and operations of *pf*-sets is discussed.

2. Preliminaries

In this section, we present the concepts of fuzzy sets [1], intuitionistic fuzzy sets [3], and picture fuzzy sets (*pf*-sets) [5] and some of *pf*-sets' operators and properties provided in [5] by considering the notations used throughout this paper.

Across this paper, let E be a parameter set, $F(E)$ be the set of all fuzzy sets over E, and $\mu \in F(E)$. Here, a fuzzy set is denoted by $\{{}^{\mu(x)}x : x \in E\}$ instead of $\{(x,\mu(x)) : x \in E\}$.

Definition 2.1. [3] Let f is a function from E to $[0,1]\times[0,1]$. Then, the set $\left\{ {}^{\mu(x)}_{\nu(x)}x : x \in E \right\}$ being the graphic of f is called an intuitionistic fuzzy set (*if*-set) over E.

Here, for all $x \in E$, $\mu(x)+\nu(x) \leq 1$. Moreover, μ and ν are called the membership function and non-membership function, respectively, and $\pi(x) = 1-(\mu(x)+\nu(x))$ is called the degree of indeterminacy of the element $x \in E$. Obviously, each ordinary fuzzy set can be written as $\left\{ {}^{\mu(x)}_{1-\mu(x)}x : x \in E \right\}$.

Definition 2.2. [5] Let f is a function from E to $[0,1]\times[0,1]\times[0,1]$. Then, the set $\{(x,\mu(x),\eta(x),\nu(x)) : x \in E\}$ being the graphic of f is called a picture fuzzy set (*pf*-set) over E. Here, for all $x \in E$, $\mu(x)+\eta(x)+\nu(x) \leq 1$ and a *pf*-set is denoted by $\left\{ \left\langle \begin{smallmatrix} \mu(x) \\ \eta(x) \\ \nu(x) \end{smallmatrix} \right\rangle x : x \in E \right\}$ instead of $\{(x,\mu(x),\eta(x),\nu(x)) : x \in E\}$.

Moreover, μ, η, and ν are called the membership function, neutral membership function, and non-membership function, respectively, and $\pi(x) = 1-(\mu(x)+\eta(x)+\nu(x))$ is called the degree of indeterminacy of the element $x \in E$.

In the present paper, the set of all *pf*-sets over E is denoted by $PF(E)$.

Definition 2.3. [5] Let $f_1, f_2 \in PF(E)$. For all $x \in E$, if $\mu_1(x) \leq \mu_2(x)$, $\eta_1(x) \leq \eta_2(x)$, and $\nu_1(x) \geq \nu_2(x)$, then f_1 is called a subset of f_2 and is denoted by $f_1 \tilde{\subseteq} f_2$.

Definition 2.4. [5] Let $f_1, f_2 \in PF(E)$. If $f_1 \tilde{\subseteq} f_2$ and $f_2 \tilde{\subseteq} f_1$, then f_1 and f_2 are called equal *pf*-sets and is denoted by $f_1 = f_2$.

Definition 2.5. [5] Let $f_1, f_2, f_3 \in PF(E)$. For all $x \in E$, if $\mu_3(x) = \max\{\mu_1(x),\mu_2(x)\}$, $\eta_3(x) = \min\{\eta_1(x),\eta_2(x)\}$, and $\nu_3(x) = \min\{\nu_1(x),\nu_2(x)\}$, then f_3 is called union of f_1 and f_2, and is denoted by $f_3 = f_1 \tilde{\cup} f_2$.

Definition 2.6. [5] Let $f_1, f_2, f_3 \in PF(E)$. For all $x \in E$, if $mu_3(x) = \min\{\mu_1(x),\mu_2(x)\}$, $\eta_3(x) = \min\{\eta_1(x),\eta_2(x)\}$, and $\nu_3(x) = \max\{\nu_1(x),\nu_2(x)\}$, then f_3 is called intersection of f_1 and f_2, and is denoted by $f_3 = f_1 \tilde{\cap} f_2$.

Definition 2.7. [5] Let $f_1, f_2 \in PF(E)$. For all $x \in E$, if $\mu_2(x) = \nu_1(x)$, $\eta_2(x) = \eta_1(x)$, and $\nu_2(x) = \mu_1(x)$, then f_2 is called complement of f_1 and is denoted by $f_2 = f_1^{\tilde{c}}$.

3. Motivations of the Redefining of Picture Fuzzy Sets

In this section, several counter-examples concerning Cuong's definition of *pf*-sets [5] and their inclusion, complement, and union definitions are presented. According to Definition 2.3, the definitions of empty and universal *pf*-sets should be as in Definition 3.1 and Definition 3.2, respectively, to be held the following conditions:

- Empty *pf*-set over E is a subset of all the *pf*-set over E.
- All *pf*-sets over E are the subset of universal *pf*-set over E.

Definition 3.1. Let $f \in PF(E)$. For all $x \in E$, if $\mu(x) = 0$, $\eta(x) = 0$, and $\nu(x) = 1$, then f is called empty *pf*-set and is denoted by $\left\langle \begin{smallmatrix} 0 \\ 0 \\ 1 \end{smallmatrix} \right\rangle E$ or 0_E.

Definition 3.2. Let $f \in PF(E)$. For all $x \in E$, if $\mu(x) = 1$, $\eta(x) = 1$, and $\nu(x) = 0$, then f is called universal *pf*-set and is denoted by $\left\langle \begin{smallmatrix} 1 \\ 1 \\ 0 \end{smallmatrix} \right\rangle E$ or 1_E.

Example 3.3. There is a contradiction in Definition 3.2 since $1 + 1 + 0 \nleq 1$, i.e., $1_E \notin PF(E)$. On the other hand, even if $1_E \in PF(E)$, $(1_E)^{\tilde{c}} \neq 0_E$.

Example 3.4. Let $f \in PF(E)$ such that $f = \left\{ \left\langle \begin{smallmatrix} 0.1 \\ 0.2 \\ 0.3 \end{smallmatrix} \right\rangle x \right\}$. Then, $f \tilde{\cup} 0_E \neq f$ and $f \tilde{\cup} 1_E \neq 1_E$.

The concept of *pf*-sets and their operations should be redefined to overcome the inconsistencies in Example 3.3 and 3.4.

4. Picture Fuzzy Sets and Some of Their Properties

In this section, we redefine the concepts of *pf*-sets and investigate some of their properties according to new definition herein by considering the notations used across this study.

Definition 4.1. Let f is a function from E to $[0,1] \times [0,1] \times [0,1]$. Then, the set $\{(x, \mu(x), \eta(x), \nu(x)) : x \in E\}$ being the graphic of f is called a picture fuzzy set (*pf*-set) over E. Here, for all $x \in E$, $\mu(x) + \eta(x) + \nu(x) \leq 2$ and a *pf*-set is denoted by $\left\{ \left\langle \begin{smallmatrix} \mu(x) \\ \eta(x) \\ \nu(x) \end{smallmatrix} \right\rangle x : x \in E \right\}$ instead of $\{(x, \mu(x), \eta(x), \nu(x)) : x \in E\}$.

Moreover, μ, η, and ν are called the membership function, neutral membership function, and non-membership function, respectively, and $\pi(x) = 2 - (\mu(x) + \eta(x) + \nu(x))$ is called the degree of indeterminacy-membership of the element $x \in E$. Manifestly, each ordinary fuzzy set can be written as $\left\{ \left\langle \begin{smallmatrix} \mu(x) \\ 1 \\ 1-\mu(x) \end{smallmatrix} \right\rangle x : x \in E \right\}$ and each intuitionistic fuzzy set can be written as $\left\{ \left\langle \begin{smallmatrix} \mu(x) \\ 1 \\ \nu(x) \end{smallmatrix} \right\rangle x : x \in E \right\}$.

In the present paper, the set of all *pf*-sets over E is denoted by $PF(E)$ and $f \in PF(E)$. In $PF(E)$, since the *graph*(f) and f have generated each other uniquely, the notations are interchangeable. Thus, we denote a *pf*-set *graph*(f) by f as long as it does not cause any confusion.

Example 4.2. Let $E = \{x_1, x_2, x_3, x_4\}$. Then,

$$f = \left\{ \left\langle \begin{smallmatrix} 0.7 \\ 0.5 \\ 0.3 \end{smallmatrix} \right\rangle x_1, \left\langle \begin{smallmatrix} 0.4 \\ 1 \\ 0.5 \end{smallmatrix} \right\rangle x_2, \left\langle \begin{smallmatrix} 0.8 \\ 0 \\ 0.1 \end{smallmatrix} \right\rangle x_3, \left\langle \begin{smallmatrix} 0 \\ 0 \\ 1 \end{smallmatrix} \right\rangle x_4 \right\}$$

and

$$g = \left\{ \left\langle \begin{smallmatrix} 0.6 \\ 0.8 \\ 0.2 \end{smallmatrix} \right\rangle x_1, \left\langle \begin{smallmatrix} 0.2 \\ 0.1 \\ 1 \end{smallmatrix} \right\rangle x_2, \left\langle \begin{smallmatrix} 0.3 \\ 0.9 \\ 0.4 \end{smallmatrix} \right\rangle x_3, \left\langle \begin{smallmatrix} 0.9 \\ 1 \\ 0 \end{smallmatrix} \right\rangle x_4 \right\}$$

are two *pf*-sets over E.

Definition 4.3. Let $f \in PF(E)$. For all $x \in E$, if $\mu(x) = \lambda$, $\eta(x) = \varepsilon$, and $\nu(x) = \omega$, then f is called $(\lambda, \varepsilon, \omega)$-*pf*-set and is denoted by $\left\langle \begin{smallmatrix} \lambda \\ \varepsilon \\ \omega \end{smallmatrix} \right\rangle E$.

Definition 4.4. Let $f \in PF(E)$. For all $x \in E$, if $\mu(x) = 0$, $\eta(x) = 1$, and $\nu(x) = 1$, then f is called empty *pf*-set and is denoted by $\left\langle \begin{smallmatrix} 0 \\ 1 \\ 1 \end{smallmatrix} \right\rangle E$ or 0_E.

Definition 4.5. Let $f \in PF(E)$. For all $x \in E$, if $\mu(x) = 1$, $\eta(x) = 0$, and $\nu(x) = 0$, then f is called universal *pf*-set and is denoted by $\left\langle \begin{smallmatrix} 1 \\ 0 \\ 0 \end{smallmatrix} \right\rangle E$ or 1_E.

Definition 4.6. Let $f_1, f_2 \in PF(E)$. For all $x \in E$, if $\mu_1(x) \leq \mu_2(x)$, $\eta_1(x) \geq \eta_2(x)$ and $\nu_1(x) \geq \nu_2(x)$, then f_1 is called a subset of f_2 and is denoted by $f_1 \tilde{\subseteq} f_2$.

Proposition 4.7. Let $f, f_1, f_2, f_3 \in PF(E)$. Then,

i. $f \tilde{\subseteq} 1_E$
ii. $0_E \tilde{\subseteq} f$
iii. $f \tilde{\subseteq} f$
iv. $\left[f_1 \tilde{\subseteq} f_2 \wedge f_2 \tilde{\subseteq} f_3 \right] \Rightarrow f_1 \tilde{\subseteq} f_3$

Remark 4.8. From Proposition 4.7, it can be seen that the inclusion relation is a partial ordering relation in $PF(E)$.

Definition 4.9. Let $f_1, f_2 \in PF(E)$. For all $x \in E$, if $\mu_1(x) = \mu_2(x)$, $\eta_1(x) = \eta_2(x)$ and $\nu_2(x) = \nu_1(x)$, then f_1 and f_2 are called equal *pf*-sets and is denoted by $f_1 = f_2$.

Definition 4.10. Let $f_1, f_2 \in PF(E)$. if $f_1 \tilde{\subseteq} f_2$ and $f_1 \neq f_2$, then f_1 is called a proper subset of f_2 and is denoted by $f_1 \tilde{\subsetneq} f_2$.

Proposition 4.11. Let $f_1, f_2, f_3 \in PF(E)$. Then,

i. $\left[f_1 \tilde{\subseteq} f_2 \wedge f_2 \tilde{\subseteq} f_1 \right] \Rightarrow f_1 = f_2$
ii. $[f_1 = f_2 \wedge f_2 = f_3] \Rightarrow f_1 = f_3$

Definition 4.12. Let $f_1, f_2, f_3 \in PF(E)$. For all $x \in E$, if $\mu_3(x) = \max\{\mu_1(x), \mu_2(x)\}$, $\eta_3(x) = \min\{\eta_1(x), \eta_2(x)\}$, and $\nu_3(x) = \min\{\nu_1(x), \nu_2(x)\}$, then f_3 is called union of f_1 and f_2, and is denoted by $f_3 = f_1 \tilde{\cup} f_2$.

Proposition 4.13. Let $f, f_1, f_2, f_3 \in PF(E)$. Then,

i. $f \tilde{\cup} f = f$
ii. $f \tilde{\cup} 1_E = 1_E$
iii. $f \tilde{\cup} 0_E = f$
iv. $f_1 \tilde{\cup} f_2 = f_2 \tilde{\cup} f_1$
v. $f_1 \tilde{\cup} (f_2 \tilde{\cup} f_3) = (f_1 \tilde{\cup} f_2) \tilde{\cup} f_3$
vi. $\left[f_1 \tilde{\subseteq} f_2 \Rightarrow f_1 \tilde{\cup} f_2 = f_2 \right]$

Definition 4.14. Let $f_1, f_2, f_3 \in PF(E)$. For all $x \in E$, if $\mu_3(x) = \min\{\mu_1(x), \mu_2(x)\}$, $\eta_3(x) = \max\{\eta_1(x), \eta_2(x)\}$, and $\nu_3(x) = \max\{\nu_1(x), \nu_2(x)\}$, then f_3 is called intersection of f_1 and f_2, and is denoted by $f_3 = f_1 \tilde{\cap} f_2$.

Proposition 4.15. Let $f, f_1, f_2, f_3 \in PF(E)$. Then,

i. $f \tilde{\cap} f = f$
ii. $f \tilde{\cap} 1_E = f$
iii. $f \tilde{\cap} 0_E = 0_E$
iv. $f_1 \tilde{\cap} f_2 = f_2 \tilde{\cap} f_1$
v. $f_1 \tilde{\cap} (f_2 \tilde{\cap} f_3) = (f_1 \tilde{\cap} f_2) \tilde{\cap} f_3$
vi. $\left[f_1 \tilde{\subseteq} f_2 \Rightarrow f_1 \tilde{\cap} f_2 = f_1\right]$

Proposition 4.16. Let $f, f_1, f_2, f_3 \in PF(E)$. Then,

i. $f_1 \tilde{\cup} (f_2 \tilde{\cap} f_3) = (f_1 \tilde{\cup} f_2) \tilde{\cap} (f_1 \tilde{\cup} f_3)$
ii. $f_1 \tilde{\cap} (f_2 \tilde{\cup} f_3) = (f_1 \tilde{\cap} f_2) \tilde{\cup} (f_1 \tilde{\cup} f_3)$

Proof. *i.* Let $f_1, f_2, f_3 \in PF(E)$. Then, for all $x \in E$,

$$\begin{aligned}
f_1 \tilde{\cup} (f_2 \tilde{\cap} f_3) &= f_1 \tilde{\cup} \left\{ \left\langle \begin{matrix} \min\{\mu_2(x), \mu_3(x)\} \\ \max\{\eta_2(x), \eta_3(x)\} \\ \max\{\nu_2(x), \nu_3(x)\} \end{matrix} \right\rangle x \right\} \\
&= \left\{ \left\langle \begin{matrix} \max\{\mu_1(x), \min\{\mu_2(x), \mu_3(x)\}\} \\ \min\{\eta_1(x), \max\{\eta_2(x), \eta_3(x)\}\} \\ \min\{\nu_1(x), \max\{\nu_2(x), \nu_3(x)\}\} \end{matrix} \right\rangle x \right\} \\
&= \left\{ \left\langle \begin{matrix} \min\{\max\{\mu_1(x), \mu_2(x)\}, \max\{\mu_1(x), \mu_3(x)\}\} \\ \max\{\min\{\eta_1(x), \eta_2(x)\}, \min\{\eta_1(x), \eta_3(x)\}\} \\ \max\{\min\{\nu_1(x), \nu_2(x)\}, \min\{\nu_1(x), \nu_3(x)\}\} \end{matrix} \right\rangle x \right\} \\
&= \left\{ \left\langle \begin{matrix} \max\{\mu_1(x), \mu_2(x)\} \\ \min\{\eta_1(x), \eta_2(x)\} \\ \min\{\nu_1(x), \nu_2(x)\} \end{matrix} \right\rangle x \right\} \tilde{\cap} \left\{ \left\langle \begin{matrix} \max\{\mu_1(x), \mu_3(x)\} \\ \min\{\eta_1(x), \eta_3(x)\} \\ \min\{\nu_1(x), \nu_3(x)\} \end{matrix} \right\rangle x \right\} \\
&= (f_1 \tilde{\cup} f_2) \tilde{\cap} (f_1 \tilde{\cup} f_3)
\end{aligned}$$

□

Definition 4.17. Let $f_1, f_2 \in PF(E)$. For all $x \in E$, if $\mu_2(x) = \nu_1(x)$, $\eta_2(x) = 1 - \eta_1(x)$, and $\nu_2(x) = \mu_1(x)$, then f_2 is called complement of f_1 and is denoted by $f_2 = f_1^{\tilde{c}}$.

Proposition 4.18. Let $f, f_1, f_2 \in PF(E)$.

i. $\left(f^{\tilde{c}}\right)^{\tilde{c}} = f$
ii. $0_E^{\tilde{c}} = 1_E$
iii. $1_E^{\tilde{c}} = 0_E$
iv. $f_1 \tilde{\subseteq} f_2 = f_2^{\tilde{c}} \tilde{\subseteq} f_1^{\tilde{c}}$

Definition 4.19. Let $f_1, f_2, f_3 \in PF(E)$. For all $x \in E$, if $\mu_3(x) = \min\{\mu_1(x), \nu_2(x)\}$, $\eta_3(x) = \max\{\eta_1(x), 1 - \eta_2(x)\}$, and $\nu_3(x) = \max\{\nu_1(x), \mu_2(x)\}$, then f_3 is called difference between f_1 and f_2, and is denoted by $f_3 = f_1 \tilde{\setminus} f_2$.

Proposition 4.20. Let $f, f_1, f_2 \in PF(E)$.

i. $f \tilde{\setminus} 0_E = f$
ii. $f \tilde{\setminus} 1_E = 0_E$
iii. $f_1 \tilde{\setminus} f_2 = f_1 \tilde{\cap} f_2^{\tilde{c}}$

Remark 4.21. It must be noted that the difference is non-commutative and non-associative. For example, Let $f_1 = \left\{\left\langle\begin{smallmatrix}0.3\\0.2\\0.4\end{smallmatrix}\right\rangle x\right\}$, $f_2 = \left\{\left\langle\begin{smallmatrix}0.7\\0.6\\0.1\end{smallmatrix}\right\rangle x\right\}$, and $f_3 = \left\{\left\langle\begin{smallmatrix}0.5\\0.8\\0.2\end{smallmatrix}\right\rangle x\right\}$. Then,

i. $f_1 \tilde{\setminus} f_2 = \left\{\left\langle\begin{smallmatrix}0.1\\0.4\\0.7\end{smallmatrix}\right\rangle x\right\}$ and $f_2 \tilde{\setminus} f_1 = \left\{\left\langle\begin{smallmatrix}0.4\\0.8\\0.3\end{smallmatrix}\right\rangle x\right\} \Rightarrow f_1 \tilde{\setminus} f_2 \neq f_2 \tilde{\setminus} f_1$

ii. $f_1 \tilde{\setminus} \left(f_2 \tilde{\setminus} f_3\right) = \left\{\left\langle\begin{smallmatrix}0.3\\0.4\\0.4\end{smallmatrix}\right\rangle x\right\}$ and $\left(f_1 \tilde{\setminus} f_2\right) \tilde{\setminus} f_3 = \left\{\left\langle\begin{smallmatrix}0.1\\0.4\\0.7\end{smallmatrix}\right\rangle x\right\} \Rightarrow f_1 \tilde{\setminus} \left(f_2 \tilde{\setminus} f_3\right) \neq \left(f_1 \tilde{\setminus} f_2\right) \tilde{\setminus} f_3$

Proposition 4.22. Let $f_1, f_2 \in PF(E)$. Then, the De Morgan's Laws are valid,

i. $(f_1 \tilde{\cup} f_2)^{\tilde{c}} = f_1^{\tilde{c}} \tilde{\cap} f_2^{\tilde{c}}$

ii. $(f_1 \tilde{\cap} f_2)^{\tilde{c}} = f_1^{\tilde{c}} \tilde{\cup} f_2^{\tilde{c}}$

Proof. i. Let $f_1, f_2, f_3 \in PF(E)$. Then,

$$\begin{aligned}
(f_1 \tilde{\cup} f_2)^{\tilde{c}} &= \left(\left\{\left\langle\begin{smallmatrix}\mu_1(x)\\ \eta_1(x)\\ \nu_1(x)\end{smallmatrix}\right\rangle x\right\} \tilde{\cup} \left\{\left\langle\begin{smallmatrix}\mu_2(x)\\ \eta_2(x)\\ \nu_2(x)\end{smallmatrix}\right\rangle x\right\}\right)^{\tilde{c}} \\
&= \left(\left\{\left\langle\begin{smallmatrix}\max\{\mu_1(x),\mu_2(x)\}\\ \min\{\eta_1(x),\eta_2(x)\}\\ \min\{\nu_1(x),\nu_2(x)\}\end{smallmatrix}\right\rangle x\right\}\right)^{\tilde{c}} \\
&= \left\{\left\langle\begin{smallmatrix}\min\{\nu_1(x),\nu_2(x)\}\\ 1-\min\{\eta_1(x),\eta_2(x)\}\\ \max\{\mu_1(x),\mu_2(x)\}\end{smallmatrix}\right\rangle x\right\} \\
&= \left\{\left\langle\begin{smallmatrix}\min\{\nu_1(x),\nu_2(x)\}\\ \max\{1-\eta_1(x),1-\eta_2(x)\}\\ \max\{\mu_1(x),\mu_2(x)\}\end{smallmatrix}\right\rangle x\right\} \\
&= \left\{\left\langle\begin{smallmatrix}\nu_1(x)\\ 1-\eta_1(x)\\ \mu_1(x)\end{smallmatrix}\right\rangle x\right\} \tilde{\cap} \left\{\left\langle\begin{smallmatrix}\nu_2(x)\\ 1-\eta_2(x)\\ \mu_2(x)\end{smallmatrix}\right\rangle x\right\} \\
&= \left(\left\{\left\langle\begin{smallmatrix}\mu_1(x)\\ \eta_1(x)\\ \nu_1(x)\end{smallmatrix}\right\rangle x\right\}\right)^{\tilde{c}} \tilde{\cap} \left(\left\{\left\langle\begin{smallmatrix}\mu_2(x)\\ \eta_2(x)\\ \nu_2(x)\end{smallmatrix}\right\rangle x\right\}\right)^{\tilde{c}} \\
&= f_1^{\tilde{c}} \tilde{\cap} f_2^{\tilde{c}}
\end{aligned}$$

□

Definition 4.23. Let $f_1, f_2, f_3 \in PF(E)$. For all $x \in E$, if $\mu_3(x) = \max\{\min\{\mu_1(x), \nu_2(x)\}, \min\{\mu_2(x), \nu_1(x)\}\}$, $\eta_3(x) = \min\{\max\{\eta_1(x), 1 - \eta_2(x)\}, \max\{\eta_2(x), 1 - \eta_1(x)\}\}$, and $\nu_3(x) = \min\{\max\{\nu_1(x), \mu_2(x)\}, \max\{\nu_2(x), \mu_1(x)\}\}$, then f_3 is called symmetric difference between f_1 and f_2, and is denoted by $f_3 = f_1 \tilde{\triangle} f_2$.

Proposition 4.24. Let $f, f_1, f_2 \in PF(E)$.

i. $f \tilde{\triangle} 0_E = f$

ii. $f \tilde{\triangle} 1_E = f^{\tilde{c}}$

iii. $f_1 \tilde{\triangle} f_2 = f_2 \tilde{\triangle} f_1$

iv. $f_1 \tilde{\triangle} f_2 = (f_1 \tilde{\setminus} f_2) \tilde{\cup} (f_2 \tilde{\setminus} f_1)$

Remark 4.25. It must be noted that the symmetric difference is non-associative. For example, Let $f_1 = \left\{\left\langle\begin{array}{c}0.3\\0.2\\0.4\end{array}\right\rangle x\right\}$, $f_2 = \left\{\left\langle\begin{array}{c}0.7\\0.6\\0.1\end{array}\right\rangle x\right\}$, and $f_3 = \left\{\left\langle\begin{array}{c}0.5\\0.8\\0.2\end{array}\right\rangle x\right\}$. Since $f_1 \tilde{\triangle} \left(f_2 \tilde{\triangle} f_3\right) = \left\{\left\langle\begin{array}{c}0.3\\0.4\\0.5\end{array}\right\rangle x\right\}$ and $\left(f_1 \tilde{\triangle} f_2\right) \tilde{\triangle} f_3 = \left\{\left\langle\begin{array}{c}0.3\\0.4\\0.4\end{array}\right\rangle x\right\}$, then $f_1 \tilde{\triangle} \left(f_2 \tilde{\triangle} f_3\right) \neq \left(f_1 \tilde{\triangle} f_2\right) \tilde{\triangle} f_3$.

Definition 4.26. Let $f_1, f_2 \in PF(E)$. If $f_1 \tilde{\cap} f_2 = 0_E$, then f_1 and f_2 are called disjoint *pf*-sets.

5. Conclusion

In this study, concept of *pf*-sets was redefined for its theoretical consistency. Then, some of their basic properties were investigated and it was corroborated that *pf*-sets became more functional.

Cuong has also offered the concept of *pfs*-sets [5], a hybrid version of *pf*-sets and soft sets [6], immediately afterwards his defining of *pf*-sets in the same study. After a while,*pfs*-sets have been redefined [7] without mentioned the Cuong's defining of *pfs*-sets and studied the properties of them. Yet, in these two study, concept of *pfs*-sets has been introduced based on Cuong's definition of *pf*-sets and they have some inconsistencies just as *pf*-sets have. Therefore, examining the aforesaid studies, redefining the concept of *pfs*-sets, and investigating their properties are worthwhile to study.

6. Appendix

In the present study, the latex command utilised for the picture fuzzy environments, an example, and its output are as follows:

Command

```
\newcommand{\pfe}[3]{{\scriptsize\arraycolsep=3pt\def\arraystretch{1}
\left<\begin{array}{c}
\hspace{-0.1cm}#1\\
\hspace{-0.1cm}#2\\
\hspace{-0.1cm}#3
\end{array}\hspace{-0.1cm}\right>}\hspace{-0.025cm}}
```

Example	**Output**
`$f=\left\{\pfe{0.8}{0.1}{0.5}x\right\}$`	$f = \left\{\left\langle\begin{array}{c}0.8\\0.1\\0.5\end{array}\right\rangle x\right\}$

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ON MIDPOINT TYPE INEQUALITIES FOR CO-ORDINATED CONVEX FUNCTIONS VIA GENERALIZED FRACTIONAL INTEGRALS

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Abstract. In this study firstly, we prove an identity for twice partially differentiable mappings involving the double generalized fractional integral. By using the this obtained identity, we establish some midpoint type inequalities for differentiable co-ordinated convex functions. Furthermore, by special cases of our main results, we obtain several new inequalities for Riemann-Liouville fractional integrals and k-Riemann-Liouville fractional integrals.

1. Introduction

The inequalities, introduced by C. Hermite and J. Hadamard for convex functions, are considerable topic in the literature. These inequalities state that if $\sigma : I \to \mathbb{R}$ is a convex function on the interval I of real numbers and $\rho_1, \rho_2 \in I$ with $\rho_1 < \rho_2$, then

$$\sigma\left(\frac{\rho_1+\rho_2}{2}\right) \leq \frac{1}{\rho_2-\rho_1}\int_{\rho_1}^{\rho_2} \sigma(\kappa)d\kappa \leq \frac{\sigma(\rho_1)+\sigma(\rho_2)}{2}. \tag{1.1}$$

if σ is concave, then both inequalities in (1.1) hold to the reverse direction.

Over the years, considerable number of studies have been focused on obtaining trapezoid and midpoint type inequalities which give bounds for the right-hand side and left-hand side of the inequality (1.1), respectively. For example, Dragomir and Agarwal first obtained trapezoid inequalities for convex functions in [6] and Kirmacı first established midpoint inequalities for convex functions in [12]. In [18], Sarikaya et al. generalized the inequalities (1.1) for fractional integrals and the authors also proved some corresponding trapezoid type inequalities. Iqbal et al. presented some fractional midpoint type inequalities for convex functions in [9]. On the other hand, Dragomir proved Hermite-Hadamard inequalities for co-ordinated convex

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mappings in [5]. The midpoint and trapezoid type inequalities for co-ordinated convex functions were established in the papers [13] and [17], respectively. Moreover, Sarikaya obtained fractional Hermite-Hadamard inequalities and fractional trapezoid for functions with two variables in [20]. Tunç et al. presented some fractional midpoint type inequalities for co-ordinated convex functions in [22]. In [16], Sarikaya and Ertuğral first introduced new fractional integrals which are called generalized fractional integrals. In addition, they proved Hermite-Hadamard inequalities and several trapezoids and midpoint type inequalities for generalized fractional integrals. Furthermore, Turkay et al. described the generalized fractional integrals for functions with two variables. These authors presented Hermite-Hadamard and trapezoid type inequalities for this kind of fractional integrals in [23]. For the other similar inequalities, please refer to [3, 4, 11, 14, 15, 19, 21].

2. Generalized fractional Integrals

In this section, we summarize the generalized fractional integrals defined by Sarikaya and Ertuğral in [16].

Let us define a function $\varphi : [0,\infty) \to [0,\infty)$ satisfying the following condition:

$$\int_0^1 \frac{\varphi(\tau)}{\tau} d\tau < \infty.$$

We consider the following left-sided and right-sided generalized fractional integral operators

$$_{\rho_1+}I_\varphi \sigma(\kappa) = \int_{\rho_1}^{\kappa} \frac{\varphi(\kappa-\tau)}{\kappa-\tau} \sigma(\tau) d\tau, \quad \kappa > \rho_1 \tag{2.1}$$

and

$$_{\rho_2-}I_\varphi \sigma(\kappa) = \int_{\kappa}^{\rho_2} \frac{\varphi(\tau-\kappa)}{\tau-\kappa} \sigma(\tau) d\tau, \quad \kappa < \rho_2, \tag{2.2}$$

respectively.

Some forms of fractional integrals, namely, Riemann-Liouville fractional integral, k-Riemann-Liouville fractional integral, Katugampola fractional integrals, conformable fractional integral, Hadamard fractional integrals, etc are generalized as the most significant feature of generalized fractional integrals. These important special cases of the integral operators (2.1) and (2.2) are mentioned below:

(1) If we choose $\varphi(\tau) = \tau$, the operators (2.1) and (2.2) reduce to the Riemann integral.
(2) Considering $\varphi(\tau) = \frac{\tau^\alpha}{\Gamma(\alpha)}$ and $\alpha > 0$, the operators (2.1) and (2.2) reduce to the Riemann-Liouville fractional integrals $J_{\rho_1+}^\alpha \sigma(\kappa)$ and $J_{\rho_2-}^\alpha \sigma(\kappa)$, respectively. Here, Γ is Gamma function.
(3) For $\varphi(\tau) = \frac{1}{k\Gamma_k(\alpha)} \tau^{\frac{\alpha}{k}}$ and $\alpha, k > 0$, the operators (2.1) and (2.2) reduce to the k-Riemann-Liouville fractional integrals $J_{\rho_1+,k}^\alpha \sigma(\kappa)$ and $J_{\rho_2-,k}^\alpha \sigma(\kappa)$, respectively. Here, Γ_k is k-Gamma function.

There are several papers on inequalities for generalized fractional integrals in the literature. In [16], Sarikaya and Ertuğral also proved Hermite-Hadamard inequalities for generalized fractional integrals. In addition, Budak et al. proved Midpoint type inequalities and extensions of Hermite-Hadamard inequalities in the papers [1] and [2], respectively. In [7], Ertuğral and Sarikaya presented some Simpson type inequalities for these fractional integral operators. For some of other papers on inequalities for generalized fractional integrals, please refer to [7, 8, 10, 24, 25].

Generalized double fractional integrals are given by Turkay et al. in [23], as follows:

Definition 2.1. The Generalized double fractional integrals ${}_{\rho_1+,\rho_3+}I_{\varphi,\psi}$, ${}_{\rho_1+,\rho_4-}I_{\varphi,\psi}$, ${}_{\rho_2-,\rho_3+}I_{\varphi,\psi}$, ${}_{\rho_2-,\rho_4-}I_{\varphi,\psi}$ are defined by

$$ {}_{\rho_1+,\rho_3+}I_{\varphi,\psi}\sigma\left(\kappa,\gamma\right)=\int_{\rho_1}^{\kappa}\int_{\rho_3}^{\gamma}\frac{\varphi\left(\kappa-\tau\right)}{\kappa-\tau}\frac{\psi\left(\gamma-\xi\right)}{\gamma-\xi}\sigma\left(\tau,\xi\right)d\xi d\tau,\ \kappa>\rho_1,\ \gamma>\rho_3, \tag{2.3}$$

$$ {}_{\rho_1+,\rho_4-}I_{\varphi,\psi}\sigma\left(\kappa,\gamma\right)=\int_{\rho_1}^{\kappa}\int_{\gamma}^{\rho_4}\frac{\varphi\left(\kappa-\tau\right)}{\kappa-\tau}\frac{\psi\left(\xi-\gamma\right)}{\xi-\gamma}\sigma\left(\tau,\xi\right)d\xi d\tau,\ \kappa>\rho_1,\ \gamma<\rho_4, \tag{2.4}$$

$$ {}_{\rho_2-,\rho_3+}I_{\varphi,\psi}\sigma\left(\kappa,\gamma\right)=\int_{\kappa}^{\rho_2}\int_{\rho_3}^{\gamma}\frac{\varphi\left(\tau-\kappa\right)}{\tau-\kappa}\frac{\psi\left(\gamma-\xi\right)}{\gamma-\xi}\sigma\left(\tau,\xi\right)d\xi d\tau,\ \kappa<\rho_2,\ \gamma>\rho_3, \tag{2.5}$$

and

$$ {}_{\rho_2-,\rho_4-}I_{\varphi,\psi}\sigma\left(\kappa,\gamma\right)=\int_{\kappa}^{\rho_2}\int_{\gamma}^{\rho_4}\frac{\varphi\left(\tau-\kappa\right)}{\tau-\kappa}\frac{\psi\left(\xi-\gamma\right)}{\xi-\gamma}\sigma\left(\tau,\xi\right)d\xi d\tau,\ \kappa<\rho_2,\ \gamma<\rho_4. \tag{2.6}$$

Here, $\sigma\in L_1([\rho_1,\rho_2]\times[\rho_3,\rho_4])$ and the functions $\varphi,\psi:[0,\infty)\to[0,\infty)$ satisfy $\int_0^1\frac{\varphi\left(\tau\right)}{\tau}d\tau<\infty$ and $\int_0^1\frac{\psi\left(\xi\right)}{\xi}d\xi<\infty$, respectively.

By using Definition 2.1, well-known fractional integrals can be obtained by some special choices. For example;

(1) If we choose $\varphi\left(\tau\right)=\tau$ and $\psi\left(\xi\right)=\xi$, the operators (2.3), (2.4), (2.5) and (2.6) reduce to the double Riemann integral.
(2) Considering $\varphi\left(\tau\right)=\frac{\tau^{\alpha}}{\Gamma(\alpha)}$, $\psi\left(\xi\right)=\frac{\xi^{\beta}}{\Gamma(\beta)}$, then for $\alpha,\beta>0$, the operators (2.3), (2.4), (2.5) and (2.6) reduce to the Riemann-Liouville fractional integrals $J^{\alpha,\beta}_{\rho_1+,\rho_3+}\sigma\left(\kappa,\gamma\right)$, $J^{\alpha,\beta}_{\rho_1+,\rho_4-}\sigma\left(\kappa,\gamma\right)$ $J^{\alpha,\beta}_{\rho_2-,\rho_3+}\sigma\left(\kappa,\gamma\right)$ and $J^{\alpha,\beta}_{\rho_2-,\rho_4-}\sigma\left(\kappa,\gamma\right)$, respectively.
(3) For $\varphi\left(\tau\right)=\frac{\tau^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$ and $\psi\left(\xi\right)=\frac{\xi^{\frac{\beta}{k}}}{k\Gamma_k(\beta)}$, for $\alpha,\beta,k>0$, the operators (2.3), (2.4), (2.5) and (2.6) reduce to the k-Riemann-Liouville fractional integrals $J^{\alpha,\beta,k}_{\rho_1+,\rho_3+}\sigma\left(\kappa,\gamma\right)$, $J^{\alpha,\beta,k}_{\rho_1+,\rho_4-}\sigma\left(\kappa,\gamma\right)$, $J^{\alpha,\beta,k}_{\rho_2-,\rho_3+}\sigma\left(\kappa,\gamma\right)$ and $J^{\alpha,\beta,k}_{\rho_2-,\rho_4-}\sigma\left(\kappa,\gamma\right)$, respectively.

3. An identity for Generalized double fractional integrals

Throughout this study for brevity, we define

$$\Lambda_1(\kappa,\tau)=\int_{\tau}^{1}\frac{\varphi\left(\left(\rho_2-\kappa\right)u\right)}{u}du,\quad \Delta_1(\kappa,\tau)=\int_{\tau}^{1}\frac{\varphi\left(\left(\kappa-\rho_1\right)u\right)}{u}du, \tag{3.1}$$

and

$$\Lambda_2(\gamma,\xi)=\int_{\xi}^{1}\frac{\psi\left(\left(\rho_4-\gamma\right)u\right)}{u}du,\quad \Delta_2(\gamma,\xi)=\int_{\xi}^{1}\frac{\psi\left(\left(\gamma-\rho_3\right)u\right)}{u}du. \tag{3.2}$$

Moreover, we denote

$$\begin{aligned}\Xi_1(\kappa) &= \Lambda_1(\kappa,0)+\Delta_1(\kappa,0)\\ \Xi_2(\gamma) &= \Lambda_2(\gamma,0)+\Delta_2(\gamma,0)\end{aligned}$$
$$\Upsilon(\kappa,\gamma)=\Xi_1(\kappa)\Xi_2(\gamma).$$

Lemma 3.1. *Let $\sigma:\Delta:=[\rho_1,\rho_2]\times[\rho_3,\rho_4]\to\mathbb{R}$ be an absolutely continuous function on Δ such that the partial derivative of order $\frac{\partial^2\sigma(\tau,\xi)}{\partial\tau\partial\xi}$ exist for all $(\tau,\xi)\in\Delta$. Then, the following equality for generalized fractional integrals holds:*

$$\begin{aligned}&\Omega(\rho_1,\rho_2,\kappa;\rho_3,\rho_4,\gamma)\\
=\;&\frac{(\kappa-\rho_1)(\gamma-\rho_3)}{\Upsilon(\kappa,\gamma)}\int_0^1\int_0^1\Lambda_1(\kappa,\tau)\Lambda_2(\gamma,\xi)\\
&\times\frac{\partial^2}{\partial\tau\partial\xi}\sigma\left(\tau\rho_2+(1-\tau)(\rho_1+\rho_2-\kappa),\xi\rho_4+(1-\xi)(\rho_3+\rho_4-\gamma)\right)d\xi d\tau\\
&-\frac{(\kappa-\rho_1)(\rho_4-\gamma)}{\Upsilon(\kappa,\gamma)}\int_0^1\int_0^1\Lambda_1(\kappa,\tau)\Delta_2(\gamma,\xi)\\
&\times\frac{\partial^2}{\partial\tau\partial\xi}\sigma\left(\tau\rho_2+(1-\tau)(\rho_1+\rho_2-\kappa),\xi\rho_3+(1-\xi)(\rho_3+\rho_4-\gamma)\right)d\xi d\tau\\
&-\frac{(\rho_2-\kappa)(\gamma-\rho_3)}{\Upsilon(\kappa,\gamma)}\int_0^1\int_0^1\Delta_1(\kappa,\tau)\Lambda_2(\gamma,\xi)\\
&\times\frac{\partial^2}{\partial\tau\partial\xi}\sigma\left(\tau\rho_1+(1-\tau)(\rho_1+\rho_2-\kappa),\xi\rho_4+(1-\xi)(\rho_3+\rho_4-\gamma)\right)d\xi d\tau\\
&+\frac{(\rho_2-\kappa)(\rho_4-\gamma)}{\Upsilon(\kappa,\gamma)}\int_0^1\int_0^1\Delta_1(\kappa,\tau)\Delta_2(\gamma,\xi)\\
&\times\frac{\partial^2}{\partial\tau\partial\xi}\sigma\left(\tau\rho_1+(1-\tau)(\rho_1+\rho_2-\kappa),\xi\rho_3+(1-\xi)(\rho_3+\rho_4-\gamma)\right)d\xi d\tau,\end{aligned}$$

where

$$\begin{aligned}&\Omega(\rho_1,\rho_2,\kappa;\rho_3,\rho_4,\gamma)\\
=\;&\sigma(\rho_1+\rho_2-\kappa,\rho_3+\rho_4-\gamma)\\
&-\frac{1}{\Xi_2(\gamma)}\left[\,{}_{\rho_4-}I_{\psi}\sigma(\rho_1+\rho_2-\kappa,\rho_3+\rho_4-\gamma)+\,{}_{\rho_3+}I_{\psi}\sigma(\rho_1+\rho_2-\kappa,\rho_3+\rho_4-\gamma)\right]\\
&-\frac{1}{\Xi_1(\kappa)}\left[\,{}_{\rho_2-}I_{\varphi}\sigma(\rho_1+\rho_2-\kappa,\rho_3+\rho_4-\gamma)+\,{}_{\rho_1+}I_{\varphi}\sigma(\rho_1+\rho_2-\kappa,\rho_3+\rho_4-\gamma)\right]\end{aligned}$$

$$+\frac{1}{\Upsilon(\kappa,\gamma)}\left[\ {}_{\rho_2-,\rho_4-}I_{\varphi,\psi}\sigma(\rho_1+\rho_2-\kappa,\rho_3+\rho_4-\gamma)+\ {}_{\rho_2-,\rho_3+}I_{\varphi,\psi}\sigma(\rho_1+\rho_2-\kappa,\rho_3+\rho_4-\gamma)\right.$$
$$\left.+\ {}_{\rho_1+,\rho_4-}I_{\varphi,\psi}\sigma(\rho_1+\rho_2-\kappa,\rho_3+\rho_4-\gamma)+\ {}_{\rho_1+,\rho_3+}I_{\varphi,\psi}\sigma(\rho_1+\rho_2-\kappa,\rho_3+\rho_4-\gamma)\right].$$

Proof. By using integration by parts, we have

$$\begin{aligned}
&H_1=\int_0^1\int_0^1\Lambda_1(\kappa,\tau)\Lambda_2(\gamma,\xi)\\
&\times\frac{\partial^2}{\partial\tau\partial\xi}\sigma\left(\tau\rho_2+(1-\tau)\left(\rho_1+\rho_2-\kappa\right),\xi\rho_4+(1-\xi)\left(\rho_3+\rho_4-\gamma\right)\right)d\xi d\tau\\
=\ &\frac{\Lambda_1(\kappa,0)\Lambda_2(\gamma,0)}{(\kappa-\rho_1)\left(\gamma-\rho_3\right)}\sigma(\rho_1+\rho_2-\kappa,\rho_3+\rho_4-\gamma)\\
&-\frac{\Lambda_1(\kappa,0)}{(\kappa-\rho_1)\left(\gamma-\rho_3\right)}\ {}_{\rho_4-}I_{\psi}\sigma(\rho_1+\rho_2-\kappa,\rho_3+\rho_4-\gamma)\\
&-\frac{\Lambda_2(\gamma,0)}{(\kappa-\rho_1)\left(\gamma-\rho_3\right)}\ {}_{\rho_2-}I_{\varphi}\sigma(\rho_1+\rho_2-\kappa,\rho_3+\rho_4-\gamma)\qquad(3.3)\\
&+\frac{1}{(\kappa-\rho_1)\left(\gamma-\rho_3\right)}\ {}_{\rho_2-,\rho_4-}I_{\varphi,\psi}\sigma(\rho_1+\rho_2-\kappa,\rho_3+\rho_4-\gamma),
\end{aligned}$$

$$\begin{aligned}
&H_2=\int_0^1\int_0^1\Lambda_1(\kappa,\tau)\Delta_2(\gamma,\xi)\\
&\times\frac{\partial^2}{\partial\tau\partial\xi}\sigma\left(\tau\rho_2+(1-\tau)\left(\rho_1+\rho_2-\kappa\right),\xi\rho_3+(1-\xi)\left(\rho_3+\rho_4-\gamma\right)\right)d\xi d\tau\\
=\ &-\frac{\Lambda_1(\kappa,0)\Delta_2(\gamma,0)}{(\kappa-\rho_1)\left(\rho_4-\gamma\right)}\sigma(\rho_1+\rho_2-\kappa,\rho_3+\rho_4-\gamma)\\
&+\frac{\Lambda_1(\kappa,0)}{(\kappa-\rho_1)\left(\rho_4-\gamma\right)}\ {}_{\rho_3+}I_{\psi}\sigma(\rho_1+\rho_2-\kappa,\rho_3+\rho_4-\gamma)\qquad(3.4)\\
&+\frac{\Delta_2(\gamma,0)}{(\kappa-\rho_1)\left(\rho_4-\gamma\right)}\ {}_{\rho_2-}I_{\varphi}\sigma(\rho_1+\rho_2-\kappa,\rho_3+\rho_4-\gamma)\\
&-\frac{1}{(\kappa-\rho_1)\left(\rho_4-\gamma\right)}\ {}_{\rho_2-,\rho_3+}I_{\varphi,\psi}\sigma(\rho_1+\rho_2-\kappa,\rho_3+\rho_4-\gamma),
\end{aligned}$$

$$\begin{aligned}
&H_3=\int_0^1\int_0^1\Delta_1(\kappa,\tau)\Lambda_2(\gamma,\xi)\\
&\times\frac{\partial^2}{\partial\tau\partial\xi}\sigma\left(\tau\rho_1+(1-\tau)\left(\rho_1+\rho_2-\kappa\right),\xi\rho_4+(1-\xi)\left(\rho_3+\rho_4-\gamma\right)\right)d\xi d\tau\\
=\ &-\frac{\Delta_1(\kappa,0)\Lambda_2(\gamma,0)}{(\rho_2-\kappa)\left(\gamma-\rho_3\right)}\sigma(\rho_1+\rho_2-\kappa,\rho_3+\rho_4-\gamma)\\
&+\frac{\Delta_1(\kappa,0)}{(\rho_2-\kappa)\left(\gamma-\rho_3\right)}\ {}_{\rho_4-}I_{\psi}\sigma(\rho_1+\rho_2-\kappa,\rho_3+\rho_4-\gamma)
\end{aligned}$$

$$+\frac{\Lambda_2(\gamma,0)}{(\rho_2-\kappa)(\gamma-\rho_3)}\ {}_{\rho_1+}I_{\varphi}\sigma(\rho_1+\rho_2-\kappa,\rho_3+\rho_4-\gamma) \tag{3.5}$$
$$-\frac{1}{(\rho_2-\kappa)(\gamma-\rho_3)}\ {}_{\rho_1+,\rho_4-}I_{\varphi,\psi}\sigma(\rho_1+\rho_2-\kappa,\rho_3+\rho_4-\gamma),$$

and

$$\begin{aligned} H_4 &= \int_0^1\int_0^1 \Delta_1(\kappa,\tau)\Delta_2(\gamma,\xi) \\ &\times\frac{\partial^2}{\partial\tau\partial\xi}\sigma\left(\tau\rho_1+(1-\tau)\left(\rho_1+\rho_2-\kappa\right),\xi\rho_3+(1-\xi)\left(\rho_3+\rho_4-\gamma\right)\right)d\xi d\tau \\ &= \frac{\Delta_1(\kappa,0)\Delta_2(\gamma,0)}{(\rho_2-\kappa)(\rho_4-\gamma)}\sigma(\rho_1+\rho_2-\kappa,\rho_3+\rho_4-\gamma) \end{aligned}$$
$$-\frac{\Delta_1(\kappa,0)}{(\rho_2-\kappa)(\rho_4-\gamma)}\ {}_{\rho_3+}I_{\psi}\sigma(\rho_1+\rho_2-\kappa,\rho_3+\rho_4-\gamma) \tag{3.6}$$
$$-\frac{\Delta_2(\gamma,0)}{(\rho_2-\kappa)(\rho_4-\gamma)}\ {}_{\rho_1+}I_{\varphi}\sigma(\rho_1+\rho_2-\kappa,\rho_3+\rho_4-\gamma)$$
$$+\frac{1}{(\rho_2-\kappa)(\rho_4-\gamma)}\ {}_{\rho_1+,\rho_3+}I_{\varphi,\psi}\sigma(\rho_1+\rho_2-\kappa,\rho_3+\rho_4-\gamma).$$

By using the Equations (3.3)-(3.6), we have

$$\begin{aligned} &\frac{(\kappa-\rho_1)(\gamma-\rho_3)}{\Upsilon(\kappa,\gamma)}H_1-\frac{(\kappa-\rho_1)(\rho_4-\gamma)}{\Upsilon(\kappa,\gamma)}H_2 \\ &-\frac{(\rho_2-\kappa)(\gamma-\rho_3)}{\Upsilon(\kappa,\gamma)}H_3+\frac{(\rho_2-\kappa)(\rho_4-\gamma)}{\Upsilon(\kappa,\gamma)}H_4 \\ =\ &\Omega(\rho_1,\rho_2,\kappa;\rho_3,\rho_4,\gamma) \end{aligned}$$

which completes the proof of Lemma 3.1. □

4. New midpoint type inequalities for Generalized Fractional Integrals

Theorem 4.1. *Assume that the assumptions of Lemma 3.1 hold. Assume also that the mapping* $\left|\frac{\partial^2\sigma}{\partial\tau\partial\xi}\right|$ *is co-ordinated convex on* Δ*. Then, we obtain the following inequality for generalized fractional integrals*

$$\begin{aligned} &|\Omega(\rho_1,\rho_2,\kappa;\rho_3,\rho_4,\gamma)| \\ \leq\ &\frac{(\kappa-\rho_1)(\gamma-\rho_3)}{\Upsilon(\kappa,\gamma)}\left[A_1B_1\left|\frac{\partial^2}{\partial\tau\partial\xi}\sigma\left(\rho_2,\rho_4\right)\right|+A_1B_2\left|\frac{\partial^2}{\partial\tau\partial\xi}\sigma\left(\rho_2,\rho_3+\rho_4-\gamma\right)\right|\right. \\ &\left.+A_2B_1\left|\frac{\partial^2}{\partial\tau\partial\xi}\sigma\left(\rho_1+\rho_2-\kappa,\rho_4\right)\right|+A_2B_2\left|\frac{\partial^2}{\partial\tau\partial\xi}\sigma\left(\rho_1+\rho_2-\kappa,\rho_3+\rho_4-\gamma\right)\right|\right] \\ &+\frac{(\kappa-\rho_1)(\rho_4-\gamma)}{\Upsilon(\kappa,\gamma)}\left[A_1B_4\left|\frac{\partial^2}{\partial\tau\partial\xi}\sigma\left(\rho_2,\rho_3\right)\right|+A_1B_3\left|\frac{\partial^2}{\partial\tau\partial\xi}\sigma\left(\rho_2,\rho_3+\rho_4-\gamma\right)\right|\right. \\ &\left.+A_2B_4\left|\frac{\partial^2}{\partial\tau\partial\xi}\sigma\left(\rho_1+\rho_2-\kappa,\rho_3\right)\right|+A_2B_3\left|\frac{\partial^2}{\partial\tau\partial\xi}\sigma\left(\rho_1+\rho_2-\kappa,\rho_3+\rho_4-\gamma\right)\right|\right] \end{aligned}$$

$$+\frac{(\rho_2-\kappa)(\gamma-\rho_3)}{\Upsilon(\kappa,\gamma)}\left[A_4B_1\left|\frac{\partial^2}{\partial\tau\partial\xi}\sigma(\rho_1,\rho_4)\right|+A_4B_2\left|\frac{\partial^2}{\partial\tau\partial\xi}\sigma(\rho_1,\rho_3+\rho_4-\gamma)\right|\right.$$
$$\left.+A_3B_1\left|\frac{\partial^2}{\partial\tau\partial\xi}\sigma(\rho_1+\rho_2-\kappa,\rho_4)\right|+A_3B_2\left|\frac{\partial^2}{\partial\tau\partial\xi}\sigma(\rho_1+\rho_2-\kappa,\rho_3+\rho_4-\gamma)\right|\right]$$
$$+\frac{(\rho_2-\kappa)(\rho_4-\gamma)}{\Upsilon(\kappa,\gamma)}\left[A_4B_4\left|\frac{\partial^2}{\partial\tau\partial\xi}\sigma(\rho_1,\rho_3)\right|+A_4B_3\left|\frac{\partial^2}{\partial\tau\partial\xi}\sigma(\rho_1,\rho_3+\rho_4-\gamma)\right|\right.$$
$$\left.+A_3B_4\left|\frac{\partial^2}{\partial\tau\partial\xi}\sigma(\rho_1+\rho_2-\kappa,\rho_3)\right|+A_3B_3\left|\frac{\partial^2}{\partial\tau\partial\xi}\sigma(\rho_1+\rho_2-\kappa,\rho_3+\rho_4-\gamma)\right|\right].$$

Here,

$$\begin{cases} A_1=\int_0^1 \tau\Lambda_1(\kappa,\tau)d\tau, & A_2=\int_0^1 (1-\tau)\Lambda_1(\kappa,\tau)d\tau, \\ A_3=\int_0^1 (1-\tau)\Delta_1(\kappa,\tau)d\tau, & A_4=\int_0^1 \tau\Delta_1(\kappa,\tau)d\tau, \end{cases} \tag{4.1}$$

and

$$\begin{cases} B_1=\int_0^1 \xi\Lambda_2(\gamma,\xi)d\xi, & B_2=\int_0^1 (1-\xi)\Lambda_2(\gamma,\xi)d\xi, \\ B_3=\int_0^1 (1-\xi)\Delta_2(\gamma,\xi)d\xi, & B_4=\int_0^1 \xi\Delta_2(\gamma,\xi)d\xi. \end{cases} \tag{4.2}$$

Proof. By taking modulus in Lemma 3.1, we have

$$\begin{aligned} &|\Omega(\rho_1,\rho_2,\kappa;\rho_3,\rho_4,\gamma)| \\ \le\ & \frac{(\kappa-\rho_1)(\gamma-\rho_3)}{\Upsilon(\kappa,\gamma)}\int_0^1\int_0^1 \Lambda_1(\kappa,\tau)\Lambda_2(\gamma,\xi) \\ &\times\left|\frac{\partial^2}{\partial\tau\partial\xi}\sigma(\tau\rho_2+(1-\tau)(\rho_1+\rho_2-\kappa),\xi\rho_4+(1-\xi)(\rho_3+\rho_4-\gamma))\right|d\xi d\tau \\ &+\frac{(\kappa-\rho_1)(\rho_4-\gamma)}{\Upsilon(\kappa,\gamma)}\int_0^1\int_0^1 \Lambda_1(\kappa,\tau)\Delta_2(\gamma,\xi) \\ &\times\left|\frac{\partial^2}{\partial\tau\partial\xi}\sigma(\tau\rho_2+(1-\tau)(\rho_1+\rho_2-\kappa),\xi\rho_3+(1-\xi)(\rho_3+\rho_4-\gamma))\right|d\xi d\tau \\ &+\frac{(\rho_2-\kappa)(\gamma-\rho_3)}{\Upsilon(\kappa,\gamma)}\int_0^1\int_0^1 \Delta_1(\kappa,\tau)\Lambda_2(\gamma,\xi) \\ &\times\left|\frac{\partial^2}{\partial\tau\partial\xi}\sigma(\tau\rho_1+(1-\tau)(\rho_1+\rho_2-\kappa),\xi\rho_4+(1-\xi)(\rho_3+\rho_4-\gamma))\right|d\xi d\tau \\ &+\frac{(\rho_2-\kappa)(\rho_4-\gamma)}{\Upsilon(\kappa,\gamma)}\int_0^1\int_0^1 \Delta_1(\kappa,\tau)\Delta_2(\gamma,\xi) \\ &\times\left|\frac{\partial^2}{\partial\tau\partial\xi}\sigma(\tau\rho_1+(1-\tau)(\rho_1+\rho_2-\kappa),\xi\rho_3+(1-\xi)(\rho_3+\rho_4-\gamma))\right|d\xi d\tau. \end{aligned} \tag{4.3}$$

Since $\left|\frac{\partial^2\sigma}{\partial\tau\partial\xi}\right|$ is co-ordinated convex, we get

$$\begin{aligned}
(4.4)\quad & \int_0^1\int_0^1 \Lambda_1(\kappa,\tau)\Lambda_2(\gamma,\xi) \\
& \times\left|\frac{\partial^2}{\partial\tau\partial\xi}\sigma\left(\tau\rho_2+(1-\tau)\left(\rho_1+\rho_2-\kappa\right),\xi\rho_4+(1-\xi)\left(\rho_3+\rho_4-\gamma\right)\right)\right|d\xi d\tau \\
\leq\; & \int_0^1\int_0^1 \Lambda_1(\kappa,\tau)\Lambda_2(\gamma,\xi)\left(\tau\xi\left|\frac{\partial^2}{\partial\tau\partial\xi}\sigma\left(\rho_2,\rho_4\right)\right|+\tau(1-\xi)\left|\frac{\partial^2}{\partial\tau\partial\xi}\sigma\left(\rho_2,\rho_3+\rho_4-\gamma\right)\right|\right. \\
& +(1-\tau)\xi\left|\frac{\partial^2}{\partial\tau\partial\xi}\sigma\left(\rho_1+\rho_2-\kappa,\rho_4\right)\right| \\
& \left.+(1-\tau)(1-\xi)\left|\frac{\partial^2}{\partial\tau\partial\xi}\sigma\left(\rho_1+\rho_2-\kappa,\rho_3+\rho_4-\gamma\right)\right|\right)d\xi d\tau \\
=\; & A_1B_1\left|\frac{\partial^2}{\partial\tau\partial\xi}\sigma\left(\rho_2,\rho_4\right)\right|+A_1B_2\left|\frac{\partial^2}{\partial\tau\partial\xi}\sigma\left(\rho_2,\rho_3+\rho_4-\gamma\right)\right| \\
& +A_2B_1\left|\frac{\partial^2}{\partial\tau\partial\xi}\sigma\left(\rho_1+\rho_2-\kappa,\rho_4\right)\right|+A_2B_2\left|\frac{\partial^2}{\partial\tau\partial\xi}\sigma\left(\rho_1+\rho_2-\kappa,\rho_3+\rho_4-\gamma\right)\right|.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
(4.5)\quad & \int_0^1\int_0^1 \Lambda_1(\kappa,\tau)\Delta_2(\gamma,\xi) \\
& \times\left|\frac{\partial^2}{\partial\tau\partial\xi}\sigma\left(\tau\rho_2+(1-\tau)\left(\rho_1+\rho_2-\kappa\right),\xi\rho_3+(1-\xi)\left(\rho_3+\rho_4-\gamma\right)\right)\right|d\xi d\tau \\
\leq\; & A_1B_4\left|\frac{\partial^2}{\partial\tau\partial\xi}\sigma\left(\rho_2,\rho_3\right)\right|+A_1B_3\left|\frac{\partial^2}{\partial\tau\partial\xi}\sigma\left(\rho_2,\rho_3+\rho_4-\gamma\right)\right| \\
& +A_2B_4\left|\frac{\partial^2}{\partial\tau\partial\xi}\sigma\left(\rho_1+\rho_2-\kappa,\rho_3\right)\right|+A_2B_3\left|\frac{\partial^2}{\partial\tau\partial\xi}\sigma\left(\rho_1+\rho_2-\kappa,\rho_3+\rho_4-\gamma\right)\right|,
\end{aligned}$$

$$\begin{aligned}
(4.6)\quad & \int_0^1\int_0^1 \Delta_1(\kappa,\tau)\Lambda_2(\gamma,\xi) \\
& \times\left|\frac{\partial^2}{\partial\tau\partial\xi}\sigma\left(\tau\rho_1+(1-\tau)\left(\rho_1+\rho_2-\kappa\right),\xi\rho_4+(1-\xi)\left(\rho_3+\rho_4-\gamma\right)\right)\right|d\xi d\tau \\
\leq\; & A_4B_1\left|\frac{\partial^2}{\partial\tau\partial\xi}\sigma\left(\rho_1,\rho_4\right)\right|+A_4B_2\left|\frac{\partial^2}{\partial\tau\partial\xi}\sigma\left(\rho_1,\rho_3+\rho_4-\gamma\right)\right| \\
& +A_3B_1\left|\frac{\partial^2}{\partial\tau\partial\xi}\sigma\left(\rho_1+\rho_2-\kappa,\rho_4\right)\right|+A_3B_2\left|\frac{\partial^2}{\partial\tau\partial\xi}\sigma\left(\rho_1+\rho_2-\kappa,\rho_3+\rho_4-\gamma\right)\right|,
\end{aligned}$$

and

$$(4.7)\quad \int_0^1\int_0^1 \Delta_1(\kappa,\tau)\Delta_2(\gamma,\xi)$$

$$\times \left|\frac{\partial^2}{\partial\tau\partial\xi}\sigma\left(\tau\rho_1+(1-\tau)\left(\rho_1+\rho_2-\kappa\right),\xi\rho_3+(1-\xi)\left(\rho_3+\rho_4-\gamma\right)\right)\right|d\xi d\tau$$

$$\leq A_4B_4\left|\frac{\partial^2}{\partial\tau\partial\xi}\sigma\left(\rho_1,\rho_3\right)\right|+A_4B_3\left|\frac{\partial^2}{\partial\tau\partial\xi}\sigma\left(\rho_1,\rho_3+\rho_4-\gamma\right)\right|$$

$$+A_3B_4\left|\frac{\partial^2}{\partial\tau\partial\xi}\sigma\left(\rho_1+\rho_2-\kappa,\rho_3\right)\right|+A_3B_3\left|\frac{\partial^2}{\partial\tau\partial\xi}\sigma\left(\rho_1+\rho_2-\kappa,\rho_3+\rho_4-\gamma\right)\right|.$$

If we substitute the inequalities (4.4)-(4.7) in (4.3), we obtain the desired result. This ends the proof of Theorem 4.1. □

Remark 4.2. In Theorem 4.1, if we assign $\varphi(\tau)=\tau$ and $\psi(\xi)=\xi$ for all $(\tau,\xi)\in\triangle$ and if we choose $\kappa=\frac{\rho_1+\rho_2}{2}$ and $\gamma=\frac{\rho_3+\rho_4}{2}$, then Theorem 4.1 reduces to [13, Theorem 2].

Corollary 1. In Theorem 4.1, if we assign $\varphi(\tau)=\frac{\tau^\alpha}{\Gamma(\alpha)}$ and $\psi(\xi)=\frac{\xi^\beta}{\Gamma(\beta)}$ for all $(\tau,\xi)\in\triangle$ and if we choose $\kappa=\frac{\rho_1+\rho_2}{2}$ and $\gamma=\frac{\rho_3+\rho_4}{2}$, then we have the following midpoint type inequality for Riemann-Liouville fractional integrals

$$\left|\sigma\left(\frac{\rho_1+\rho_2}{2},\frac{\rho_3+\rho_4}{2}\right)\right.$$
$$-\frac{2^{\beta-1}\Gamma(\beta+1)}{(\rho_4-\rho_3)^\beta}\left[J^\beta_{\rho_4-}\sigma\left(\frac{\rho_1+\rho_2}{2},\frac{\rho_3+\rho_4}{2}\right)+J^\beta_{\rho_3+}\sigma\left(\frac{\rho_1+\rho_2}{2},\frac{\rho_3+\rho_4}{2}\right)\right]$$
$$-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\rho_2-\rho_1)^\alpha}\left[J^\alpha_{\rho_2-}\sigma\left(\frac{\rho_1+\rho_2}{2},\frac{\rho_3+\rho_4}{2}\right)+J^\alpha_{\rho_1+}\sigma\left(\frac{\rho_1+\rho_2}{2},\frac{\rho_3+\rho_4}{2}\right)\right]$$
$$+\frac{2^{\alpha+\beta-2}\Gamma(\alpha+1)\Gamma(\beta+1)}{(\rho_2-\rho_1)^\alpha(\rho_4-\rho_3)^\beta}$$
$$\times\left[J^{\alpha,\beta}_{\rho_2-,\rho_4-}\sigma\left(\frac{\rho_1+\rho_2}{2},\frac{\rho_3+\rho_4}{2}\right)+J^{\alpha,\beta}_{\rho_2-,\rho_3+}\sigma\left(\frac{\rho_1+\rho_2}{2},\frac{\rho_3+\rho_4}{2}\right)\right.$$
$$\left.\left.+J^{\alpha,\beta}_{\rho_1+,\rho_4-}\sigma\left(\frac{\rho_1+\rho_2}{2},\frac{\rho_3+\rho_4}{2}\right)+J^{\alpha,\beta}_{\rho_1+,\rho_3+}\sigma\left(\frac{\rho_1+\rho_2}{2},\frac{\rho_3+\rho_4}{2}\right)\right]\right|$$
$$\leq\frac{\alpha\beta(\rho_2-\rho_1)(\rho_4-\rho_3)}{16(\alpha+1)(\beta+1)}$$
$$\times\left[\left|\frac{\partial^2}{\partial\tau\partial\xi}\sigma(\rho_2,\rho_4)\right|+\left|\frac{\partial^2}{\partial\tau\partial\xi}\sigma(\rho_2,\rho_3)\right|+\left|\frac{\partial^2}{\partial\tau\partial\xi}\sigma(\rho_1,\rho_4)\right|+\left|\frac{\partial^2}{\partial\tau\partial\xi}\sigma(\rho_1,\rho_3)\right|\right].$$

Corollary 2. In Theorem 4.1, if we assign $\varphi(\tau)=\frac{\tau^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$ and $\psi(\xi)=\frac{\xi^{\frac{\beta}{k}}}{k\Gamma_k(\beta)}$ for all $(\tau,\xi)\in\triangle$ and if we choose $\kappa=\frac{\rho_1+\rho_2}{2}$ and $\gamma=\frac{\rho_3+\rho_4}{2}$, then we have the following midpoint type inequality for k-Riemann-Liouville fractional integrals

$$\left|\sigma\left(\frac{\rho_1+\rho_2}{2},\frac{\rho_3+\rho_4}{2}\right)\right.$$
$$-\frac{2^{\frac{\beta}{k}-1}\Gamma_k(\beta+k)}{(\rho_4-\rho_3)^{\frac{\beta}{k}}}\left[J^\beta_{\rho_4-,k}\sigma\left(\frac{\rho_1+\rho_2}{2},\frac{\rho_3+\rho_4}{2}\right)+J^\beta_{\rho_3+,k}\sigma\left(\frac{\rho_1+\rho_2}{2},\frac{\rho_3+\rho_4}{2}\right)\right]$$

$$-\frac{2^{\frac{\alpha}{k}-1}\Gamma_k(\alpha+k)}{(\rho_2-\rho_1)^{\frac{\alpha}{k}}}\left[J^{\alpha}_{\rho_2-,k}\sigma\left(\frac{\rho_1+\rho_2}{2},\frac{\rho_3+\rho_4}{2}\right)+J^{\alpha}_{\rho_1+,k}\sigma\left(\frac{\rho_1+\rho_2}{2},\frac{\rho_3+\rho_4}{2}\right)\right]$$

$$+\frac{2^{\frac{\alpha+\beta}{k}-2}\Gamma_k(\alpha+k)\Gamma_k(\beta+k)}{(\rho_2-\rho_1)^{\frac{\alpha}{k}}(\rho_4-\rho_3)^{\frac{\beta}{k}}}$$

$$\times\left[J^{\alpha,\beta,k}_{\rho_2-,\rho_4-}\sigma\left(\frac{\rho_1+\rho_2}{2},\frac{\rho_3+\rho_4}{2}\right)+J^{\alpha,\beta,k}_{\rho_2-,\rho_3+}\sigma\left(\frac{\rho_1+\rho_2}{2},\frac{\rho_3+\rho_4}{2}\right)\right.$$

$$\left.\left.+J^{\alpha,\beta,k}_{\rho_1+,\rho_4-}\sigma\left(\frac{\rho_1+\rho_2}{2},\frac{\rho_3+\rho_4}{2}\right)+J^{\alpha,\beta,k}_{\rho_1+,\rho_3+}\sigma\left(\frac{\rho_1+\rho_2}{2},\frac{\rho_3+\rho_4}{2}\right)\right]\right|$$

$$\leq \frac{\alpha\beta(\rho_2-\rho_1)(\rho_4-\rho_3)}{16(\alpha+k)(\beta+k)}$$

$$\times\left[\left|\frac{\partial^2}{\partial\tau\partial\xi}\sigma(\rho_2,\rho_4)\right|+\left|\frac{\partial^2}{\partial\tau\partial\xi}\sigma(\rho_2,\rho_3)\right|+\left|\frac{\partial^2}{\partial\tau\partial\xi}\sigma(\rho_1,\rho_4)\right|+\left|\frac{\partial^2}{\partial\tau\partial\xi}\sigma(\rho_1,\rho_3)\right|\right].$$

Theorem 4.3. *Suppose that the assumptions of Lemma 3.1 hold. Suppose also that the mapping* $\left|\frac{\partial^2\sigma}{\partial\tau\partial\xi}\right|^q$, $q>1$ *is co-ordinated convex on* Δ*. Then, we get the following inequality for generalized fractional integrals,*

$$|\Omega(\rho_1,\rho_2,\kappa;\rho_3,\rho_4,\gamma)|$$

$$\leq \frac{(\kappa-\rho_1)(\gamma-\rho_3)}{2^{\frac{2}{q}}\Upsilon(\kappa,\gamma)}\left(\int_0^1\int_0^1[\Lambda_1(\kappa,\tau)\Lambda_2(\gamma,\xi)]^p\,d\xi d\tau\right)^{\frac{1}{p}}$$

$$\times\left(\left|\frac{\partial^2}{\partial\tau\partial\xi}\sigma(\rho_2,\rho_4)\right|^q+\left|\frac{\partial^2}{\partial\tau\partial\xi}\sigma(\rho_2,\rho_3+\rho_4-\gamma)\right|^q\right.$$

$$\left.+\left|\frac{\partial^2}{\partial\tau\partial\xi}\sigma(\rho_1+\rho_2-\kappa,\rho_4)\right|^q+\left|\frac{\partial^2}{\partial\tau\partial\xi}\sigma(\rho_1+\rho_2-\kappa,\rho_3+\rho_4-\gamma)\right|^q\right)^{\frac{1}{q}}$$

$$+\frac{(\kappa-\rho_1)(\rho_4-\gamma)}{2^{\frac{2}{q}}\Upsilon(\kappa,\gamma)}\left(\int_0^1\int_0^1[\Lambda_1(\kappa,\tau)\Delta_2(\gamma,\xi)]^p\,d\xi d\tau\right)^{\frac{1}{p}}$$

$$\times\left(\left|\frac{\partial^2}{\partial\tau\partial\xi}\sigma(\rho_2,\rho_3)\right|^q+\left|\frac{\partial^2}{\partial\tau\partial\xi}\sigma(\rho_2,\rho_3+\rho_4-\gamma)\right|^q\right.$$

$$\left.+\left|\frac{\partial^2}{\partial\tau\partial\xi}\sigma(\rho_1+\rho_2-\kappa,\rho_3)\right|^q+\left|\frac{\partial^2}{\partial\tau\partial\xi}\sigma(\rho_1+\rho_2-\kappa,\rho_3+\rho_4-\gamma)\right|^q\right)^{\frac{1}{q}}$$

$$+\frac{(\rho_2-\kappa)(\gamma-\rho_3)}{2^{\frac{2}{q}}\Upsilon(\kappa,\gamma)}\left(\int_0^1\int_0^1[\Delta_1(\kappa,\tau)\Lambda_2(\gamma,\xi)]^p\,d\xi d\tau\right)^{\frac{1}{p}}$$

$$\times\left(\left|\frac{\partial^2}{\partial\tau\partial\xi}\sigma(\rho_1,\rho_4)\right|^q+\left|\frac{\partial^2}{\partial\tau\partial\xi}\sigma(\rho_1,\rho_3+\rho_4-\gamma)\right|^q\right.$$

$$\left.+\left|\frac{\partial^2}{\partial\tau\partial\xi}\sigma(\rho_1+\rho_2-\kappa,\rho_4)\right|^q+\left|\frac{\partial^2}{\partial\tau\partial\xi}\sigma(\rho_1+\rho_2-\kappa,\rho_3+\rho_4-\gamma)\right|^q\right)^{\frac{1}{q}}$$

$$+\frac{(\rho_2-\kappa)(\rho_4-\gamma)}{2^{\frac{2}{q}}\Upsilon(\kappa,\gamma)}\left(\int_0^1\int_0^1[\Delta_1(\kappa,\tau)\Delta_2(\gamma,\xi)]^p\,d\xi d\tau\right)^{\frac{1}{p}}$$
$$\times\left(\left|\frac{\partial^2}{\partial\tau\partial\xi}\sigma(\rho_1,\rho_3)\right|^q+\left|\frac{\partial^2}{\partial\tau\partial\xi}\sigma(\rho_1,\rho_3+\rho_4-\gamma)\right|^q\right.$$
$$\left.+\left|\frac{\partial^2}{\partial\tau\partial\xi}\sigma(\rho_1+\rho_2-\kappa,\rho_3)\right|^q+\left|\frac{\partial^2}{\partial\tau\partial\xi}\sigma(\rho_1+\rho_2-\kappa,\rho_3+\rho_4-\gamma)\right|^q\right)^{\frac{1}{q}}$$

where $\frac{1}{p}+\frac{1}{q}=1$ *and* $\Omega(\rho_1,\rho_2,\kappa;\rho_3,\rho_4,\gamma)$ *are defined as in Lemma 3.1.*

Proof. With the help of Hölder inequality and co-ordinated convexity of $\left|\frac{\partial^2\sigma}{\partial\tau\partial\xi}\right|^q$, we have

$$\int_0^1\int_0^1\Lambda_1(\kappa,\tau)\Lambda_2(\gamma,\xi) \tag{4.8}$$
$$\times\left|\frac{\partial^2}{\partial\tau\partial\xi}\sigma(\tau\rho_2+(1-\tau)(\rho_1+\rho_2-\kappa),\xi\rho_4+(1-\xi)(\rho_3+\rho_4-\gamma))\right|d\xi d\tau$$
$$\leq\left(\int_0^1\int_0^1[\Lambda_1(\kappa,\tau)\Lambda_2(\gamma,\xi)]^p\,d\xi d\tau\right)^{\frac{1}{p}}$$
$$\times\left(\int_0^1\int_0^1\left|\frac{\partial^2}{\partial\tau\partial\xi}\sigma(\tau\rho_2+(1-\tau)(\rho_1+\rho_2-\kappa),\xi\rho_4+(1-\xi)(\rho_3+\rho_4-\gamma))\right|^q d\xi d\tau\right)^{\frac{1}{q}}$$
$$\leq\frac{1}{2^{\frac{2}{q}}}\left(\int_0^1\int_0^1[\Lambda_1(\kappa,\tau)\Lambda_2(\gamma,\xi)]^p\,d\xi d\tau\right)^{\frac{1}{p}}$$
$$\times\left|\frac{\partial^2}{\partial\tau\partial\xi}\sigma(\rho_2,\rho_4)\right|^q+\left|\frac{\partial^2}{\partial\tau\partial\xi}\sigma(\rho_2,\rho_3+\rho_4-\gamma)\right|^q\Bigg)$$
$$\left.+\left|\frac{\partial^2}{\partial\tau\partial\xi}\sigma(\rho_1+\rho_2-\kappa,\rho_4)\right|^q+\left|\frac{\partial^2}{\partial\tau\partial\xi}\sigma(\rho_1+\rho_2-\kappa,\rho_3+\rho_4-\gamma)\right|^q\right)^{\frac{1}{q}}.$$

Similarly, we get

$$\int_0^1\int_0^1\Lambda_1(\kappa,\tau)\Delta_2(\gamma,\xi) \tag{4.9}$$
$$\times\left|\frac{\partial^2}{\partial\tau\partial\xi}\sigma(\tau\rho_2+(1-\tau)(\rho_1+\rho_2-\kappa),\xi\rho_3+(1-\xi)(\rho_3+\rho_4-\gamma))\right|d\xi d\tau$$
$$\leq\frac{1}{2^{\frac{2}{q}}}\left(\int_0^1\int_0^1[\Lambda_1(\kappa,\tau)\Delta_2(\gamma,\xi)]^p\,d\xi d\tau\right)^{\frac{1}{p}}$$
$$\times\left(\left|\frac{\partial^2}{\partial\tau\partial\xi}\sigma(\rho_2,\rho_3)\right|^q+\left|\frac{\partial^2}{\partial\tau\partial\xi}\sigma(\rho_2,\rho_3+\rho_4-\gamma)\right|^q\right.$$

$$+\left|\frac{\partial^2}{\partial\tau\partial\xi}\sigma\left(\rho_1+\rho_2-\kappa,\rho_3\right)\right|^q+\left|\frac{\partial^2}{\partial\tau\partial\xi}\sigma\left(\rho_1+\rho_2-\kappa,\rho_3+\rho_4-\gamma\right)\right|^q\Bigg)^{\frac{1}{q}},$$

$$\begin{aligned}
&\int_0^1\int_0^1 \Delta_1(\kappa,\tau)\Lambda_2(\gamma,\xi) \\
&\times\left|\frac{\partial^2}{\partial\tau\partial\xi}\sigma\left(\tau\rho_1+(1-\tau)\left(\rho_1+\rho_2-\kappa\right),\xi\rho_4+(1-\xi)\left(\rho_3+\rho_4-\gamma\right)\right)\right|d\xi d\tau \\
\leq\ & \frac{1}{2^{\frac{2}{q}}}\left(\int_0^1\int_0^1\left[\Delta_1(\kappa,\tau)\Lambda_2(\gamma,\xi)\right]^p d\xi d\tau\right)^{\frac{1}{p}} \\
&\times\Bigg(\left|\frac{\partial^2}{\partial\tau\partial\xi}\sigma\left(\rho_1,\rho_4\right)\right|^q+\left|\frac{\partial^2}{\partial\tau\partial\xi}\sigma\left(\rho_1,\rho_3+\rho_4-\gamma\right)\right|^q \\
&+\left|\frac{\partial^2}{\partial\tau\partial\xi}\sigma\left(\rho_1+\rho_2-\kappa,\rho_4\right)\right|^q+\left|\frac{\partial^2}{\partial\tau\partial\xi}\sigma\left(\rho_1+\rho_2-\kappa,\rho_3+\rho_4-\gamma\right)\right|^q\Bigg)^{\frac{1}{q}},
\end{aligned} \tag{4.10}$$

and

$$\begin{aligned}
&\int_0^1\int_0^1 \Delta_1(\kappa,\tau)\Delta_2(\gamma,\xi) \\
&\times\left|\frac{\partial^2}{\partial\tau\partial\xi}\sigma\left(\tau\rho_1+(1-\tau)\left(\rho_1+\rho_2-\kappa\right),\xi\rho_3+(1-\xi)\left(\rho_3+\rho_4-\gamma\right)\right)\right|d\xi d\tau \\
\leq\ & \frac{1}{2^{\frac{2}{q}}}\left(\int_0^1\int_0^1\left[\Delta_1(\kappa,\tau)\Delta_2(\gamma,\xi)\right]^p d\xi d\tau\right)^{\frac{1}{p}} \\
&\times\Bigg(\left|\frac{\partial^2}{\partial\tau\partial\xi}\sigma\left(\rho_1,\rho_3\right)\right|^q+\left|\frac{\partial^2}{\partial\tau\partial\xi}\sigma\left(\rho_1,\rho_3+\rho_4-\gamma\right)\right|^q \\
&+\left|\frac{\partial^2}{\partial\tau\partial\xi}\sigma\left(\rho_1+\rho_2-\kappa,\rho_3\right)\right|^q+\left|\frac{\partial^2}{\partial\tau\partial\xi}\sigma\left(\rho_1+\rho_2-\kappa,\rho_3+\rho_4-\gamma\right)\right|^q\Bigg)^{\frac{1}{q}}.
\end{aligned} \tag{4.11}$$

By substituting the inequalities (4.8)-(4.11) in (4.3), we establish required result. This is the end of the proof of Theorem 4.3. □

Theorem 4.4. *Assume that the assumptions of Lemma 3.1 hold. If the mapping* $\left|\frac{\partial^2\sigma}{\partial\tau\partial\xi}\right|^q$, $q\geq 1$ *is co-ordinated convex on* Δ, *then we get the following inequality for generalized fractional integrals*

$$\begin{aligned}
&\left|\Omega(\rho_1,\rho_2,\kappa;\rho_3,\rho_4,\gamma)\right| \\
\leq\ & \frac{\left(\kappa-\rho_1\right)\left(\gamma-\rho_3\right)}{\Upsilon(\kappa,\gamma)}\left(\int_0^1\int_0^1\Lambda_1(\kappa,\tau)\Lambda_2(\gamma,\xi)d\xi d\tau\right)^{1-\frac{1}{q}} \\
&\times\left[A_1B_1\left|\frac{\partial^2}{\partial\tau\partial\xi}\sigma\left(\rho_2,\rho_4\right)\right|^q+A_1B_2\left|\frac{\partial^2}{\partial\tau\partial\xi}\sigma\left(\rho_2,\rho_3+\rho_4-\gamma\right)\right|^q\right.
\end{aligned}$$

$$+A_2B_1\left|\frac{\partial^2}{\partial\tau\partial\xi}\sigma\left(\rho_1+\rho_2-\kappa,\rho_4\right)\right|^q+A_2B_2\left|\frac{\partial^2}{\partial\tau\partial\xi}\sigma\left(\rho_1+\rho_2-\kappa,\rho_3+\rho_4-\gamma\right)\right|^q\Bigg]^{\frac{1}{q}}$$

$$+\frac{\left(\kappa-\rho_1\right)\left(\rho_4-\gamma\right)}{\Upsilon(\kappa,\gamma)}\left(\int_0^1\int_0^1\Lambda_1(\kappa,\tau)\Delta_2(\gamma,\xi)d\xi d\tau\right)^{1-\frac{1}{q}}$$

$$\times\Bigg[A_1B_4\left|\frac{\partial^2}{\partial\tau\partial\xi}\sigma\left(\rho_2,\rho_3\right)\right|^q+A_1B_3\left|\frac{\partial^2}{\partial\tau\partial\xi}\sigma\left(\rho_2,\rho_3+\rho_4-\gamma\right)\right|^q$$

$$+A_2B_4\left|\frac{\partial^2}{\partial\tau\partial\xi}\sigma\left(\rho_1+\rho_2-\kappa,\rho_3\right)\right|^q+A_2B_3\left|\frac{\partial^2}{\partial\tau\partial\xi}\sigma\left(\rho_1+\rho_2-\kappa,\rho_3+\rho_4-\gamma\right)\right|^q\Bigg]^{\frac{1}{q}}$$

$$+\frac{\left(\rho_2-\kappa\right)\left(\gamma-\rho_3\right)}{\Upsilon(\kappa,\gamma)}\left(\int_0^1\int_0^1\Delta_1(\kappa,\tau)\Lambda_2(\gamma,\xi)d\xi d\tau\right)^{1-\frac{1}{q}}$$

$$\times\Bigg[A_4B_1\left|\frac{\partial^2}{\partial\tau\partial\xi}\sigma\left(\rho_1,\rho_4\right)\right|^q+A_4B_2\left|\frac{\partial^2}{\partial\tau\partial\xi}\sigma\left(\rho_1,\rho_3+\rho_4-\gamma\right)\right|^q$$

$$+A_3B_1\left|\frac{\partial^2}{\partial\tau\partial\xi}\sigma\left(\rho_1+\rho_2-\kappa,\rho_4\right)\right|^q+A_3B_2\left|\frac{\partial^2}{\partial\tau\partial\xi}\sigma\left(\rho_1+\rho_2-\kappa,\rho_3+\rho_4-\gamma\right)\right|^q\Bigg]^{\frac{1}{q}}$$

$$+\frac{\left(\rho_2-\kappa\right)\left(\rho_4-\gamma\right)}{\Upsilon(\kappa,\gamma)}\left(\int_0^1\int_0^1\Delta_1(\kappa,\tau)\Delta_2(\gamma,\xi)d\xi d\tau\right)^{1-\frac{1}{q}}$$

$$\times\Bigg[A_4B_4\left|\frac{\partial^2}{\partial\tau\partial\xi}\sigma\left(\rho_1,\rho_3\right)\right|^q+A_4B_3\left|\frac{\partial^2}{\partial\tau\partial\xi}\sigma\left(\rho_1,\rho_3+\rho_4-\gamma\right)\right|^q$$

$$+A_3B_4\left|\frac{\partial^2}{\partial\tau\partial\xi}\sigma\left(\rho_1+\rho_2-\kappa,\rho_3\right)\right|^q+A_3B_3\left|\frac{\partial^2}{\partial\tau\partial\xi}\sigma\left(\rho_1+\rho_2-\kappa,\rho_3+\rho_4-\gamma\right)\right|^q\Bigg]^{\frac{1}{q}}.$$

Here, $\Omega(\rho_1,\rho_2,\kappa;\rho_3,\rho_4,\gamma)$ is defined as in Lemma 3.1, A_i, $i=1,2,3,4$ are defined as in (4.1) and B_i, $i=1,2,3,4$ are defined as in (4.2).

Proof. Power mean inequality and co-ordinated convexity of $\left|\frac{\partial^2\sigma}{\partial\tau\partial\xi}\right|^q$ yield

$$\int_0^1\int_0^1\Lambda_1(\kappa,\tau)\Lambda_2(\gamma,\xi) \tag{4.12}$$

$$\times\left|\frac{\partial^2}{\partial\tau\partial\xi}\sigma\left(\tau\rho_2+(1-\tau)\left(\rho_1+\rho_2-\kappa\right),\xi\rho_4+(1-\xi)\left(\rho_3+\rho_4-\gamma\right)\right)\right|d\xi d\tau$$

$$\leq\left(\int_0^1\int_0^1\Lambda_1(\kappa,\tau)\Lambda_2(\gamma,\xi)d\xi d\tau\right)^{1-\frac{1}{q}}\left(\int_0^1\int_0^1\Lambda_1(\kappa,\tau)\Lambda_2(\gamma,\xi)\right.$$

$$\left.\times\left|\frac{\partial^2}{\partial\tau\partial\xi}\sigma\left(\tau\rho_2+(1-\tau)\left(\rho_1+\rho_2-\kappa\right),\xi\rho_4+(1-\xi)\left(\rho_3+\rho_4-\gamma\right)\right)\right|^q d\xi d\tau\right)^{\frac{1}{q}}$$

$$\leq \left(\int_0^1\int_0^1 \Lambda_1(\kappa,\tau)\Lambda_2(\gamma,\xi)d\xi d\tau\right)^{1-\frac{1}{q}}$$
$$\left(\int_0^1\int_0^1 \Lambda_1(\kappa,\tau)\Lambda_2(\gamma,\xi)\right.$$
$$\times\left[\tau\xi\left|\frac{\partial^2}{\partial\tau\partial\xi}\sigma\left(\rho_2,\rho_4\right)\right|^q+\tau\left(1-\xi\right)\left|\frac{\partial^2}{\partial\tau\partial\xi}\sigma\left(\rho_2,\rho_3+\rho_4-\gamma\right)\right|^q\right.$$
$$+\left(1-\tau\right)\xi\left|\frac{\partial^2}{\partial\tau\partial\xi}\sigma\left(\rho_1+\rho_2-\kappa,\rho_4\right)\right|^q$$
$$\left.\left.+\left(1-\tau\right)\left(1-\xi\right)\left|\frac{\partial^2}{\partial\tau\partial\xi}\sigma\left(\rho_1+\rho_2-\kappa,\rho_3+\rho_4-\gamma\right)\right|^q\right]d\xi d\tau\right)^{\frac{1}{q}}$$
$$= \left(\int_0^1\int_0^1 \Lambda_1(\kappa,\tau)\Lambda_2(\gamma,\xi)d\xi d\tau\right)^{1-\frac{1}{q}}$$
$$\times\left(A_1B_1\left|\frac{\partial^2}{\partial\tau\partial\xi}\sigma\left(\rho_2,\rho_4\right)\right|^q+A_1B_2\left|\frac{\partial^2}{\partial\tau\partial\xi}\sigma\left(\rho_2,\rho_3+\rho_4-\gamma\right)\right|^q\right.$$
$$\left.+A_2B_1\left|\frac{\partial^2}{\partial\tau\partial\xi}\sigma\left(\rho_1+\rho_2-\kappa,\rho_4\right)\right|^q+A_2B_2\left|\frac{\partial^2}{\partial\tau\partial\xi}\sigma\left(\rho_1+\rho_2-\kappa,\rho_3+\rho_4-\gamma\right)\right|^q\right)^{\frac{1}{q}}.$$

Similarly, we have

$$\int_0^1\int_0^1 \Lambda_1(\kappa,\tau)\Delta_2(\gamma,\xi) \tag{4.13}$$
$$\times\left|\frac{\partial^2}{\partial\tau\partial\xi}\sigma\left(\tau\rho_2+\left(1-\tau\right)\left(\rho_1+\rho_2-\kappa\right),\xi\rho_3+\left(1-\xi\right)\left(\rho_3+\rho_4-\gamma\right)\right)\right|d\xi d\tau$$
$$\leq \left(\int_0^1\int_0^1 \Lambda_1(\kappa,\tau)\Delta_2(\gamma,\xi)d\xi d\tau\right)^{1-\frac{1}{q}}$$
$$\times\left(A_1B_4\left|\frac{\partial^2}{\partial\tau\partial\xi}\sigma\left(\rho_2,\rho_3\right)\right|^q+A_1B_3\left|\frac{\partial^2}{\partial\tau\partial\xi}\sigma\left(\rho_2,\rho_3+\rho_4-\gamma\right)\right|^q\right.$$
$$\left.+A_2B_4\left|\frac{\partial^2}{\partial\tau\partial\xi}\sigma\left(\rho_1+\rho_2-\kappa,\rho_3\right)\right|^q+A_2B_3\left|\frac{\partial^2}{\partial\tau\partial\xi}\sigma\left(\rho_1+\rho_2-\kappa,\rho_3+\rho_4-\gamma\right)\right|^q\right)^{\frac{1}{q}},$$

$$\int_0^1\int_0^1 \Delta_1(\kappa,\tau)\Lambda_2(\gamma,\xi) \tag{4.14}$$
$$\times\left|\frac{\partial^2}{\partial\tau\partial\xi}\sigma\left(\tau\rho_1+\left(1-\tau\right)\left(\rho_1+\rho_2-\kappa\right),\xi\rho_4+\left(1-\xi\right)\left(\rho_3+\rho_4-\gamma\right)\right)\right|d\xi d\tau$$
$$\leq \left(\int_0^1\int_0^1 \Delta_1(\kappa,\tau)\Lambda_2(\gamma,\xi)d\xi d\tau\right)^{1-\frac{1}{q}}$$

$$\left(A_4B_1\left|\frac{\partial^2}{\partial\tau\partial\xi}\sigma\left(\rho_1,\rho_4\right)\right|^q+A_4B_2\left|\frac{\partial^2}{\partial\tau\partial\xi}\sigma\left(\rho_1,\rho_3+\rho_4-\gamma\right)\right|^q\right.$$
$$\left.+A_3B_1\left|\frac{\partial^2}{\partial\tau\partial\xi}\sigma\left(\rho_1+\rho_2-\kappa,\rho_4\right)\right|^q+A_3B_2\left|\frac{\partial^2}{\partial\tau\partial\xi}\sigma\left(\rho_1+\rho_2-\kappa,\rho_3+\rho_4-\gamma\right)\right|^q\right)^{\frac{1}{q}},$$

and

$$\int_0^1\int_0^1\Delta_1(\kappa,\tau)\Delta_2(\gamma,\xi) \tag{4.15}$$
$$\times\left|\frac{\partial^2}{\partial\tau\partial\xi}\sigma\left(\tau\rho_1+(1-\tau)\left(\rho_1+\rho_2-\kappa\right),\xi\rho_3+(1-\xi)\left(\rho_3+\rho_4-\gamma\right)\right)\right|d\xi d\tau$$
$$\leq\left(\int_0^1\int_0^1\Delta_1(\kappa,\tau)\Delta_2(\gamma,\xi)d\xi d\tau\right)^{1-\frac{1}{q}}$$
$$\left(A_4B_4\left|\frac{\partial^2}{\partial\tau\partial\xi}\sigma\left(\rho_1,\rho_3\right)\right|^q+A_4B_3\left|\frac{\partial^2}{\partial\tau\partial\xi}\sigma\left(\rho_1,\rho_3+\rho_4-\gamma\right)\right|^q\right.$$
$$\left.+A_3B_4\left|\frac{\partial^2}{\partial\tau\partial\xi}\sigma\left(\rho_1+\rho_2-\kappa,\rho_3\right)\right|^q+A_3B_3\left|\frac{\partial^2}{\partial\tau\partial\xi}\sigma\left(\rho_1+\rho_2-\kappa,\rho_3+\rho_4-\gamma\right)\right|^q\right)^{\frac{1}{q}}.$$

If we substitute the inequalities (4.12)-(4.15) in (4.3), then we establish desired result. This ends the proof of Theorem 4.4. □

5. Conclusions

We proved an identity for twice partially differentiable mappings involving the double generalized fractional integral. This will lead to new research. The special cases obtained show how valuable this study is. The new identity here can be used in different studies. Researchers can do new studies with different types of convexity.

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ON NEW INEQUALITIES OF HERMITE-HADAMARD TYPE FOR FRACTIONAL INTEGRALS

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Abstract. In the present paper, we prove a new version of the Hermite-Hadamard inequality for generalized fractional integrals. We also establish a new identity for generalized fractional integrals. Furthermore, the fractional integral operators have been applied to Hermite Hadamard type integral inequalities to provide their generalized properties.

1. Introduction & Preliminaries

The inequalities discovered by C. Hermite and J. Hadamard for convex functions are very significant in the literature (see, e.g.,[11, p.137], [5]). These inequalities state that if $\varphi : I \to \mathbb{R}$ is a convex function on the interval I of real numbers and $\kappa_1, \kappa_2 \in I$ with $\kappa_1 < \kappa_2$, then

$$\varphi\left(\frac{\kappa_1+\kappa_2}{2}\right) \leq \frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\varphi(\tau)d\tau \leq \frac{\varphi(\kappa_1)+\varphi(\kappa_2)}{2}.$$

if φ is concave, then both inequalities hold in the reversed direction. Let us note that Hadamard's inequality may be regarded as a refinement of the concept of convexity and it follows easily from Jensen's inequality. Moreover, Hadamard's inequality for convex functions has been received considerable attention in recent years and a remarkable variety of refinements and generalizations have been studied extensively (see, for example, [1, 2, 5, 6, 11, 15, 16]).

On the other hand, a number of mathematicians have studied the fractional integral inequalities and their applications using Riemann–Liouville fractional integrals. For more information and result about Hermite–Hadamard type inequalities involving fractional integrals, we refer the reader to [3, 4, 8, 12, 13, 14, 17, 19, 20] and the references therein. In the following, we will give a brief synopsis of all necessary definitions and results that will be required. More details, one can consult [7, 9, 10].

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Definition 1.1. Let us consider $\varphi \in L_1[\kappa_1, \kappa_2]$. The Riemann-Liouville fractional integrals $J^{\alpha}_{\kappa_1+}\varphi$ and $J^{\alpha}_{\kappa_2-}\varphi$ of order $\alpha > 0$ with $\kappa_1 \geq 0$ are defined by

$$J^{\alpha}_{\kappa_1+}\varphi(\tau) = \frac{1}{\Gamma(\alpha)} \int_{\kappa_1}^{\tau} (\tau - t)^{\alpha-1} \varphi(t)dt, \quad \tau > \kappa_1$$

and

$$J^{\alpha}_{\kappa_2-}\varphi(\tau) = \frac{1}{\Gamma(\alpha)} \int_{\tau}^{\kappa_2} (t - \tau)^{\alpha-1} \varphi(t)dt, \quad \tau < \kappa_2,$$

respectively. Here, $\Gamma(\alpha)$ is the Gamma function and $J^{0}_{\kappa_1+}\varphi(\tau) = J^{0}_{\kappa_2-}\varphi(\tau) = \varphi(\tau)$.

In [13], Sarikaya et al. proved a variant of Hermite–Hadamard's inequalities in Riemann-Liouville fractional integral forms as follows:

Theorem 1.2. *Let $\varphi : [\kappa_1, \kappa_2] \to \mathbb{R}$ be a positive function with $0 \leq \kappa_1 < \kappa_2$ and $\varphi \in L_1[\kappa_1, \kappa_2]$. If φ is a convex function on $[\kappa_1, \kappa_2]$, then the following inequalities for fractional integrals*

$$\varphi\left(\frac{\kappa_1 + \kappa_2}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(\kappa_2 - \kappa_1)^{\alpha}} \left[J^{\alpha}_{\kappa_1+}\varphi(\kappa_2) + J^{\alpha}_{\kappa_2-}\varphi(\kappa_1)\right] \leq \frac{\varphi(\kappa_1) + \varphi(\kappa_2)}{2}$$

is valid with $\alpha > 0$.

Sarikaya et al. introduce the following generalized fractional integrals and they also prove the corresponding Hermite-Hadamard inequality in [18].

Definition 1.3. [18] Let $u : [\kappa_1, \kappa_2] \to \mathbb{R}$ be an increasing and positive monotone function on (κ_1, κ_2) and $\varphi, u \in L[\kappa_1, \kappa_2]$ with $\kappa_1 < \kappa_2$. The generalized Riemann-Liouville fractional integrals $J^{\alpha,k}_{\kappa_1+,u}\varphi$ and $J^{\alpha,k}_{\kappa_2-,u}\varphi$ of order $\alpha > 0$ with $\kappa_1 \geq 0$ are defined by

$$J^{\alpha,k}_{\kappa_1+,u}(\varphi)(\tau) = \frac{1}{\Gamma(\alpha)} \int_{\kappa_1}^{\tau} (\tau - t)^{\alpha-1} (u(\tau) - u(t))^k \varphi(t)dt, \quad \tau > \kappa_1$$

and

$$J^{\alpha,k}_{\kappa_2-,u}(\varphi)(\tau) = \frac{1}{\Gamma(\alpha)} \int_{\tau}^{\kappa_2} (t - \tau)^{\alpha-1} (u(t) - u(\tau))^k \varphi(t)dt, \quad \tau < \kappa_2$$

provided that the integrals exist, respectively, $k \in N \cup \{0\}$.

Theorem 1.4. [18] *Let $\varphi : [\kappa_1, \kappa_2] \to \mathbb{R}$ be a convex function on $[\kappa_1, \kappa_2]$ and $u : [\kappa_1, \kappa_2] \to \mathbb{R}$ be an increasing and positive monotone function on (κ_1, κ_2) and $\varphi, u \in L[\kappa_1, \kappa_2]$ with $\kappa_1 < \kappa_2$. Then, Θ is also integrable and the following inequalities for fractional integral operators*

$$\begin{aligned} \varphi\left(\frac{\kappa_1 + \kappa_2}{2}\right) &\left[J^{\alpha,k}_{\kappa_1+,u}(1)(\kappa_2) + J^{\alpha,k}_{\kappa_2-,u}(1)(\kappa_1)\right] \\ \leq\ & \frac{1}{2}\left[J^{\alpha,k}_{\kappa_1+,u}(\Theta)(\kappa_2) + J^{\alpha,k}_{\kappa_2-,u}(\Theta)(\kappa_1)\right] \\ \leq\ & \left[J^{\alpha,k}_{\kappa_1+,u}(1)(\kappa_2) + J^{\alpha,k}_{\kappa_2-,u}(1)(\kappa_1)\right] \frac{\varphi(\kappa_1) + \varphi(\kappa_2)}{2} \end{aligned} \tag{1.1}$$

is valid with $\alpha > 0$ and $k \in N \cup \{0\}$.

The aim of this paper is to establish new variant of the inequality (1.1) and to obtaint some corresponding midpoint type inequalities.

2. New Hermite-Hadamard type inequalities

Let us start with some notations given in [8] for obtaining our results. Let $\varphi : I^{\circ} \to \mathbb{R}$ be a function such that $\kappa_1, \kappa_2 \in I^{\circ}$ and $0 < \kappa_1 < \kappa_2 < \infty$. Throughout this article, we suppose that $\Theta(\tau) = \varphi(\tau) + \varphi(\kappa_1 + \kappa_2 - \tau)$ for $\tau \in [\kappa_1, \kappa_2]$. Then it is easy to show that if φ is a convex function, then Θ is also convex function.

Now, we prove a new version of the Hermite-Hadamard inequality (1.1).

Theorem 2.1. *Suppose $\varphi : [\kappa_1, \kappa_2] \to \mathbb{R}$ is a convex function on $[\kappa_1, \kappa_2]$ and $u : [\kappa_1, \kappa_2] \to \mathbb{R}$ is an increasing, positive monotone function on (κ_1, κ_2). Suppose also $\varphi, u \in L[\kappa_1, \kappa_2]$ with $\kappa_1 < \kappa_2$. Then, Θ is also integrable and we have the following Hermite-Hadamard inequalities for generalized fractional integral operators*

$$\varphi\left(\frac{\kappa_1+\kappa_2}{2}\right)\left[J^{\alpha,k}_{\left(\frac{\kappa_1+\kappa_2}{2}\right)+,u}(1)(\kappa_2) + J^{\alpha,k}_{\left(\frac{\kappa_1+\kappa_2}{2}\right)-,u}(1)(\kappa_1)\right] \tag{2.1}$$

$$\leq \frac{1}{2}\left[J^{\alpha,k}_{\left(\frac{\kappa_1+\kappa_2}{2}\right)+,u}(\Theta)(\kappa_2) + J^{\alpha,k}_{\left(\frac{\kappa_1+\kappa_2}{2}\right)-,u}(\Theta)(\kappa_1)\right]$$

$$\leq \left[J^{\alpha,k}_{\left(\frac{\kappa_1+\kappa_2}{2}\right)+,u}(1)(\kappa_2) + J^{\alpha,k}_{\left(\frac{\kappa_1+\kappa_2}{2}\right)-,u}(1)(\kappa_1)\right]\frac{\varphi(\kappa_1)+\varphi(\kappa_2)}{2}$$

for $\alpha > 0$ and $k \in N \cup \{0\}$.

Proof. Since φ is an convex mapping on $[\kappa_1, \kappa_2]$, we have

$$\varphi\left(\frac{\tau+y}{2}\right) \leq \frac{\varphi(\tau)+\varphi(y)}{2}$$

for $\tau, y \in [\kappa_1, \kappa_2]$. Now, for $t \in [0,1]$, let us note that $\tau = \frac{t}{2}\kappa_1 + \frac{2-t}{2}\kappa_2$ and $y = \frac{2-t}{2}\kappa_1 + \frac{t}{2}\kappa_2$. Then, we find that

$$2\varphi\left(\frac{\kappa_1+\kappa_2}{2}\right) \leq \varphi\left(\frac{t}{2}\kappa_1 + \frac{2-t}{2}\kappa_2\right) + \varphi\left(\frac{2-t}{2}\kappa_1 + \frac{t}{2}\kappa_2\right). \tag{2.2}$$

If we multiply both sides of inequality (2.2) by $t^{\alpha-1}\left(u(\kappa_2) - u\left(\frac{t}{2}\kappa_1 + \frac{2-t}{2}\kappa_2\right)\right)^k$ and integrate the resulting inequality with respect to t over $[0,1]$, then the following inequality holds:

$$\varphi\left(\frac{\kappa_1+\kappa_2}{2}\right)\int_0^1 t^{\alpha-1}\left(u(\kappa_2) - u\left(\frac{t}{2}\kappa_1 + \frac{2-t}{2}\kappa_2\right)\right)^k dt$$

$$\leq \frac{1}{2}\left[\int_0^1 t^{\alpha-1}\left(u(\kappa_2) - u\left(\frac{t}{2}\kappa_1 + \frac{2-t}{2}\kappa_2\right)\right)^k \varphi\left(\frac{t}{2}\kappa_1 + \frac{2-t}{2}\kappa_2\right)dt\right.$$

$$\left.+\int_0^1 t^{\alpha-1}\left(u(\kappa_2) - u\left(\frac{t}{2}\kappa_1 + \frac{2-t}{2}\kappa_2\right)\right)^k \varphi\left(\frac{2-t}{2}\kappa_1 + \frac{t}{2}\kappa_2\right)dt\right].$$

Using the change of variable $y = \frac{t}{2}\kappa_1 + \frac{2-t}{2}\kappa_2$, we obtain

$$\varphi\left(\frac{\kappa_1+\kappa_2}{2}\right)\left(\frac{2}{\kappa_2-\kappa_1}\right)^{\alpha}\int_{\frac{\kappa_1+\kappa_2}{2}}^{\kappa_2}(\kappa_2 - y)^{\alpha-1}\left(u(\kappa_2) - u(y)\right)^k dy$$

$$\leq \frac{1}{2}\left[\left(\frac{2}{\kappa_2-\kappa_1}\right)^{\alpha}\int_{\frac{\kappa_1+\kappa_2}{2}}^{\kappa_2}(\kappa_2-y)^{\alpha-1}(u(\kappa_2)-u(y))^k\varphi(y)\,dy\right.$$

$$\left.+\left(\frac{2}{\kappa_2-\kappa_1}\right)^{\alpha}\int_{\frac{\kappa_1+\kappa_2}{2}}^{\kappa_2}(\kappa_2-y)^{\alpha-1}(u(\kappa_2)-u(y))^k\varphi(\kappa_1+\kappa_2-y)\,dy\right]$$

$$=\frac{1}{2}\left(\frac{2}{\kappa_2-\kappa_1}\right)^{\alpha}\int_{\frac{\kappa_1+\kappa_2}{2}}^{\kappa_2}(\kappa_2-y)^{\alpha-1}(u(\kappa_2)-u(y))^k\left[\varphi(y)+\varphi(\kappa_1+\kappa_2-y)\right]dy.$$

That is

$$\varphi\left(\frac{\kappa_1+\kappa_2}{2}\right)J^{\alpha,k}_{\left(\frac{\kappa_1+\kappa_2}{2}\right)+,u}(1)(\kappa_2)\leq\frac{1}{2}J^{\alpha,k}_{\left(\frac{\kappa_1+\kappa_2}{2}\right)+,u}(\Theta)(\kappa_2). \tag{2.3}$$

Similarly, if we multiply both sides of (2.2) by $t^{\alpha-1}\left(u\left(\frac{2-t}{2}\kappa_1+\frac{t}{2}\kappa_2\right)-u(\kappa_1)\right)^k$ and integrate the resulting inequality with respect to t over $[0,1]$, we have

$$\varphi\left(\frac{\kappa_1+\kappa_2}{2}\right)\int_0^1 t^{\alpha-1}\left(u\left(\frac{2-t}{2}\kappa_1+\frac{t}{2}\kappa_2\right)-u(\kappa_1)\right)^k dt$$

$$\leq\frac{1}{2}\left[\int_0^1 t^{\alpha-1}\left(u\left(\frac{2-t}{2}\kappa_1+\frac{t}{2}\kappa_2\right)-u(\kappa_1)\right)^k\varphi\left(\frac{t}{2}\kappa_1+\frac{2-t}{2}\kappa_2\right)dt\right.$$

$$\left.+\int_0^1 t^{\alpha-1}\left(u\left(\frac{2-t}{2}\kappa_1+\frac{t}{2}\kappa_2\right)-u(\kappa_1)\right)^k\varphi\left(\frac{2-t}{2}\kappa_1+\frac{t}{2}\kappa_2\right)dt\right].$$

Using the change of variable $y=\frac{2-t}{2}\kappa_1+\frac{t}{2}\kappa_2$, we get

$$\varphi\left(\frac{\kappa_1+\kappa_2}{2}\right)J^{\alpha,k}_{\left(\frac{\kappa_1+\kappa_2}{2}\right)-,u}(1)(\kappa_1)\leq\frac{1}{2}J^{\alpha,k}_{\left(\frac{\kappa_1+\kappa_2}{2}\right)-,u}(\Theta)(\kappa_1). \tag{2.4}$$

If we collect the inequalities (2.3) and (2.4), then the following inequality holds:

$$\varphi\left(\frac{\kappa_1+\kappa_2}{2}\right)\left[J^{\alpha,k}_{\left(\frac{\kappa_1+\kappa_2}{2}\right)+,u}(1)(\kappa_2)+J^{\alpha,k}_{\left(\frac{\kappa_1+\kappa_2}{2}\right)-,u}(1)(\kappa_1)\right]$$

$$\leq\frac{1}{2}\left[J^{\alpha,k}_{\left(\frac{\kappa_1+\kappa_2}{2}\right)+,u}(\Theta)(\kappa_2)+J^{\alpha,k}_{\left(\frac{\kappa_1+\kappa_2}{2}\right)-,u}(\Theta)(\kappa_1)\right].$$

This completes the proof of first part of inequality in (2.1).

For the proof of the second part of inequality in (2.1), since φ is convex, we have

$$\varphi\left(\frac{t}{2}\kappa_1+\frac{2-t}{2}\kappa_2\right)+\varphi\left(\frac{2-t}{2}\kappa_1+\frac{t}{2}\kappa_2\right)\leq\left[\varphi(\kappa_1)+\varphi(\kappa_2)\right]. \tag{2.5}$$

If we multiply both sides of inequality (2.5) by $t^{\alpha-1}\left(u(\kappa_2)-u\left(\frac{t}{2}\kappa_1+\frac{2-t}{2}\kappa_2\right)\right)^k$ and integrate the resulting inequality with respect to t over $[0,1]$, then we obtain

$$\begin{aligned}
&\int_0^1 t^{\alpha-1}\left(u(\kappa_2)-u\left(\frac{t}{2}\kappa_1+\frac{2-t}{2}\kappa_2\right)\right)^k \varphi\left(\frac{t}{2}\kappa_1+\frac{2-t}{2}\kappa_2\right)dt \\
&+\int_0^1 t^{\alpha-1}\left(u(\kappa_2)-u\left(\frac{t}{2}\kappa_1+\frac{2-t}{2}\kappa_2\right)\right)^k \varphi\left(\frac{2-t}{2}\kappa_1+\frac{t}{2}\kappa_2\right)dt \\
\leq\ & [\varphi(\kappa_1)+\varphi(\kappa_2)]\int_0^1 t^{\alpha-1}\left(u(\kappa_2)-u\left(\frac{t}{2}\kappa_1+\frac{2-t}{2}\kappa_2\right)\right)^k dt.
\end{aligned}$$

Then, we get

$$J^{\alpha,k}_{\left(\frac{\kappa_1+\kappa_2}{2}\right)+,u}(\Theta)(\kappa_2) \leq [\varphi(\kappa_1)+\varphi(\kappa_2)]\, J^{\alpha,k}_{\left(\frac{\kappa_1+\kappa_2}{2}\right)+,u}(1)(\kappa_2). \tag{2.6}$$

Similarly, multiplying both sides of (2.5) by $t^{\alpha-1}\left(u\left(\frac{2-t}{2}\kappa_1+\frac{t}{2}\kappa_2\right)-u(\kappa_1)\right)^k$ and integrating the resulting inequality with respect to t over $[0,1]$, we get

$$J^{\alpha,k}_{\left(\frac{\kappa_1+\kappa_2}{2}\right)-,u}(\Theta)(\kappa_1) \leq [\varphi(\kappa_1)+\varphi(\kappa_2)]\, J^{\alpha,k}_{\left(\frac{\kappa_1+\kappa_2}{2}\right)-,u}(1)(\kappa_1). \tag{2.7}$$

By adding the inequalities (2.6) and (2.7), we have

$$\begin{aligned}
&\frac{1}{2}\left[J^{\alpha,k}_{\left(\frac{\kappa_1+\kappa_2}{2}\right)+,u}(\Theta)(\kappa_2)+J^{\alpha,k}_{\left(\frac{\kappa_1+\kappa_2}{2}\right)-,u}(\Theta)(\kappa_1)\right] \\
\leq\ & \left[J^{\alpha,k}_{\left(\frac{\kappa_1+\kappa_2}{2}\right)+,u}(1)(\kappa_2)+J^{\alpha,k}_{\left(\frac{\kappa_1+\kappa_2}{2}\right)-,u}(1)(\kappa_1)\right]\frac{\varphi(\kappa_1)+\varphi(\kappa_2)}{2}.
\end{aligned}$$

This is the end of the proof of Theorem 2.1. □

Remark 2.2. If we choose $k=0$ in Theorem 2.1, then we have the inequality

$$\varphi\left(\frac{\kappa_1+\kappa_2}{2}\right) \leq \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\kappa_2-\kappa_1)^{\alpha}}\left[J^{\alpha}_{\left(\frac{\kappa_1+\kappa_2}{2}\right)+}\varphi(\kappa_2)+J^{\alpha}_{\left(\frac{\kappa_1+\kappa_2}{2}\right)-}\varphi(\kappa_1)\right] \leq \frac{\varphi(\kappa_1)+\varphi(\kappa_2)}{2}$$

which is proved by Sarikaya and Yıldırım in [12].

Corollary 1. Let us consider $u(t)=t$ in Theorem 2.1. Then, we have the following inequality for Riemann-Liouville fractional integrals

$$\varphi\left(\frac{\kappa_1+\kappa_2}{2}\right) \leq \frac{2^{\alpha+k-1}\Gamma(\alpha+k+1)}{(\kappa_2-\kappa_1)^{\alpha+k}}\left[J^{\alpha+k}_{\left(\frac{\kappa_1+\kappa_2}{2}\right)+}\varphi(\kappa_2)+J^{\alpha+k}_{\left(\frac{\kappa_1+\kappa_2}{2}\right)-}\varphi(\kappa_1)\right] \leq \frac{\varphi(\kappa_1)+\varphi(\kappa_2)}{2}.$$

Proof. From Definition 1.3 with $u(t)=t$, we have

$$\begin{aligned}
J^{\alpha,k}_{\left(\frac{\kappa_1+\kappa_2}{2}\right)+,u}(\Theta)(\kappa_2) &= \frac{1}{\Gamma(\alpha)}\int_{\frac{\kappa_1+\kappa_2}{2}}^{\kappa_2}(\kappa_2-y)^{\alpha+k-1}\,\Theta(y)\,dy \\
&= \frac{1}{\Gamma(\alpha)}\int_{\frac{\kappa_1+\kappa_2}{2}}^{\kappa_2}(\kappa_2-y)^{\alpha+k-1}\,\varphi(y)\,dy
\end{aligned}$$

$$+ \frac{1}{\Gamma(\alpha)} \int_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}} (t-\kappa_1)^{\alpha+k-1} \varphi(t)\, dy$$
$$= \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)} \left[J^{\alpha+k}_{\left(\frac{\kappa_1+\kappa_2}{2}\right)+} \varphi(\kappa_2) + J^{\alpha+k}_{\left(\frac{\kappa_1+\kappa_2}{2}\right)-} \varphi(\kappa_1) \right]$$

and similarly,

$$J^{\alpha,k}_{\left(\frac{\kappa_1+\kappa_2}{2}\right)-,u} (\Theta)(\kappa_1) = \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)} \left[J^{\alpha+k}_{\left(\frac{\kappa_1+\kappa_2}{2}\right)+} \varphi(\kappa_2) + J^{\alpha+k}_{\left(\frac{\kappa_1+\kappa_2}{2}\right)-} \varphi(\kappa_1) \right]. \tag{2.8}$$

On the other hand, we get

$$J^{\alpha,k}_{\left(\frac{\kappa_1+\kappa_2}{2}\right)+,u} (1)(\kappa_2) = \frac{1}{(\alpha+k)\,\Gamma(\alpha)} \left(\frac{\kappa_2-\kappa_1}{2} \right)^{\alpha+k} \tag{2.9}$$

and

$$J^{\alpha,k}_{\left(\frac{\kappa_1+\kappa_2}{2}\right)-,u} (1)(\kappa_1) = \frac{1}{(\alpha+k)\,\Gamma(\alpha)} \left(\frac{\kappa_2-\kappa_1}{2} \right)^{\alpha+k}. \tag{2.10}$$

By using the equalities (2.8)-(2.10), we obtain the desired result. □

3. New Midpoint type inequalities

Now, we give a new identity for generalized fractional integrals and we present some new midpoint type inequalities.

Lemma 3.1. *Assume $\varphi : [\kappa_1, \kappa_2] \to \mathbb{R}$ is a differentiable function on (κ_1, κ_2) and assume also $u : [\kappa_1, \kappa_2] \to \mathbb{R}$ is an increasing and positive monotone function on (κ_1, κ_2) with $\kappa_1 < \kappa_2$. If $\varphi', u \in L[\kappa_1, \kappa_2]$, then Θ is also differentiable and $\Theta \in L[\kappa_1, \kappa_2]$. Then the following equality*

$$\begin{aligned} &\frac{1}{2} \left[J^{\alpha,k}_{\left(\frac{\kappa_1+\kappa_2}{2}\right)+,u} (\Theta)(\kappa_2) + J^{\alpha,k}_{\left(\frac{\kappa_1+\kappa_2}{2}\right)-,u} (\Theta)(\kappa_1) \right] \\ &- \left[J^{\alpha,k}_{\left(\frac{\kappa_1+\kappa_2}{2}\right)+,u} (1)(\kappa_2) + J^{\alpha,k}_{\left(\frac{\kappa_1+\kappa_2}{2}\right)-,u} (1)(\kappa_1) \right] \varphi\left(\frac{\kappa_1+\kappa_2}{2} \right) \\ = &\frac{(\kappa_2-\kappa_1)^{\alpha+1}}{2^{\alpha+2}\Gamma(\alpha)} \int_0^1 G_k(u;t) \Theta' \left(\frac{t}{2}\kappa_1 + \frac{2-t}{2}\kappa_2 \right) dt \end{aligned}$$

is valid. Here, $\Theta'(y) = \varphi'(y) - \varphi'(\kappa_1+\kappa_2-y)$ and

$$\begin{aligned} G_k(u;t) = &\int_0^t s^{\alpha-1} \left[\left(u(\kappa_2) - u\left(\frac{s}{2}\kappa_1 + \frac{2-s}{2}\kappa_2 \right) \right)^k \right. \\ &\left. + \left(u\left(\frac{2-s}{2}\kappa_1 + \frac{s}{2}\kappa_2 \right) - u(\kappa_1) \right)^k \right] ds. \end{aligned} \tag{3.1}$$

Proof. With the help of the integration by parts, we have

$$\int_0^1 G_k(u;t) \Theta' \left(\frac{t}{2}\kappa_1 + \frac{2-t}{2}\kappa_2 \right) dt \tag{3.2}$$

$$\begin{aligned}
&= -\frac{2}{\kappa_2-\kappa_1}G_k(u;t)\Theta\left(\frac{t}{2}\kappa_1+\frac{2-t}{2}\kappa_2\right)\Big|_0^1 \\
&+\frac{2}{\kappa_2-\kappa_1}\int_0^1 t^{\alpha-1}\left[\left(u(\kappa_2)-u\left(\frac{t}{2}\kappa_1+\frac{2-t}{2}\kappa_2\right)\right)^k\right. \\
&\left.+\left(u\left(\frac{2-t}{2}\kappa_1+\frac{t}{2}\kappa_2\right)-u(\kappa_1)\right)^k\right]\Theta\left(\frac{t}{2}\kappa_1+\frac{2-t}{2}\kappa_2\right)dt \\
&= -\frac{2}{\kappa_2-\kappa_1}G_k(u;1)\Theta\left(\frac{\kappa_1+\kappa_2}{2}\right) \\
&+\frac{2}{\kappa_2-\kappa_1}\int_0^1 t^{\alpha-1}\left(u(\kappa_2)-u\left(\frac{t}{2}\kappa_1+\frac{2-t}{2}\kappa_2\right)\right)^k\Theta\left(\frac{t}{2}\kappa_1+\frac{2-t}{2}\kappa_2\right)dt \\
&+\frac{2}{\kappa_2-\kappa_1}\int_0^1 t^{\alpha-1}\left(u\left(\frac{2-t}{2}\kappa_1+\frac{t}{2}\kappa_2\right)-u(\kappa_1)\right)^k\Theta\left(\frac{t}{2}\kappa_1+\frac{2-t}{2}\kappa_2\right)dt.
\end{aligned}$$

By Definition 1.3, we obtain

$$\begin{aligned}
G_k(u;1) &= \int_0^1 s^{\alpha-1}\left(u(\kappa_2)-u\left(\frac{s}{2}\kappa_1+\frac{2-s}{2}\kappa_2\right)\right)^k ds \\
&+\int_0^1 s^{\alpha-1}\left(u\left(\frac{2-s}{2}\kappa_1+\frac{s}{2}\kappa_2\right)-u(\kappa_1)\right)^k ds \\
&= \left(\frac{2}{\kappa_2-\kappa_1}\right)^\alpha\int_{\frac{\kappa_1+\kappa_2}{2}}^{\kappa_2}(\kappa_2-t)^{\alpha-1}(u(\kappa_2)-u(t))^k dt \\
&+\left(\frac{2}{\kappa_2-\kappa_1}\right)^\alpha\int_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}}(t-\kappa_1)^{\alpha-1}(u(t)-u(\kappa_1))^k dt \\
&= \Gamma(\alpha)\left(\frac{2}{\kappa_2-\kappa_1}\right)^\alpha\left[J^{\alpha,k}_{\left(\frac{\kappa_1+\kappa_2}{2}\right)+,u}(1)(\kappa_2)+J^{\alpha,k}_{\left(\frac{\kappa_1+\kappa_2}{2}\right)-,u}(1)(\kappa_1)\right].
\end{aligned} \tag{3.3}$$

On the other hand, we get

$$\begin{aligned}
&\int_0^1 t^{\alpha-1}\left(u(\kappa_2)-u\left(\frac{t}{2}\kappa_1+\frac{2-t}{2}\kappa_2\right)\right)^k\Theta\left(\frac{t}{2}\kappa_1+\frac{2-t}{2}\kappa_2\right)dt \\
&= \left(\frac{2}{\kappa_2-\kappa_1}\right)^\alpha\int_{\frac{\kappa_1+\kappa_2}{2}}^{\kappa_2}(\kappa_2-t)^{\alpha-1}(u(\kappa_2)-u(t))^k\Theta(t)dt \\
&= \Gamma(\alpha)\left(\frac{2}{\kappa_2-\kappa_1}\right)^\alpha J^{\alpha,k}_{\left(\frac{\kappa_1+\kappa_2}{2}\right)+,u}(\Theta)(\kappa_2).
\end{aligned} \tag{3.4}$$

Similarly, we have

$$\int_0^1 t^{\alpha-1}\left(u\left(\frac{2-t}{2}\kappa_1+\frac{t}{2}\kappa_2\right)-u(\kappa_1)\right)^k \Theta\left(\frac{t}{2}\kappa_1+\frac{2-t}{2}\kappa_2\right)dt = \Gamma(\alpha)\left(\frac{2}{\kappa_2-\kappa_1}\right)^{\alpha} J^{\alpha,k}_{\left(\frac{\kappa_1+\kappa_2}{2}\right)-,u}(\Theta)(\kappa_1). \tag{3.5}$$

If we substitute the equalities (3.3)-(3.5) in (3.2), then we have

$$\begin{aligned}\int_0^1 & G_k(u;t)\Theta'\left(\frac{t}{2}\kappa_1+\frac{2-t}{2}\kappa_2\right)dt \\ =& -\Gamma(\alpha)\left(\frac{2}{\kappa_2-\kappa_1}\right)^{\alpha+1}\left[J^{\alpha,k}_{\left(\frac{\kappa_1+\kappa_2}{2}\right)+,u}(1)(\kappa_2)+J^{\alpha,k}_{\left(\frac{\kappa_1+\kappa_2}{2}\right)-,u}(1)(\kappa_1)\right]\Theta\left(\frac{\kappa_1+\kappa_2}{2}\right) \\ &+\Gamma(\alpha)\left(\frac{2}{\kappa_2-\kappa_1}\right)^{\alpha+1}\left[J^{\alpha,k}_{\left(\frac{\kappa_1+\kappa_2}{2}\right)+,u}(\Theta)(\kappa_2)+J^{\alpha,k}_{\left(\frac{\kappa_1+\kappa_2}{2}\right)-,u}(\Theta)(\kappa_1)\right].\end{aligned} \tag{3.6}$$

By using the fact that $\Theta\left(\frac{\kappa_1+\kappa_2}{2}\right)=2\varphi\left(\frac{\kappa_1+\kappa_2}{2}\right)$ and multiplying both sides of (3.6) by $\frac{1}{2\Gamma(\alpha)}\left(\frac{\kappa_2-\kappa_1}{2}\right)^{\alpha+1}$, we obtain the desired result. □

Theorem 3.2. *Let $\varphi:[\kappa_1,\kappa_2]\to\mathbb{R}$ be a differentiable function on (κ_1,κ_2), $u:[\kappa_1,\kappa_2]\to\mathbb{R}$ be an increasing and positive monotone function on (κ_1,κ_2) and $\varphi',u\in L[\kappa_1,\kappa_2]$ with $\kappa_1<\kappa_2$. If $|\varphi'|$ is convex on $[\kappa_1,\kappa_2]$, then the following inequality holds:*

$$\begin{aligned}&\left|\frac{1}{2}\left[J^{\alpha,k}_{\left(\frac{\kappa_1+\kappa_2}{2}\right)+,u}(\Theta)(\kappa_2)+J^{\alpha,k}_{\left(\frac{\kappa_1+\kappa_2}{2}\right)-,u}(\Theta)(\kappa_1)\right]\right. \\ &\left.-\left[J^{\alpha,k}_{\left(\frac{\kappa_1+\kappa_2}{2}\right)+,u}(1)(\kappa_2)+J^{\alpha,k}_{\left(\frac{\kappa_1+\kappa_2}{2}\right)-,u}(1)(\kappa_1)\right]\varphi\left(\frac{\kappa_1+\kappa_2}{2}\right)\right| \\ \leq&\ \frac{(\kappa_2-\kappa_1)^{\alpha+1}}{2^{\alpha+2}\Gamma(\alpha)}\left[|\varphi'(\kappa_1)|+|\varphi'(\kappa_2)|\right]\int_0^1|G_k(u;t)|\,dt.\end{aligned} \tag{3.7}$$

Proof. By taking modulus in Lemma 3.1, we obtain

$$\begin{aligned}&\left|\frac{1}{2}\left[J^{\alpha,k}_{\left(\frac{\kappa_1+\kappa_2}{2}\right)+,u}(\Theta)(\kappa_2)+J^{\alpha,k}_{\left(\frac{\kappa_1+\kappa_2}{2}\right)-,u}(\Theta)(\kappa_1)\right]\right. \\ &\left.-\left[J^{\alpha,k}_{\left(\frac{\kappa_1+\kappa_2}{2}\right)+,u}(1)(\kappa_2)+J^{\alpha,k}_{\left(\frac{\kappa_1+\kappa_2}{2}\right)-,u}(1)(\kappa_1)\right]\varphi\left(\frac{\kappa_1+\kappa_2}{2}\right)\right| \\ \leq&\ \frac{(\kappa_2-\kappa_1)^{\alpha+1}}{2^{\alpha+2}\Gamma(\alpha)}\int_0^1|G_k(u;t)|\left|\Theta'\left(\frac{t}{2}\kappa_1+\frac{2-t}{2}\kappa_2\right)\right|dt.\end{aligned}$$

Since $\Theta'(y)=\varphi'(y)-\varphi'(\kappa_1+\kappa_2-y)$ and $|\varphi'|$ is convex, we have

$$\begin{aligned}\left|\Theta'\left(\frac{t}{2}\kappa_1+\frac{2-t}{2}\kappa_2\right)\right| &= \left|\varphi'\left(\frac{t}{2}\kappa_1+\frac{2-t}{2}\kappa_2\right)-\varphi'\left(\frac{2-t}{2}\kappa_1+\frac{t}{2}\kappa_2\right)\right| \\ &\leq \left|\varphi'\left(\frac{t}{2}\kappa_1+\frac{2-t}{2}\kappa_2\right)\right|+\left|\varphi'\left(\frac{2-t}{2}\kappa_1+\frac{t}{2}\kappa_2\right)\right|\end{aligned} \tag{3.8}$$

$$\leq \quad |\varphi'(\kappa_1)| + |\varphi'(\kappa_2)|.$$

By the inequality (3.8), we get

$$\begin{aligned} &\left| \frac{1}{2}\left[J^{\alpha,k}_{\left(\frac{\kappa_1+\kappa_2}{2}\right)+,u}(\Theta)(\kappa_2) + J^{\alpha,k}_{\left(\frac{\kappa_1+\kappa_2}{2}\right)-,u}(\Theta)(\kappa_1)\right] \right. \\ &\left. - \left[J^{\alpha,k}_{\left(\frac{\kappa_1+\kappa_2}{2}\right)+,u}(1)(\kappa_2) + J^{\alpha,k}_{\left(\frac{\kappa_1+\kappa_2}{2}\right)-,u}(1)(\kappa_1)\right] \varphi\left(\frac{\kappa_1+\kappa_2}{2}\right) \right| \\ \leq\ & \frac{(\kappa_2-\kappa_1)^{\alpha+1}}{2^{\alpha+2}\Gamma(\alpha)} \left[|\varphi'(\kappa_1)| + |\varphi'(\kappa_2)| \right] \int_0^1 |G_k(u;t)|\, dt. \end{aligned}$$

This finishes the proof of Theorem 3.2. □

Remark 3.3. If we choose $k = 0$ in Theorem 3.2, then we have

$$\begin{aligned} &\left| \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\kappa_2-\kappa_1)^{\alpha}} \left[J^{\alpha}_{\left(\frac{\kappa_1+\kappa_2}{2}\right)+}\varphi(\kappa_2) + J^{\alpha}_{\left(\frac{\kappa_1+\kappa_2}{2}\right)-}\varphi(\kappa_1)\right] - \varphi\left(\frac{\kappa_1+\kappa_2}{2}\right) \right| \\ \leq\ & \frac{\kappa_2-\kappa_1}{4(\alpha+1)} \left[|\varphi'(\kappa_1)| + |\varphi'(\kappa_2)| \right], \end{aligned}$$

which is proved by Sarikaya and Yıldırım in [12].

Corollary 2. If we assign $u(t) = t$ in Theorem 3.2, then we have the inequality for Riemann-Liouville fractional integrals

$$\begin{aligned} &\left| \frac{2^{\alpha+k-1}\Gamma(\alpha+k+1)}{(\kappa_2-\kappa_1)^{\alpha+k}} \left[J^{\alpha+k}_{\left(\frac{\kappa_1+\kappa_2}{2}\right)+}\varphi(\kappa_2) + J^{\alpha+k}_{\left(\frac{\kappa_1+\kappa_2}{2}\right)-}\varphi(\kappa_1)\right] - \varphi\left(\frac{\kappa_1+\kappa_2}{2}\right) \right| \\ \leq\ & \frac{(\kappa_2-\kappa_1)}{4(\alpha+k+1)} \left[|\varphi'(\kappa_1)| + |\varphi'(\kappa_2)| \right]. \end{aligned}$$

Proof. By using equality (3.1) with $u(t) = t$, we obtain

$$\begin{aligned} G_k(u;t) &= \int_0^t s^{\alpha-1} \left[\left(s\left(\frac{\kappa_2-\kappa_1}{2}\right)\right)^k + \left(s\left(\frac{\kappa_2-\kappa_1}{2}\right)\right)^k \right] ds \\ &= 2\left(\frac{\kappa_2-\kappa_1}{2}\right)^k \int_0^t s^{\alpha+k-1} ds \\ &= \frac{2}{\alpha+k}\left(\frac{\kappa_2-\kappa_1}{2}\right)^k t^{\alpha+k} \end{aligned}$$

and

$$\int_0^1 |G_k(u;t)|\, dt = \frac{2}{(\alpha+k)(\alpha+k+1)}\left(\frac{\kappa_2-\kappa_1}{2}\right)^k.$$

This completes the proof of Corollary 2. □

4. Conclusions

In this research paper, we established new Hermite-Hadamard inequality for generalized fractional integrals which are defined Sarikaya et al. in [18]. Some midpoint type inequalities are also presented. In the future works, authors can provide some corresponding trapezoid type inequalities. It can also be studied to obtain similar inequalities for the different types of convexities.

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THE SPHERICAL IMAGES OF A CURVE ACCORDING TO TYPE-2 BISHOP FRAME IN WEYL SPACE

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Abstract. In this work, we obtained the spherical images of a curve according to type-2 Bishop frame in three dimensional Weyl space. We investigated the relations among type-2 Bishop and Frenet-Serret invariants of these spherical images. Besides, we expressed the conditions to be general helix, slant helix and spherical curve of the spherical images. For this reason, we discussed the equivalents of the above concepts in Weyl space. We have seen that, all of these concepts are expressed depending on the first and second curvatures of a curve and hence Bishop curvatures. Also, we gave the definition of circle and the condition to be circle of a curve in Weyl space, using prolonged covariant derivative. Finally, the condition to be the Chebyshev net of the first kind for the net which is generated by Frenet-Serret vector fields of the spherical images of C was obtained.

1. Introduction

Bishop frame, which is also called alternative or parallel frame of the curves, was introduced by L.R. Bishop in 1975 by means of parallel vector fields [1]. Many researchers used this frame in their papers, in the Euclidean space, see [4, 5]; in Minkowksi space, see [6–11, 20]; in Lorentzian space, see [3]; in Weyl space , see [13]. Bishop and Frenet-Serret frames have a common vector field, namely the tangent vector field of the Frenet-Serret frame. Later, Yılmaz and Turgut [19] have introduced a new version of Bishop frame and they called it as type-2 Bishop frame. This time, the common vector field of Bishop and Frenet-Serret frames was binormal vector field of Frenet-Serret frame. Yılmaz and Turgut, by using type-2 Bishop frame, obtained new spherical images of a curve in Euclidean space. This frame was used in Euclidean space, see [2, 17, 22]; in Minkowski space, see [21, 23]; in Weyl space, see [14].

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2. Preliminaries

Let C be a curve in three dimensioanl Weyl space W_3. Let $\{\underset{1}{v}, \underset{2}{v}, \underset{3}{v}, \kappa_1, \kappa_2\}$ and $\{\underset{1}{n}, \underset{2}{n}, \underset{3}{v}, k_1, k_2\}$ be the Frenet-Serret and type-2 Bishop apparatus of C, respectively. Frenet-Serret formulas of C are expressed in the following form:

$$\begin{aligned} \underset{1}{v}^k \dot{\nabla}_k \underset{1}{v}^i &= \kappa_1 \underset{2}{v}^i \\ \underset{1}{v}^k \dot{\nabla}_k \underset{2}{v}^i &= -\kappa_1 \underset{1}{v}^i + \kappa_2 \underset{3}{v}^i \\ \underset{1}{v}^k \dot{\nabla}_k \underset{3}{v}^i &= -\kappa_2 \underset{2}{v}^i \quad (i, k = 1, 2, 3) \end{aligned} \tag{2.1}$$

where $\kappa_1 = \overset{2}{\underset{1}{\mathfrak{S}}}$ and $\kappa_2 = -\overset{2}{\underset{31}{\tau}}$ [14] and also the derivatives of type-2 Bishop vector fields are:

$$\begin{aligned} \underset{1}{v}^k \dot{\nabla}_k \underset{1}{n}^i &= -k_1 \underset{3}{v}^i \\ \underset{1}{v}^k \dot{\nabla}_k \underset{2}{n}^i &= -k_2 \underset{3}{v}^i \\ \underset{1}{v}^k \dot{\nabla}_k \underset{3}{v}^i &= k_1 \underset{1}{n}^i + k_2 \underset{2}{n}^i \end{aligned} \tag{2.2}$$

where $k_1 = \overset{1}{\underset{31}{\tau}} \sin\theta + \overset{2}{\underset{31}{\tau}} \cos\theta (\theta = \sphericalangle(\underset{2}{v}^i, \underset{1}{n}^i))$ or $k_1 = \overset{p}{\underset{31p}{\tau}} v^i \underset{1}{n}^j g_{ij} = g_{ij} \underset{31}{a}^i \underset{1}{n}^j (j, p = 1, 2, 3)$ and $k_2 = -\overset{1}{\underset{31}{\tau}} \cos\theta + \overset{2}{\underset{31}{\tau}} \sin\theta$ or $k_2 = \overset{p}{\underset{31p}{\tau}} v^i \underset{2}{n}^j g_{ij} = g_{ij} \underset{31}{a}^i \underset{2}{n}^j$ [14]. The vector fields $\underset{31}{a}^i = \overset{p}{\underset{31p}{\tau}} v^i$ are named as the Chebyshev vector fields of the first kind [15].

Besides $k_1 = -\kappa_2 \cos\theta, k_2 = -\kappa_2 \sin\theta, \kappa_1 = \underset{1}{v}^k \dot{\nabla}_k \theta (\theta = \theta(s))$ and $\kappa_2 = \sqrt{k_1^2 + k_2^2}$ [14]. The relation between the vector fields of Frenet-Serret frame and type-2 Bishop frame can be expressed as

$$\begin{pmatrix} \underset{1}{v}^i \\ \underset{2}{v}^i \\ \underset{3}{v}^i \end{pmatrix} = \begin{pmatrix} \sin\theta & -\cos\theta & 0 \\ \cos\theta & \sin\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \underset{1}{n}^i \\ \underset{2}{n}^i \\ \underset{3}{v}^i \end{pmatrix}. \tag{2.3}$$

3. Some Special Curves in Weyl Space

Definition 3.1. Let $C : x^i = x^i(s)$ (s is the arc length parameter of C) be a curve in three dimensional Weyl space. C is called a general helix if the tangent vector field $\underset{1}{v}$ of C has constant angle φ with some fixed vector field u, i.e.,

$$g_{ij} \underset{1}{v}^i u^j = \cos\varphi = \text{constant} \tag{3.1}$$

where $g_{ij} \underset{1}{v}^i \underset{1}{v}^j = 1$ and $g_{ij} u^i u^j = 1$.

Theorem 3.2. *C is a general helix if and only if*

$$\frac{\kappa_2}{\kappa_1} = \textit{constant}$$

where κ_1 and κ_2 are the first and second curvatures of C.

Proof. Let C be a general helix in W_3. By means of Definition 3.1, $g_{ij}\underset{1}{v}^i u^j = \cos\varphi =$ constant can be written. Taking prolonged covariant derivative of this equality in the direction of $\underset{1}{v}$, we get

$$g_{ij}(\underset{1}{v}^k \dot{\nabla}_k \underset{1}{v}^i)u^j = 0 \tag{3.2}$$

$$g_{ij}\underset{2}{v}^i u^j = 0 \quad (\kappa_1 \neq 0) \tag{3.3}$$

where $\underset{2}{v}$ is the principal normal vector field of C and $g_{ij}\underset{2}{v}^i\underset{2}{v}^j = 1$. From (3.3), we can write

$$u^j = \alpha\underset{1}{v}^j + \beta\underset{3}{v}^j \tag{3.4}$$

where $\alpha = g_{jh}u^j\underset{1}{v}^h = \cos\varphi$ and $\beta = g_{jh}u^j\underset{3}{v}^h = \cos(\frac{\pi}{2} - \varphi) = \sin\varphi$ $(h = 1, 2, 3)$. Here $\underset{3}{v}$ is the binormal vector field of C and $g_{ij}\underset{3}{v}^i\underset{3}{v}^j = 1$.

Using α and β in (3.4), we get

$$u^j = \cos\varphi\underset{1}{v}^j + \sin\varphi\underset{3}{v}^j. \tag{3.5}$$

Taking prolonged covariant derivative of (3.5) in the direction of $\underset{1}{v}$, we obtain

$$(\kappa_1\cos\varphi - \kappa_2\sin\varphi)\underset{2}{v}^j = 0 \tag{3.6}$$

or

$$\kappa_1\cos\varphi - \kappa_2\sin\varphi = 0 \tag{3.7}$$

or

$$\frac{\kappa_2}{\kappa_1} = \frac{\cos\varphi}{\sin\varphi} = \cot\varphi = \text{constant}. \tag{3.8}$$

Its converse is also true.

Using [8], the following proposition can be given

Proposition 3.3. If C is a slant helix,

$$\frac{\kappa_1^2}{(\kappa_1^2 + \kappa_2^2)^{3/2}}\left(\underset{1}{v}^k\dot{\nabla}_k\frac{\kappa_2}{\kappa_1}\right) = \mathit{constant} \tag{3.9}$$

is satisfied.

By means of [18], the following proposition can be expressed:

Proposition 3.4. If C is a spherical curve,

$$\frac{\kappa_2}{\kappa_1} + \underset{1}{v}^l\dot{\nabla}_l\left[\frac{1}{\kappa_2}\left(\underset{1}{v}^k\dot{\nabla}_k\frac{1}{\kappa_1}\right)\right] = 0 \tag{3.10}$$

is satisfied.

With the help of [16], we can express the following definition and proposition:

Definition 3.5. C is called a circle if there exists a vector field z^i and a positive constant k such that

$$\underset{1}{v}^k\dot{\nabla}_k\underset{1}{v}^i = kz^i \tag{3.11}$$

$$\underset{1}{v}^k\dot{\nabla}_k z^i = -k\underset{1}{v}^i \tag{3.12}$$

where $g_{ij}z^iz^j = 1$.

Proposition 3.6. If C is a circle, the equation

$$\underset{1}{v}^l \dot{\nabla}_l \left(\underset{1}{v}^k \dot{\nabla}_k \underset{1}{v}^i \right) + \mathrm{g}_{ij} \left(\underset{1}{v}^k \dot{\nabla}_k \underset{1}{v}^i \right) \left(\underset{1}{v}^k \dot{\nabla}_k \underset{1}{v}^j \right) \underset{1}{v}^i = 0 \tag{3.13}$$

is satisfied ($l = 1, 2, 3$). Conversely, if C satisfies (3.13), C is either a geodesic or a circle.

4. The Spherical Images of a Curve in Weyl Space

Definition 4.1. Let $C : x^i = x^i(s)$ be a curve in W_3. If we translate of the first vector field of type-2 Bishop frame to the center O of the unit sphere S^2, we obtain a spherical image $\bar{C} : y^i = y^i(\bar{s})$ ($\bar{s}$ is the arc length parameter of $\bar{C}$). $\bar{C}$ is called $\underset{1}{n}$ Bishop spherical image or indicatrix of the curve C.

Let us investigate the relations between type-2 Bishop and Frenet-Serret invariants:

Taking prolonged covariant derivative of y^i in the direction of $\underset{1}{v}$, we get

$$\underset{1}{v}^k \dot{\nabla}_k y^i = -k_1 \underset{3}{v}^i \tag{4.1}$$

$$\left(\underset{1}{\bar{v}}^k \dot{\bar{\nabla}}_k y^i \right) A = -k_1 \underset{3}{v}^i \tag{4.2}$$

$$\underset{1}{\bar{v}}^i A = -k_1 \underset{3}{v}^i \tag{4.3}$$

where $\underset{1}{\bar{v}}$ is the tangent vector field of $\bar{C}, g_{ij} \underset{1}{\bar{v}}^i \underset{1}{\bar{v}}^j = 1$ and $A = A(s)$.

Taking norm of both sides of (4.3), we obtain

$$A = \mp k_1. \tag{4.4}$$

Let us take $A = -k_1$. Then we get

$$\underset{1}{\bar{v}}^i = \underset{3}{v}^i. \tag{4.5}$$

Taking prolonged covariant derivative of (4.5) in the direction of $\underset{1}{v}$, we get

$$\underset{1}{v}^k \dot{\nabla}_k \underset{1}{\bar{v}}^i = \left(\underset{1}{\bar{v}}^k \dot{\bar{\nabla}}_k \underset{1}{\bar{v}}^i \right) A = \underset{1}{v}^k \dot{\nabla}_k \underset{3}{v}^i \tag{4.6}$$

$$\bar{\kappa}_1 \underset{2}{\bar{v}}^i (-k_1) = k_1 \underset{1}{n}^i + k_2 \underset{2}{n}^i \tag{4.7}$$

$$\bar{\kappa}_1 \underset{2}{\bar{v}}^i = -\underset{1}{n}^i - \frac{k_2}{k_1} \underset{2}{n}^i \tag{4.8}$$

Taking norm of both sides of (4.8), we have

$$\bar{\kappa}_1 = \sqrt{1 + \left(\frac{k_2}{k_1} \right)^2} = \sqrt{1 + \left[\frac{g_{ij} \underset{31}{a}^i \underset{2}{n}^j}{g_{ij} \underset{31}{a}^i \underset{1}{n}^j} \right]^2} \tag{4.9}$$

and

$$\underset{2}{\bar{v}}^i = -\frac{1}{\bar{\kappa}_1} \underset{1}{n}^i - \frac{1}{\bar{\kappa}_1} \frac{k_2}{k_1} \underset{2}{n}^i \tag{4.10}$$

where $\underset{2}{\bar{v}}$ is the principal normal vector field of $\bar{C}, g_{ij}\underset{2}{\bar{v}}^i\underset{2}{\bar{v}}^j = 1$ and $\bar{\kappa}_1$ is the first curvature of $\bar{C}$.

$\underset{3}{\bar{v}}$ is the binormal vector field of $\bar{C}$ and we know that

$$\underset{3}{\bar{v}}^i = \varepsilon_{ijk}\underset{1}{\bar{v}}^j\underset{2}{\bar{v}}^k. \tag{4.11}$$

Using (4.5) and (4.10) in (4.11), we have

$$\underset{3}{\bar{v}}^i = \frac{1}{\bar{\kappa}_1}\frac{k_2}{k_1}\underset{1}{n}^i - \frac{1}{\bar{\kappa}_1}\underset{2}{n}^i. \tag{4.12}$$

Taking prolonged covariant derivative of (4.12) in the direction of $\underset{1}{v}$, we get

$$\begin{aligned} \underset{1}{v}^k\dot{\bar{\nabla}}_k\underset{3}{\bar{v}}^i &= \left(\underset{1}{\bar{v}}^k\dot{\nabla}_k\underset{3}{\bar{v}}^i\right)(A) = -\bar{\kappa}_2\underset{2}{\bar{v}}^i(-k_1) = \\ &= \left(\underset{1}{v}^k\dot{\nabla}_k\frac{1}{\bar{\kappa}_1}\right)\frac{k_2}{k_1}\underset{1}{n}^i + \frac{1}{\bar{\kappa}_1}\left(\underset{1}{v}^k\dot{\nabla}_k\frac{k_2}{k_1}\right)\underset{1}{n}^i - \left(\underset{1}{v}^k\dot{\nabla}_k\frac{1}{\bar{\kappa}_1}\right)\underset{2}{n}^i \end{aligned} \tag{4.13}$$

and multiplying (4.13) by $g_{ij}\underset{2}{\bar{v}}^j$, we obtain

$$\bar{\kappa}_2 = -\frac{k_1\left(\underset{1}{v}^k\dot{\nabla}_k\frac{k_2}{k_1}\right)}{k_1^2+k_2^2} = -\frac{\left(g_{ij}\underset{31}{a}^i\underset{1}{n}^j\right)}{\left(g_{ij}\underset{31}{a}^i\underset{1}{n}^j\right)^2 + \left(g_{ij}\underset{31}{a}^i\underset{2}{n}^j\right)^2}\left[\underset{1}{v}^k\dot{\nabla}_k\left(\frac{g_{ij}\underset{31}{a}^i\underset{2}{n}^j}{g_{ij}\underset{31}{a}^i\underset{1}{n}^j}\right)\right] \tag{4.14}$$

where $\bar{\kappa}_2$ is the second curvature of $\bar{C}$.

Corollary 4.2. Let $\bar{C}$ be $\underset{1}{n}$ Bishop spherical curve of C. If $\frac{k_2}{k_1}$ = constant, then, $\underset{1}{n}$ Bishop spherical image $y^i = y^i(\bar{s})$ is a circle.

Using Proposition 3.3, we get

$$\begin{aligned} \underset{1}{v}^l\dot{\nabla}_l\left(\underset{1}{v}^k\dot{\nabla}_k\underset{1}{\bar{v}}^i\right) &= \underset{1}{v}^l\dot{\nabla}_l\left(\underset{1}{v}^k\dot{\nabla}_k\underset{3}{v}^i\right) = \\ &= \left(\underset{1}{v}^l\dot{\nabla}_l k_1\right)\underset{1}{n}^i + \left(\underset{1}{v}^l\dot{\nabla}_l k_2\right)\underset{2}{n}^i - \left(k_1^2+k_2^2\right)\underset{3}{v}^i, \end{aligned} \tag{4.15}$$

and on the other hand

$$g_{ij}\left(\underset{1}{v}^k\dot{\nabla}_k\underset{1}{\bar{v}}^i\right)\left(\underset{1}{v}^k\dot{\nabla}_k\underset{1}{\bar{v}}^j\right)\underset{1}{\bar{v}}^i = g_{ij}\left(\underset{1}{v}^k\dot{\nabla}_k\underset{3}{v}^i\right)\left(\underset{1}{v}^k\dot{\nabla}_k\underset{3}{v}^j\right)\underset{3}{v}^i = \left(k_1^2+k_2^2\right)\underset{3}{v}^i. \tag{4.16}$$

Summing (4.15) and (4.16), we get

$$\underset{1}{v}^l\dot{\nabla}_l\left(\underset{1}{v}^k\dot{\nabla}_k\underset{1}{\bar{v}}^i\right) + g_{ij}\left(\underset{1}{v}^k\dot{\nabla}_k\underset{1}{\bar{v}}^i\right)\left(\underset{1}{v}^k\dot{\nabla}_k\underset{1}{\bar{v}}^j\right)\underset{1}{\bar{v}}^i = \left(\underset{1}{v}^l\dot{\nabla}_l k_1\right)\underset{1}{n}^i + \left(\underset{1}{v}^l\dot{\nabla}_l k_2\right)\underset{2}{n}^i. \tag{4.17}$$

Using $k_1 = -\kappa_2\cos\theta$ and $k_2 = -\kappa_2\sin\theta$, we get

$$\underset{1}{v}^l\dot{\nabla}_l k_1 = \underset{1}{v}^l\dot{\nabla}_l\left(-\kappa_2\cos\theta\right) = -\left(\underset{1}{v}^l\dot{\nabla}_l\kappa_2\right)\cos\theta + \kappa_2\left(\underset{1}{v}^k\dot{\nabla}_k\theta\right)\sin\theta \tag{4.18}$$

and

$$\underset{1}{v}^l\dot{\nabla}_l k_2 = \underset{1}{v}^l\dot{\nabla}_l\left(-\kappa_2\sin\theta\right) = -\left(\underset{1}{v}^l\dot{\nabla}_l\kappa_2\right)\sin\theta - \kappa_2\left(\underset{1}{v}^l\dot{\nabla}_l\theta\right)\cos\theta \tag{4.19}$$

where $\theta = \theta(s) = \arctan\frac{k_2}{k_1}$ and $\underset{1}{v}^k\dot{\nabla}_k\theta = \frac{\underset{1}{v}^k\dot{\nabla}_k\frac{k_2}{k_1}}{1+\left(\frac{k_2}{k_1}\right)^2}$. We know that $\kappa_2 = \sqrt{k_1^2+k_2^2}$, then we have

$$\underset{1}{v}^k \dot{\nabla}_k \kappa_2 = \underset{1}{v}^k \dot{\nabla}_k \sqrt{k_1^2 \left[1 + \left(\frac{k_2}{k_1}\right)^2\right]} =$$

(4.20)
$$= \frac{1}{\sqrt{k_1^2 + k_2^2}} \left\{ k_1 \left(\underset{1}{v}^k \dot{\nabla}_k k_1\right) \left[\frac{k_1^2 + k_2^2}{k_1^2}\right] + k_1^2 \frac{k_2}{k_1} \left(\underset{1}{v}^k \dot{\nabla}_k \frac{k_2}{k_1}\right) \right\}$$

$$= \frac{1}{\sqrt{k_1^2 + k_2^2}} \left\{ -k_1 \left(\underset{1}{v}^k \dot{\nabla}_k k_1\right) \left(k_1^2 + k_2^2\right) \frac{1}{\left(\underset{1}{v}^k \dot{\nabla}_k \frac{k_1}{k_2}\right) k_2^2} \left(\underset{1}{v}^k \dot{\nabla}_k \frac{k_2}{k_1}\right) + k_1^2 \frac{k_2}{k_1} \left(\underset{1}{v}^k \dot{\nabla}_k \frac{k_2}{k_1}\right) \right\}.$$

If $\frac{k_2}{k_1}$ = constant, $\underset{1}{v}^k \dot{\nabla}_k \theta = 0$ and $\underset{1}{v}^k \dot{\nabla}_k \kappa_2 = 0$ are obtained. This means $\underset{1}{v}^l \dot{\nabla}_k k_1 = 0$ and $\underset{1}{v}^l \dot{\nabla}_k k_2 = 0$. Using these results in (4.17), we obtain

$$\underset{1}{v}^l \dot{\nabla}_l \left(\underset{1}{v}^k \dot{\nabla}_k \underset{1}{\bar{v}}^i\right) + g_{ij} \left(\underset{1}{v}^k \dot{\nabla}_k \underset{1}{\bar{v}}^i\right) \left(\underset{1}{v}^k \dot{\nabla}_k \underset{1}{\bar{v}}^j\right) \underset{1}{\bar{v}}^i = 0,$$

i.e. $\bar{C} : y^i = y^i(\bar{s})$ is a circle. Besides, let us note that $\bar{\kappa}_1$ = constant and $\bar{\kappa}_2 = 0$.

Theorem 4.3. *Let $\bar{C} : y^i = y^i(\bar{s})$ be $\underset{1}{n}$ Bishop spherical image of C. If $y^i = y^i(\bar{s})$ is a general helix, then*

$$\frac{k_1^2 \left(\underset{1}{v}^k \dot{\nabla}_k \frac{k_2}{k_1}\right)}{\left(k_1^2 + k_2^2\right)^{3/2}} = constant$$

is satisfied.

Theorem 4.4. *Let $\bar{C} : y^i = y^i(\bar{s})$ be $\underset{1}{n}$ Bishop spherical image of C. If $y^i = y^i(\bar{s})$ is a slant helix, then*

$$\frac{k_1 (k_1^2 + k_2^2)^4}{(k_1^2 + k_2^2)^3 + k_1^4 \left[\underset{1}{v}^k \dot{\nabla}_k \frac{k_2}{k_1}\right]^2} \underset{1}{v}^l \dot{\nabla}_l \left[\frac{k_1^2 \left(\underset{1}{v}^k \dot{\nabla}_k \frac{k_2}{k_1}\right)}{\left(k_1^2 + k_2^2\right)^{3/2}}\right] = constant$$

Since $\bar{C} : y^i = y^i(\bar{s})$ is a spherical curve, by means of Proposition 3.4, we can express the following theorem:

Theorem 4.5. *Let $\bar{C} : y^i = y^i(\bar{s})$ be $\underset{1}{n}$ Bishop spherical image of C. The following equation*

$$\frac{k_1^2 \left(\underset{1}{v}^k \dot{\nabla}_k \frac{k_2}{k_1}\right)}{\left(k_1^2 + k_2^2\right)^{3/2}} + \underset{1}{v}^k \dot{\nabla}_k \frac{k_1 k_2}{\sqrt{k_1^2 + k_2^2}} = constant$$

is satisfied.

Theorem 4.6. *Let $\bar{C} : y^i = y^i(\bar{s})$ be $\underset{1}{n}$ Bishop spherical image of C. If $\frac{k_2}{k_1}$ = constant, then the the net $\left(\underset{1}{\bar{v}}, \underset{2}{\bar{v}}, \underset{3}{\bar{v}}\right)$ is the Chebyshev net of the first kind.*

Proof. We know that

$$\underset{1}{v}^k \dot{\nabla}_k \underset{3}{\bar{v}}^i = \left(\underset{1}{\bar{v}}^k \dot{\bar{\nabla}}_k \underset{3}{\bar{v}}^k\right)(-k_1) =$$

$$= \left(\underset{1}{v}^k \dot{\nabla}_k \frac{k_2}{k_1}\right)\frac{1}{\bar{\kappa}_1}\underset{1}{n}^i + \frac{k_2}{k_1}\left(\underset{1}{v}^k \dot{\nabla}_k \frac{1}{\bar{\kappa}_1}\right)\underset{1}{n}^i - \left(\underset{1}{v}^k \dot{\nabla}_k \frac{1}{\bar{\kappa}_1}\right)\underset{2}{n}^i \tag{4.21}$$

or

$$\underset{1}{\bar{v}}^k \dot{\bar{\nabla}}_k \underset{3}{\bar{v}}^i = \underset{31}{\bar{a}}^i =$$

$$= -\frac{1}{k_1}\left(\underset{1}{\bar{v}}^k \dot{\bar{\nabla}}_k \frac{k_2}{k_1}\right)\frac{1}{\bar{\kappa}_1}\underset{1}{n}^i - \frac{k_2}{k_1^2}\left(\underset{1}{v}^k \dot{\nabla}_k \frac{1}{\bar{\kappa}_1}\right)\underset{1}{n}^i - \frac{1}{k_1}\left(\underset{1}{v}^k \dot{\nabla}_k \frac{1}{\bar{\kappa}_1}\right)\underset{2}{n}^i. \tag{4.22}$$

Using

$$\underset{1}{v}^k \dot{\nabla}_k \frac{1}{\bar{\kappa}_1} = -\frac{k_1 k_2}{(k_1^2 + k_2^2)^{3/2}}\left(\underset{1}{v}^k \dot{\nabla}_k \frac{k_2}{k_1}\right), \tag{4.23}$$

under the condition $\frac{k_2}{k_1} =$ constant, we obtain from (4.22)

$$\underset{31}{\bar{a}}^i = 0. \tag{4.24}$$

From the equation (4.24), we have seen that the net $\left(\underset{1}{\bar{v}}, \underset{2}{\bar{v}}, \underset{3}{\bar{v}}\right)$ is the Chebyshev net of the first kind.

Definition 4.7. Let $C : x^i = x^i(s)$ be a curve in W_3. If we translate of the second vector field of type-2 Bishop frame to the center O of the unit sphere S^2, we obtain a spherical image $\bar{D} : z^i = z^i(\bar{s})$ ($\bar{s}$ is the arc length parameter of $\bar{D}$). $\bar{D}$ is called $\underset{2}{n}$ Bishop spherical image or indicatrix of the curve C.

Let us investigate the relations between type-2 Bishop and Frenet-Serret invariants:

Taking prolonged covariant derivative of z^i in the direction of $\underset{1}{v}$, we get

$$\underset{1}{v}^k \dot{\bar{\nabla}}_k z^i = -k_2 \underset{3}{v}^i \tag{4.25}$$

$$\left(\underset{1}{\bar{v}}^k \dot{\bar{\nabla}}_k z^i\right) B = -k_2 \underset{3}{v}^i \tag{4.26}$$

$$\underset{1}{\bar{v}}^i B = -k_2 \underset{3}{v}^i \tag{4.27}$$

where $\underset{1}{\bar{v}}$ is the tangent vector field of $\bar{D}, g_{ij}\underset{1}{\bar{v}}^i \underset{1}{\bar{v}}^j = 1$ and $B = B(s)$. Taking norm of both sides of (4.27), we obtain

$$B = \mp k_2. \tag{4.28}$$

Let us take $B = -k_2$. Then we get

$$\underset{1}{\bar{v}}^i = \underset{3}{v}^i. \tag{4.29}$$

Taking prolonged covariant derivative of (4.29) in the direction of $\underset{1}{v}$, we get

$$\underset{1}{v}^k \dot{\nabla}_k \underset{1}{\bar{v}}^i = \left(\underset{1}{\bar{v}}^k \dot{\bar{\nabla}}_k \underset{1}{\bar{v}}^i \right) B = \underset{1}{v}^k \dot{\nabla}_k \underset{3}{v}^i \tag{4.30}$$

$$\bar{L}_1 \underset{2}{\bar{v}}^i (-k_2) = k_1 \underset{1}{n}^i + k_2 \underset{2}{n}^i \tag{4.31}$$

$$\bar{L}_1 \underset{2}{\bar{v}}^i = -\frac{k_1}{k_2} \underset{1}{n}^i - \underset{2}{n}^i. \tag{4.32}$$

Taking norm of both sides of (4.32), we have

$$\bar{L}_1 = \sqrt{\left(\frac{k_1}{k_2}\right)^2 + 1} = \sqrt{\left[\frac{g_{ij} \underset{31}{a}^i \underset{1}{n}^j}{g_{ij} \underset{31}{a}^i \underset{2}{n}^j}\right]^2 + 1} \tag{4.33}$$

and

$$\underset{2}{\bar{v}}^i = -\frac{1}{\bar{L}_1} \frac{k_1}{k_2} \underset{1}{n}^i - \frac{1}{\bar{L}_1} \underset{2}{n}^i \tag{4.34}$$

where $\underset{2}{\bar{v}}$ is the principal normal vector field of $\bar{D}, g_{ij} \underset{2}{\bar{v}}^i \underset{2}{\bar{v}}^j = 1$ and $\bar{L}_1$ is the first curvature of $\bar{D}$.

$\underset{3}{\bar{v}}$ is the binormal vector field of $\bar{D}$ and we know that

$$\underset{3}{\bar{v}}^i = \varepsilon_{ijk} \underset{1}{\bar{v}}^j \underset{2}{\bar{v}}^k. \tag{4.35}$$

Using (4.29) and (4.34) in (4.35), we have

$$\underset{3}{\bar{v}}^i = -\frac{1}{\bar{L}_1} \frac{k_1}{k_2} \underset{2}{n}^i + \frac{1}{\bar{L}_1} \underset{1}{n}^i. \tag{4.36}$$

Taking prolonged covariant derivative of (4.36) in the direction of $\underset{1}{v}$, we get

$$\begin{aligned} \underset{1}{v}^k \dot{\nabla}_k \underset{3}{\bar{v}}^i &= \left(\underset{1}{\bar{v}}^k \dot{\bar{\nabla}}_k \underset{3}{\bar{v}}^i \right) (B) = -\bar{L}_2 \underset{2}{\bar{v}}^i (-k_2) = \\ &= -\left(\underset{1}{v}^k \dot{\nabla}_k \frac{1}{\bar{L}_1} \right) \frac{k_1}{k_2} \underset{2}{n}^i - \frac{1}{\bar{L}_1} \left(\underset{1}{v}^k \dot{\nabla}_k \frac{k_1}{k_2} \right) \underset{2}{n}^i + \left(\underset{1}{v}^k \dot{\nabla}_k \frac{1}{\bar{L}_1} \right) \underset{1}{n}^i \end{aligned} \tag{4.37}$$

and multiplying (4.37) by $g_{ij} \underset{2}{\bar{v}}^j$, we obtain

$$\bar{L}_2 = \frac{k_2}{k_1^2 + k_2^2} \left(\underset{1}{v}^k \dot{\nabla}_k \frac{k_1}{k_2} \right) = \frac{g_{ij} \underset{31}{a}^i \underset{2}{n}^j}{\left(g_{ij} \underset{31}{a}^i \underset{1}{n}^j \right)^2 + \left(g_{ij} \underset{31}{a}^i \underset{2}{n}^j \right)^2} \underset{1}{v}^k \dot{\nabla}_k \left[\frac{g_{ij} \underset{31}{a}^i \underset{1}{n}^j}{g_{ij} \underset{31}{a}^i \underset{2}{n}^j} \right] \tag{4.38}$$

where $\bar{L}_2$ is the second curvature of $\bar{D}$.

Corollary 4.8. Let $\bar{D}$ be $\underset{2}{n}$ Bishop spherical image of C. If $\frac{k_1}{k_2} = constant$, then, $\underset{2}{n}$ Bishop spherical image $z^i = z^i(\bar{s})$ is a circle.

Let us note that $\bar{L}_1 = \text{constant}$ and $\bar{L}_2 = 0$ under the condition $\frac{k_1}{k_2} = \text{constant}$.

Theorem 4.9. *Let $\bar{D} : z^i = z^i(\bar{s})$ be $\underset{2}{n}$ Bishop spherical image of C. If $z^i = z^i(\bar{s})$ is a general helix, then*

$$\frac{k_2^2 \left(\underset{1}{v}^k \dot{\nabla}_k \frac{k_1}{k_2} \right)}{\left(k_1^2 + k_2^2\right)^{3/2}} = constant$$

is satisfied.

Theorem 4.10. *Let $\bar{D} : z^i = z^i(\bar{s})$ be $\underset{z}{n}$ Bishop spherical image of C. If $z^i = z^i(\bar{s})$ is a slant helix, then*

$$\frac{k_1(k_1^2+k_2^2)^4}{(k_1^2+k_2^2)^3 + k_2^4\left[\underset{1}{v}^k\dot{\nabla}_k\frac{k_1}{k_2}\right]^2}\underset{1}{v}^l\dot{\nabla}_l\left[\frac{k_2^2\left(\underset{1}{v}^k\dot{\nabla}_k\frac{k_1}{k_2}\right)}{(k_1^2+k_2^2)^{3/2}}\right] = constant$$

is satisfied.

Since $\bar{D} : z^i = z^i(\bar{s})$ is a spherical curve, by means of Proposition 3.4, we can express the following theorem:

Theorem 4.11. *Let $\bar{D} : z^i = z^i(\bar{s})$ be $\underset{2}{n}$ Bishop spherical image of C. The following equation*

$$\frac{k_2^2\left(\underset{1}{v}^k\dot{\nabla}_k\frac{k_1}{k_2}\right)}{(k_1^2+k_2^2)^{3/2}} - \underset{1}{v}^k\dot{\nabla}_k\frac{k_1k_2}{\sqrt{k_1^2+k_2^2}} = constant$$

is satisfied.

Theorem 4.12. *Let $\bar{D} : z^i = z^i(\bar{s})$ be $\underset{2}{n}$ Bishop spherical image of C. If $\frac{k_1}{k_2} =$ constant, then the net $\left(\underset{1}{\bar{v}}, \underset{2}{\bar{v}}, \underset{3}{\bar{v}}\right)$ is the Chebyshev net of the first kind.*

Proof. We know that

$$\underset{1}{v}^k\dot{\nabla}_k\underset{3}{\bar{v}}^i = \left(\underset{1}{\bar{v}}^k\dot{\bar{\nabla}}_k\underset{3}{\bar{v}}^k\right)(-k_2) =$$

$$= -\left(\underset{1}{v}^k\dot{\nabla}_k\frac{1}{\bar{L}_1}\right)\frac{k_1}{k_2}\underset{2}{n}^i - \frac{1}{\bar{L}_1}\left(\underset{1}{v}^k\dot{\nabla}_k\frac{k_1}{k_2}\right)\underset{2}{n}^i + \left(\underset{1}{v}^k\dot{\nabla}_k\frac{1}{\bar{L}_1}\right)\underset{1}{n}^i \tag{4.39}$$

or

(4.40)

$$\underset{1}{\bar{v}}^k\dot{\bar{\nabla}}_k\underset{3}{\bar{v}}^i = \underset{31}{\bar{a}}^i =$$

$$= \frac{1}{k_2}\left(\underset{1}{v}^k\dot{\nabla}_k\frac{1}{\bar{L}_1}\right)\frac{k_1}{k_2}\underset{2}{n}^i + \frac{1}{k_2}\frac{1}{\bar{L}_1}\left(\underset{1}{v}^k\dot{\nabla}_k\frac{k_1}{k_2}\right)\underset{2}{n}^i - \frac{1}{k_2}\left(\underset{1}{v}^k\dot{\nabla}_k\frac{1}{\bar{L}_1}\right)\underset{1}{n}^i.$$

Using

$$\underset{1}{v}^k\dot{\nabla}_k\frac{1}{\bar{L}_1} = -\frac{k_1k_2}{(k_1^2+k_2^2)^{3/2}}\left(\underset{1}{v}^k\dot{\nabla}_k\frac{k_1}{k_2}\right), \tag{4.41}$$

under the condition $\frac{k_1}{k_2} =$ constant, we obtain from (4.40)

$$\underset{31}{\bar{a}}^i = 0. \tag{4.42}$$

From (4.42), the net $\left(\underset{1}{\bar{v}}, \underset{2}{\bar{v}}, \underset{3}{\bar{v}}\right)$ is the Chebyshev net of the first kind.

Definition 4.13. Let $C : x^i = x^i(s)$ be a curve in W_3. If we translate of the third vector field of type-2 Bishop frame to the center O of the unit sphere S^2, we obtain a spherical image $\bar{E} : \omega^i = \omega^i(\bar{s})$ ($\bar{s}$ is the arc length parameter of $\bar{E}$). $\bar{E}$ is called binormal Bishop spherical image or indicatrix of the curve C.

Let us investigate the relations between type-2 Bishop and Frenet-Serret invariants:

Taking prolonged covariant derivative of ω^i in the direction of $\underset{1}{v}$, we get

$$\underset{1}{v}^k \dot{\nabla}_k \omega^i = k_1 \underset{1}{n}^i + k_2 \underset{2}{n}^i \tag{4.43}$$

$$\left(\underset{1}{\bar{v}}^k \dot{\bar{\nabla}}_k \omega^i\right) F = k_1 \underset{1}{n}^i + k_2 \underset{2}{n}^i \tag{4.44}$$

$$\underset{1}{\bar{v}}^i F = k_1 \underset{1}{n}^i + k_2 \underset{2}{n}^i \tag{4.45}$$

where $\underset{1}{\bar{v}}$ is the tangent vector field of $\bar{E}$, $g_{ij} \underset{1}{\bar{v}}^i \underset{1}{\bar{v}}^j = 1$ and $F = F(s)$.

Taking norm of both sides of (4.45), we obtain

$$F = \mp\sqrt{k_1^2 + k_2^2}. \tag{4.46}$$

Let us take $F = \sqrt{k_1^2 + k_2^2}$. Then we get

$$\underset{1}{\bar{v}}^i = \frac{k_1}{\sqrt{k_1^2 + k_2^2}} \underset{1}{n}^i + \frac{k_2}{\sqrt{k_1^2 + k_2^2}} \underset{2}{n}^i. \tag{4.47}$$

Taking prolonged covariant derivative of (4.47) in the direction of $\underset{1}{v}$, we get

$$\underset{1}{v}^k \dot{\nabla}_k \underset{1}{\bar{v}}^i = \frac{k_2^3 \left(\underset{1}{v}^k \dot{\nabla}_k \frac{k_1}{k_2}\right)}{(k_1^2 + k_2^2)^{3/2}} \underset{1}{n}^i + \frac{k_1^3 \left(\underset{1}{v}^k \dot{\nabla}_k \frac{k_2}{k_1}\right)}{(k_1^2 + k_2^2)^{3/2}} \underset{2}{n}^i - \sqrt{k_1^2 + k_2^2}\, \underset{3}{v}^i \tag{4.48}$$

Using $\underset{1}{v}^k \dot{\nabla}_k \underset{1}{\bar{v}}^i = \left(\underset{1}{\bar{v}}^k \dot{\bar{\nabla}}_k \underset{1}{\bar{v}}^i\right) F$ and $F = \sqrt{k_1^2 + k_2^2}$, we get from (4.48)

$$\underset{1}{\bar{v}}^k \dot{\bar{\nabla}}_k \underset{1}{\bar{v}}^i = \bar{N}_1 \underset{2}{\bar{v}}^i = \frac{k_2^3 \left(\underset{1}{v}^k \dot{\nabla}_k \frac{k_1}{k_2}\right)}{(k_1^2 + k_2^2)^2} \underset{1}{n}^i + \frac{\left(k_1^3 \underset{1}{v}^k \dot{\nabla}_k \frac{k_2}{k_1}\right)}{(k_1^2 + k_2^2)^2} \underset{2}{n}^i - \underset{3}{v}^i \tag{4.49}$$

where $\bar{N}_1$ is the first curvature of $\bar{E}$, $\underset{2}{\bar{v}}$ is the principal normal vector field of $\bar{E}$ and $g_{ij} \underset{2}{\bar{v}}^i \underset{2}{\bar{v}}^i = 1$.

Taking norm of both sides of (4.49), we have

$$\bar{N}_1 = \sqrt{\left[\frac{k_2^3 \left(\underset{1}{v}^k \dot{\nabla}_k \frac{k_1}{k_2}\right)}{(k_1^2 + k_2^2)^2}\right]^2 + \left[\frac{k_1^3 \left(\underset{1}{v}^k \dot{\nabla}_k \frac{k_2}{k_1}\right)}{(k_1^2 + k_2^2)^2}\right]^2 + 1} \tag{4.50}$$

and hence

$$\underset{2}{\bar{v}}^i = \frac{1}{\bar{N}_1} \frac{k_2^3 \left(\underset{1}{v}^k \dot{\nabla}_k \frac{k_1}{k_2}\right)}{(k_1^2 + k_2^2)^2} \underset{1}{n}^i + \frac{1}{\bar{N}_1} \frac{k_1^3 \left(\underset{1}{v}^k \dot{\nabla}_k \frac{k_2}{k_1}\right)}{(k_1^2 + k_2^2)^2} \underset{2}{n}^i - \frac{1}{\bar{N}_1} \underset{3}{v}^i. \tag{4.51}$$

Since $\underset{3}{\bar{v}}^i = \varepsilon_{ijk} \underset{1}{\bar{v}}^j \underset{2}{\bar{v}}^k$, we have the binormal vector field of $\bar{E}$ as:

$$\begin{aligned} \underset{3}{\bar{v}}^i = {} & \frac{1}{\bar{N}_1} \left[\frac{k_1^4}{(k_1^2 + k_2^2)^{5/2}} \left(\underset{1}{v}^k \dot{\nabla}_k \frac{k_2}{k_1}\right) - \frac{k_2^4}{(k_1^2 + k_2^2)^{5/2}} \left(\underset{1}{v}^k \dot{\nabla}_k \frac{k_1}{k_2}\right)\right] \underset{3}{v}^i \\ & + \frac{1}{\bar{N}_1} \frac{k_1}{\sqrt{k_1^2 + k_2^2}} \underset{2}{n}^i - \frac{1}{\bar{N}_1} \frac{k_2}{\sqrt{k_1^2 + k_2^2}} \underset{1}{n}^i \end{aligned} \tag{4.52}$$

where $g_{ij}\underset{3}{\bar{v}}^i\underset{3}{\bar{v}}^j = 1$.

Taking prolonged covariant derivative of (4.52) in the direction of $\underset{1}{v}$ and multiplying $g_{ij}\underset{2}{\bar{v}}^j$, we have

$$\begin{aligned}\bar{N}_2 = & \frac{1}{k_1^4\left[\underset{1}{v}^k\dot{\nabla}_k\frac{k_2}{k_1}\right]^2 + (k_1^2+k_2^2)^3}\Big\{-3k_1^2\left(\underset{1}{v}^k\dot{\nabla}_k k_1\right)\left(\underset{1}{v}^k\dot{\nabla}_k k_2\right) + \\ & + 3k_1k_2\left(\underset{1}{v}^k\dot{\nabla}_k k_1\right)\left(\underset{1}{v}^k\dot{\nabla}_k k_1\right) - 3k_2k_1\left(\underset{1}{v}^k\dot{\nabla}_k k_2\right)\left(\underset{1}{v}^k\dot{\nabla}_k k_2\right) + \\ & + 3k_2^2\left(\underset{1}{v}^k\dot{\nabla}_k k_2\right)\left(\underset{1}{v}^k\dot{\nabla}_k k_1\right) + \left[\underset{1}{v}^l\dot{\nabla}_l\left(\underset{1}{v}^k\dot{\nabla}_k k_2\right)\right]k_1(k_1^2+k_2^2) + \\ & + \left[\underset{1}{v}^l\dot{\nabla}_l\left(\underset{1}{v}^k\dot{\nabla}_k k_1\right)\right]k_2(k_1^2+k_2^2)\Big\}\end{aligned} \tag{4.53}$$

where $\bar{N}_2$ is the second curvature of $\bar{E}$.

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A CONTRIBUTION TO THE FIXED-DISC RESULTS ON S-METRIC SPACES

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Abstract. In this paper, we prove some fixed-disc results using new contractions. To do this, we define the notions of Jleli-Samet type x_0-S-contraction and Li-Jiang type x_0-S-contraction. Also, we obtain an equivalent theorem using these type contractions Finally, we give an illustrative example.

1. Introduction and preliminaries

Fixed-point theory has been extensively studied with various aspects. One of these aspects is to generalize the used metric spaces. For example, an S-metric space are a generalization of a metric space [12]. After the notion of an S-metric space was introduced, many researchers have proved some fixed-point theorems on this space (for example, see [5], [6], [7], [13] and the references therein).

Recently, "Fixed-Circle Problem" has been investigated as a geometric generalization of the fixed-point theory. This problem was presented in [8]. After then, fixed-circle problem has been studied on S-metric spaces with different approaches (for example, see [4], [9], [10], [14] and the references therein).

At first, we recall some necessary notions about S-metric spaces.

Definition 1.1. [12] Let X be a nonempty set and $S : X \times X \times X \to [0, \infty)$ be a function satisfying the following conditions for all $x, y, z, a \in X$:

$(S1)$ $S(x, y, z) = 0$ if and only if $x = y = z$,

$(S2)$ $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$.

Then S is called an S-metric on X and the pair (X, S) is called an S-metric space.

Lemma 1.2. [12] *Let (X, S) be an S-metric space and $x, y \in X$. Then we have*

$$S(x, x, y) = S(y, y, x).$$

In [10] and [12], a circle and a disc are defined on an S-metric space as follows, respectively:

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$$C_{x_0,r}^S = \{x \in X : S\left(x, x, x_0\right) = r\}$$

and

$$D_{x_0,r}^S = \{x \in X : S\left(x, x, x_0\right) \leq r\}.$$

Let (X, S) be an S-metric space, $C_{x_0,r}^S$ be a circle and $g : X \to X$ be a self-mapping. If $gx = x$ for every $x \in C_{x_0,r}^S$ (resp. $x \in D_{x_0,r}^S$) then the circle $C_{x_0,r}^S$ (the disc $D_{x_0,r}^S$) is called as the fixed circle (the fixed disc) of g (see [4] and [10] for more details).

In this paper, we define the notions of Jleli-Samet type x_0-S-contraction and Li-Jiang type x_0-S-contraction on S-metric spaces modifying some known contractions (see [2], [3] and [11]). Using these notions, we prove two fixed-disc theorems and an equivalence theorem. Also, we give an illustrative example to show the validity of fixed-disc results.

2. Main results

In this section, we introduce some contractions and prove new fixed-disc results.

Definition 2.1. Let (X, S) be an S-metric space and $g : X \to X$ a self-mapping. If there exists $x_0 \in X$ such that

$$S(x, x, gx) > 0 \Longrightarrow \varphi\left(S(x, x, gx)\right) \leq \left[\varphi\left(S(x, x, x_0)\right)\right]^{\alpha},$$

for all $x \in X$, where $\alpha \in (0, 1)$ and the function $\varphi : (0, \infty) \to (1, \infty)$ is such that φ is nondecreasing, then g is called Jleli-Samet type x_0-S-contraction.

Theorem 2.2. *Let (X, S) be an S-metric space, $g : X \to X$ Jleli-Samet type x_0-S-contraction with $x_0 \in X$ and the number r defined as*

$$r = \inf\left\{S(x, x, gx) : x \neq gx,\ x \in X\right\}. \tag{2.1}$$

Then g fixes the disc $D_{x_0,r}^S$.

Proof. Let $r = 0$. Then we have $D_{x_0,r}^S = \{x_0\}$. To show $gx_0 = x_0$, we assume $x_0 \neq gx_0$, that is, $S\left(x_0, x_0, gx_0\right) > 0$. Using the Jleli-Samet type x_0-S-contraction hypothesis we get

$$\varphi\left(S(x_0, x_0, gx_0)\right) \leq \left[\varphi\left(S(x_0, x_0, x_0)\right)\right]^{\alpha} = \left[\varphi(0)\right]^{\alpha},$$

a contradiction with the definition of φ. So it should be $gx_0 = x_0$.

Now, we suppose $r > 0$ and $x \in D_{x_0,r}^S$ is an arbitrary point such that $S(x, x, gx) > 0$. From the hypothesis and the definition of r, we obtain

$$\varphi\left(S(x, x, gx)\right) \leq \left[\varphi\left(S(x, x, x_0)\right)\right]^{\alpha} \leq \left[\varphi(r)\right]^{\alpha} \leq \left[\varphi(S(x, x, gx))\right]^{\alpha},$$

a contradiction with $\alpha \in (0, 1)$. Hence it should be $x = gx$. Consequently, g fixes the disc $D_{x_0,r}^S$. □

Definition 2.3. Let (X, S) be an S-metric space and $g : X \to X$ a self-mapping. If there exists $x_0 \in X$ such that

$$S(x, x, gx) > 0 \Longrightarrow \varphi\left(S(x, x, gx)\right) \leq \left[\varphi\left(m_S(x, x_0)\right)\right]^{\alpha},$$

for all $x \in X$, where $\alpha \in (0,1)$, the function $\varphi : (0,\infty) \to (1,\infty)$ is such that φ is nondecreasing and

$$m_S(x,y) = \max\left\{S(x,x,y), S(x,x,gx), S(y,y,gy), \frac{S(x,x,gy)+S(y,y,gx)}{2}\right\},$$

then g is called Li-Jiang type x_0-S-contraction.

Theorem 2.4. *Let (X,S) be an S-metric space, $g : X \to X$ Li-Jiang type x_0-S-contraction with $x_0 \in X$ and the number r defined as in (2.1). If $S(gx,gx,x_0) \le r$, then g fixes the disc $D^S_{x_0,r}$.*

Proof. At first, we show that x_0 is a fixed point of g. To do this, we assume that x_0 is not a fixed point of g, that is, $x_0 \neq gx_0$. Using the Li-Jiang type x_0-S-contraction property, we find

$$\varphi\left(S(x_0,x_0,gx_0)\right) \le \left[\varphi\left(m_S(x_0,x_0)\right)\right]^\alpha = \left[\varphi\left(S(x_0,x_0,gx_0)\right)\right]^\alpha,$$

a contradiction with $\alpha \in (0,1)$. So it should be

$$gx_0 = x_0. \tag{2.2}$$

Let $r = 0$. Then we have $D^S_{x_0,r} = \{x_0\}$. From the equality (2.2), we say that g fixes the disc $D^S_{x_0,r}$.

Now, we suppose $r > 0$ and $x \in D^S_{x_0,r}$ is an arbitrary point such that $S(x,x,gx) > 0$. Using the hypothesis, the definition of r, Lemma 1.2 and the equality (2.2), we get

$$\varphi\left(S(x,x,gx)\right) \le \left[\varphi\left(m_S(x,x_0)\right)\right]^\alpha \le \left[\varphi\left(S(x,x,gx)\right)\right]^\alpha,$$

a contradiction. Thereby, it should be $x = gx$. Consequently, g fixes the disc $D^S_{x_0,r}$. □

In the following theorem, we see some equivalence of contractions.

Theorem 2.5. *Let $X \neq \emptyset$, the functions $S_1, S_2 : X \times X \times X \to \mathbb{R}_+$ be such that*

(i) $x = y = z$ implies $S_1(x,y,z) = 0$,

(ii) $S_2(x,y,z) = 0$ implies $x = y = z$,

and g is a self-mapping on X. Then the followings are equivalent:

(a) There exist $x_0 \in X$, a function $\varphi : (0,\infty) \to (0,\infty)$ and $\lambda \in [0,1)$ such that

$$S_1(x,x,gx) > 0 \Longrightarrow \varphi\left(S_1(x,x,gx)\right) \le \lambda\varphi\left(S_2(x,x,x_0)\right),$$

for all $x \in X$.

(b) There exist $x_0 \in X$, a function $\varphi : (0,\infty) \to (1,\infty)$ and $\alpha \in [0,1)$ such that

$$S_1(x,x,gx) > 0 \Longrightarrow \varphi\left(S_1(x,x,gx)\right) \le \left[\varphi\left(S_2(x,x,x_0)\right)\right]^\alpha,$$

for all $x \in X$.

(c) There exist $x_0 \in X$, a function $\varphi : (0,\infty) \to \mathbb{R}$ and $t > 0$ such that

$$S_1(x,x,gx) > 0 \Longrightarrow t + \varphi\left(S_1(x,x,gx)\right) \le \varphi\left(S_2(x,x,x_0)\right),$$

for all $x \in X$.

Proof. Let $S_1(x,x,gx) > 0$ for all $x \in X$.

$(a) \Longrightarrow (b)$: Assume that the condition (a) is satisfied. Then using this condition, we have

$$\exp\left[\varphi\left(S_1(x,x,gx)\right)\right] \le \exp\left[\lambda\varphi\left(S_2(x,x,x_0)\right)\right] = \exp\left[\varphi\left(S_2(x,x,x_0)\right)\right]^\lambda. \tag{2.3}$$

If we define $\alpha \in [0,1)$ by $\alpha = \lambda$ and the function $\varphi_1 : (0,\infty) \to (1,\infty)$ by $\varphi_1(t) = \exp[\varphi(t)]$, then using the inequality (2.3), we find

$$\varphi_1 \left(S_1(x,x,gx)\right) \leq \left[\varphi_1 \left(S_2(x,x,x_0)\right)\right]^{\alpha},$$

which proves the condition (b).

$(b) \Longrightarrow (c)$: Suppose that the condition (b) is satisfied. Then using this condition, we get

$$\begin{aligned} \ln\left[\ln\left(\varphi\left(S_1(x,x,gx)\right)\right)\right] &\leq \ln\left[\ln\left(\left[\varphi\left(S_2(x,x,x_0)\right)\right]^{\alpha}\right)\right] \\ &= \ln\left[\ln\left(\varphi\left(S_2(x,x,x_0)\right)\right)\right] + \ln(\alpha). \end{aligned} \tag{2.4}$$

If we define $t > 0$ by $t = -\ln(\alpha)$ and the function $\varphi_2 : (0,\infty) \to \mathbb{R}$ by $\varphi_2(t) = \ln(\ln(\varphi(t)))$, then using the inequality (2.4), we obtain

$$t + \varphi_2 \left(S_1(x,x,gx)\right) \leq \varphi_2 \left(S_2(x,x,x_0)\right),$$

which proves the condition (c).

$(c) \Longrightarrow (a)$: Assume that the condition (c) is satisfied. Then using this condition, we find

$$\exp\left[\varphi\left(S_1(x,x,gx)\right)\right] \leq \exp\left[\varphi\left(S_2(x,x,x_0)\right) - t\right] = \exp\left[\varphi\left(S_2(x,x,x_0)\right)\right]\exp\left[-t\right]. \tag{2.5}$$

If we define $\lambda \in [0,1)$ by $\lambda = \exp[-t]$ and the function $\varphi_3 : (0,\infty) \to (0,\infty)$ by $\varphi_3(t) = \exp[\varphi(t)]$, then using the inequality (2.5), we get

$$\varphi_3 \left(S_1(x,x,gx)\right) \leq \lambda\varphi_3 \left(S_2(x,x,x_0)\right),$$

which proves the condition (a). □

Then we obtain the following consequences.

Remark 2.6. (1) Theorem 2.2 and Theorem 2.4 can be considered as fixed-circle theorems. Also, they can be considered as fixed-point results in case $r = 0$.

(2) Theorem 2.5 can be consider as the equivalence of some fixed-disc or fixed-circle contractive conditions.

(3) The condition (a) of Theorem 2.5 can be considered as the Banach type contractive condition [1]. Similarly, the condition (b) can be considered as the Jleli-Samet type contractive condition [3] and finally, the condition (c) can be considered as Wardowski type contractive condition [15].

Now, we give the following example.

Example 2.7. Let $X = \mathbb{R}$ and the S-metric defined as

$$S(x,y,z) = |x-z| + |x+z-2y|,$$

for all $x,y,z \in \mathbb{R}$. Let us define the function $g : \mathbb{R} \to \mathbb{R}$ as

$$g(x) = \begin{cases} x & , \quad x \in [-3,3] \\ x+1 & , \quad x \in (-\infty,-3) \cup (3,\infty) \end{cases},$$

for all $x \in \mathbb{R}$. Then the function g is Jleli-Samet type x_0-S-contraction with $x_0 = 0$, $\alpha = 0.6$ and the function $\varphi : (0,\infty) \to (1,\infty)$ defined by $\varphi(t) = t+1$. Also, the function g is Li-Jiang type x_0-S-contraction with $x_0 = 0$, $\alpha = 0.7$ and the function $\varphi : (0,\infty) \to (1,\infty)$ defined by $\varphi(t) = t+1$. Consequently, we have $r = 2$ and so g fixes the disc $D^S_{0,2} = [-1,1]$.

3. Conclusion

In this paper, we obtain new fixed-disc results as some solutions to the "Fixed-Circle Problem". The obtained results will contribute to the literature on this subject. Some applications of these results can be investigated to the various applicable areas.

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A NUMERICAL SOLUTION OF MHD JEFFERY–HAMEL MODEL ARISING IN FLUID MECHANICS

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Abstract. This study is concerned with obtaining a numerical solution of third order MHD Jeffery–Hamel nonlinear differential equation arising in fluid dynamics, by constructing a matrix-collocation method involving the Nörlund polynomial, matrix expansions of linear and nonlinear terms, and collocation points. The method runs easily on a computer programme, which is devised specifically for the model, after gathering its all matrix compounds into a unique matrix equation. Hence, the precise numerical and graphical results are demonstrated in table and figures, respectively. These comparable tools allow us to discriminate the efficiency and accuracy of the method. One can thus observe that the method is eligible scheme to treat the equation in question.

1. Introduction

Nonlinear differential equations govern many physical phenomena occurring in mathematics, engineering, physics, fluid dynamics etc. [1]

The magneto-hydro-dynamic (MHD) Jeffery–Hamel nonlinear differential equation (JHE) appears in a cylindrical polar coordinate system in which two dimensional steady flow of a viscous incompressible fluid through a source or sink at channel walls lying on the plane and intersecting at z-axis [2, 3, 4, 5, 6]. Some details of JHE on cylindrical polar coordinate system can be viewed in [4, 5, 6]. In this study, we focus on JHE of ordinary type as the following (see [4, 5, 6]):

$$y'''(x) + (4-H)\alpha^2 y'(x) + 2\alpha Re\, y(x)y'(x) = 0,\ 0 \le x \le 1, \tag{1.1}$$

subject to the initial and boundary conditions

$$y(0) = 1,\ y'(0) = 0,\ y(1) = 0, \tag{1.2}$$

where α is the angle between two rigid plane walls, Re and H denote Reynolds and Hartman numbers, respectively [6].

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The numerical solution for Eq. (1.1) turns out to be the Nörlund polynomial form (see [7]):

$$y_N(x) = \sum_{n=0}^{N} y_n B_n^{(x)}, \tag{1.3}$$

where y_n's are the unknown coefficients to be acquired by the proposed method and $B_n^{(x)}$ is the Nörlund polynomial, which is defined to be (see [7, 8])

$$\sum_{n=0}^{\infty} B_n^{(x)} \frac{t^n}{n!} = \left[\frac{t}{e^t - 1}\right]^x,$$

and its first four bases yield

$$\left\{B_0^{(x)}, B_1^{(x)}, B_2^{(x)}, B_3^{(x)}\right\} = \left\{1, \frac{-x}{2}, \frac{x^2}{4} - \frac{x}{12}, -\frac{x^3}{8} + \frac{x^2}{8}\right\}.$$

Notice that one can refer to [7, 8] for further information about the Nörlund polynomial.

Our goal in this study is to obtain the Nörlund polynomial solution of JHE, implementing consistently matrix-collocation method at the high level of computation limit.

2. Matrix–collocation method and its outcome: Nörlund polynomial solution

In this section, the matrix–collocation method is constructed under the Nörlund polynomial base and its outcome holds the numerical solution as given in Eq. (1.3). Next, the matrix relation of solution form (1.3) is of the form (see [7])

$$y(x) = \boldsymbol{B}^{(x)} \boldsymbol{Y}, \tag{2.1}$$

where

$$\boldsymbol{B}^{(x)} = \begin{bmatrix} B_0^{(x)} & B_1^{(x)} & B_2^{(x)} & \cdots & B_N^{(x)} \end{bmatrix},$$

and

$$\boldsymbol{Y} = \begin{bmatrix} y_0 & y_1 & \cdots & y_N \end{bmatrix}^T.$$

In view of the matrix relation (2.1), the differentiated parts of Eq. (1.1) can be expanded as the matrix relations

$$\begin{aligned} y'''(x) &= \left(\boldsymbol{B}^{(x)}\right)^{(3)} \boldsymbol{Y}, \\ y'(x) &= \left(\boldsymbol{B}^{(x)}\right)^{(1)} \boldsymbol{Y}, \end{aligned} \tag{2.2}$$

where

$$\left(\boldsymbol{B}^{(x)}\right)^{(3)} = \begin{bmatrix} \left(B_0^{(x)}\right)^{(3)} & \left(B_1^{(x)}\right)^{(3)} & \cdots & \left(B_N^{(x)}\right)^{(3)} \end{bmatrix},$$

and

$$\left(\boldsymbol{B}^{(x)}\right)^{(1)} = \left[\begin{array}{cccc} \left(B_0^{(x)}\right)^{(1)} & \left(B_1^{(x)}\right)^{(1)} & \cdots & \left(B_N^{(x)}\right)^{(1)} \end{array}\right].$$

When the Chebyshev-Lobatto collocation points, which are defined to be

$$t_i = \frac{1}{2} + \frac{-1}{2}\cos\left(\frac{\pi i}{N}\right),\ i = 0, 1, 2, \ldots, N,\ \ t_0 = 0 < t_1 < \ldots < t_N = 1, \tag{2.3}$$

on [0,1], are inserted into the matrix relations (2.2), it follows that

$$\begin{aligned} y'''(x_i) &= \left(\boldsymbol{B}^{(x_i)}\right)^{(3)}\boldsymbol{Y} = \boldsymbol{B}^{(3)}\boldsymbol{Y}, \\ y'(x_i) &= \left(\boldsymbol{B}^{(x_i)}\right)^{(1)}\boldsymbol{Y} = \boldsymbol{B}^{(1)}\boldsymbol{Y}, \end{aligned} \tag{2.4}$$

where

$$\boldsymbol{B}^{(3)} = \left[\begin{array}{c} \left(\boldsymbol{B}^{(x_0)}\right)^{(3)} \\ \left(\boldsymbol{B}^{(x_1)}\right)^{(3)} \\ \vdots \\ \left(\boldsymbol{B}^{(x_N)}\right)^{(3)} \end{array}\right] = \left[\begin{array}{cccc} \left(B_0^{(x_0)}\right)^{(3)} & \left(B_1^{(x_0)}\right)^{(3)} & \cdots & \left(B_N^{(x_0)}\right)^{(3)} \\ \left(B_0^{(x_1)}\right)^{(3)} & \left(B_1^{(x_1)}\right)^{(3)} & \cdots & \left(B_N^{(x_1)}\right)^{(3)} \\ \vdots & \vdots & \ddots & \vdots \\ \left(B_0^{(x_N)}\right)^{(3)} & \left(B_1^{(x_N)}\right)^{(3)} & \cdots & \left(B_N^{(x_N)}\right)^{(3)} \end{array}\right],$$

and $\boldsymbol{B}^{(1)}$ is similarly obtained.

Now, the matrix form of linear part of Eq. (1.1) can be stated using the matrix relations (2.4) as

$$\boldsymbol{L} = \left[\boldsymbol{B}^{(3)} + (4 - H)\alpha^2\boldsymbol{B}^{(1)}\right]\boldsymbol{Y}. \tag{2.5}$$

As a next construction, we shall build the matrix relation of the nonlinear term in Eq. (1.1). By the matrix relation (2.1) and the collocation points (2.3), the matrix relation of the nonlinear term admits

$$y'(x_i)\,y(x_i) = \left(\boldsymbol{B}^{(x_i)}\right)^{(1)}\ \left(\ \overline{\boldsymbol{B}^{(x_i)}}\ \right)\overline{\boldsymbol{Y}},$$

or, equivalently,

$$y'(x_i)\,y(x_i) = \boldsymbol{B}^{(1)}\ \left(\overline{\boldsymbol{B}}\right)\overline{\boldsymbol{Y}}, \tag{2.6}$$

where

$$\boldsymbol{B} = \left[\begin{array}{cccc} B_0^{(x_0)} & B_1^{(x_0)} & \cdots & B_N^{(x_0)} \\ B_0^{(x_1)} & B_1^{(x_1)} & \cdots & B_N^{(x_1)} \\ \vdots & \vdots & \vdots & \vdots \\ B_0^{(x_N)} & B_1^{(x_N)} & \cdots & B_N^{(x_N)} \end{array}\right],\ \overline{\boldsymbol{B}} = \mathrm{diag}[\boldsymbol{B}]_{(N+1)\times(N+1)^2},$$

and

$$\overline{\boldsymbol{Y}} = \begin{bmatrix} y_0\boldsymbol{Y} & y_1\boldsymbol{Y} & \cdots & y_N\boldsymbol{Y} \end{bmatrix}^T_{1\times(N+1)^2}.$$

Then, the matrix form of the nonlinear term is implied using the matrix relation (2.6) as

$$\boldsymbol{N} = \boldsymbol{B}^{(1)}\left(\overline{\boldsymbol{B}}\right)\overline{\boldsymbol{Y}}. \tag{2.7}$$

By (2.5) and (2.7), we are now ready to reveal the fundamental matrix form as

$$\boldsymbol{L} + \boldsymbol{N} = \boldsymbol{G} \Rightarrow \boldsymbol{W}\,\boldsymbol{Y} + \boldsymbol{Z}\overline{\boldsymbol{Y}} = \boldsymbol{G} \ \text{ or } \ [\boldsymbol{W};\, \boldsymbol{Z} : \boldsymbol{G}], \tag{2.8}$$

where

$$\boldsymbol{W} = \left[\boldsymbol{B}^{(3)} + (4-H)\alpha^2\boldsymbol{B}^{(1)}\right],\ \boldsymbol{Z} = \left[\boldsymbol{B}^{(1)}\left(\overline{\boldsymbol{B}}\right)\right],$$

and

$$\boldsymbol{G} = \begin{bmatrix} 0 & 0 & \cdots & 0 \end{bmatrix}^T_{1\times(N+1)}.$$

The matrix expansions of the initial and boundary conditions (1.2) are formed using the matrix relations (2.1) and (2.2) as

(2.9)

$$y(0) = \boldsymbol{B}^{(0)}\boldsymbol{Y} \Rightarrow \begin{bmatrix} B_0^{(0)} & B_1^{(0)} & \cdots & B_N^{(0)} & : & 1 \end{bmatrix},$$

$$y'(0) = \left(\boldsymbol{B}^{(0)}\right)^{(1)}\boldsymbol{Y} \Rightarrow \begin{bmatrix} \left(B_0^{(0)}\right)^{(1)} & \left(B_1^{(0)}\right)^{(1)} & \cdots & \left(B_N^{(0)}\right)^{(1)} & : & 0 \end{bmatrix},$$

$$y(1) = \boldsymbol{B}^{(1)}\boldsymbol{Y} \Rightarrow \begin{bmatrix} B_0^{(1)} & B_1^{(1)} & \cdots & B_N^{(1)} & : & 0 \end{bmatrix}.$$

The augmented matrix system, which is ready to be treated by *Solve* command on Mathematica, after replacing the condition matrices (2.9) by the last three rows of $\boldsymbol{W}$ in the matrix form (2.8), then takes its final form

$$\left[\widehat{\boldsymbol{W}}\ ;\ \widehat{\boldsymbol{Z}}\ :\ \widehat{\boldsymbol{G}}\right].$$

We thus obtain the unknown Nörlund coefficients, which are later inserted into Eq. (1.3), eventually, the Nörlund polynomial numerical solution is appeared.

3. A MHD Jeffery–Hamel model

In this section, a MHD Jeffery–Hamel model is numerically solved by the proposed method for different values of Re, H and α. In doing so, a computer programme, which was devised to treat JHE, is deployed. Numerical and graphical results are shown in table and figures. Note that since the exact solution of JHE (1.1)-(1.2) is unknown, the present results are compared according to Mathematica solution.

An Example

Consider JHE (1.1)-(1.2) for different values of Re, H and α. After deploying the proposed method and *NDSolve* module on Mathematica, the Nörlund polynomial solutions are illustrated along with the Mathematica solution in Figs. 1 and 2, in which $\{Re, H, \alpha\}$ take their values as $\{0.1, 3, 2\}$ and $\{0.5, 15, 1.5\}$, respectively. Fig. 3 emphasises that the physical behaviours of Nörlund polynomial solutions $y_6(x)$ are modified in proportion to Hartmann number $H = 5, 7, 9, 10$. Also, Table 1 indicates the absolute error values, which are obtained by the Nörlund polynomial and Mathematica solutions, at the Chebyshev-Lobatto collocation points.

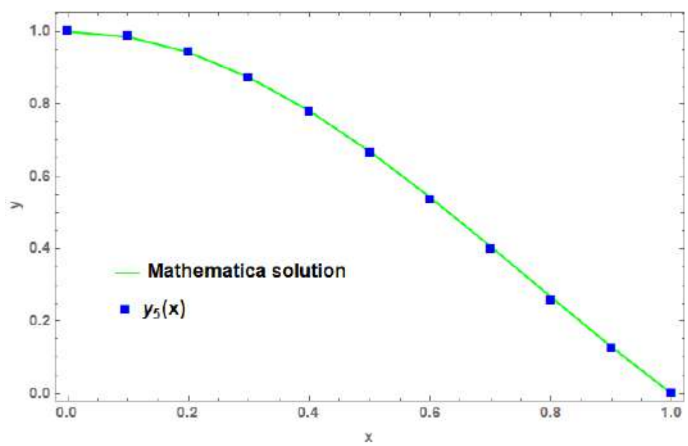

FIGURE 1. Comparison history of the Nörlund polynomial and Mathematica solutions incurred to $Re = 0.1$, $H = 3$ and $\alpha = 2$.

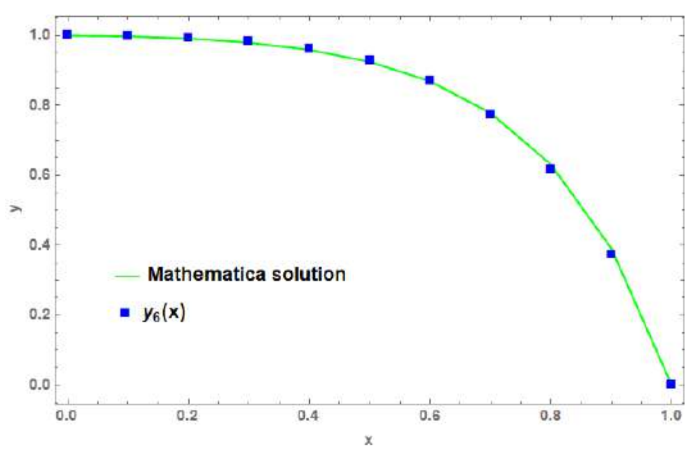

FIGURE 2. Comparison history of the Nörlund polynomial and Mathematica solutions incurred to $Re = 0.5$, $H = 15$ and $\alpha = 1.5$.

4. CONCLUDING REMARKS

A matrix-collocation method based on the Nörlund rational polynomial has been properly established to obtain the numerical solution of JHE. In doing so, the suitable and simple matrix forms at the Chebyshev-Lobatto collocation points have

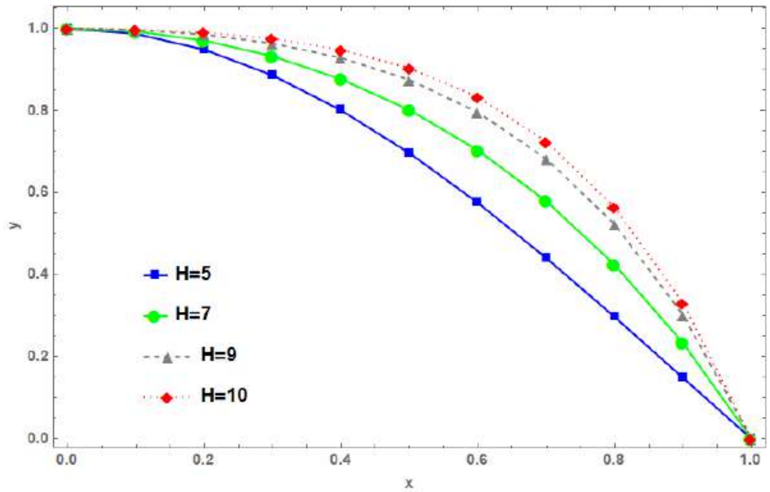

FIGURE 3. Comparison history of the Nörlund polynomial solutions $y_6(x)$ with $Re = 2$ and $\alpha = 2$ versus H.

TABLE 1. Absolute error computations at the Chebyshev-Lobatto collocation points for $Re = 0.01$, $H = 3$ and $\alpha = 0.1$.

x_i	$N = 8$
0.000000	0.00e–00
0.038060	3.80e–08
0.146447	6.80e–07
0.308658	3.00e–06
0.500000	7.43e–06
0.691342	1.17e–05
0.853553	1.09e–05
0.961940	4.29e–06
1.000000	2.37e–08

been easily coupled with a polynomial method. Upon the investigations of Figs. 1 and 2, the numerical solutions coincide suitably with Mathematica solutions. Fluctuation of Hartmann number changes the physical response of the solutions, as seen in Fig. 3. Furthermore, it can be overseen from Table 1 that the method takes six-seven decimal places precision for $N = 8$, which means that highly remarkable numerical values are obtained via the proposed method.

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NUMERICAL EXPERIMENTS WITH AN INFEASIBLE PRIMAL-DUAL ALGORITHM FOR SOLVING THE SEMIDEFINITE LEAST SQUARES PROBLEMS

CH. DAILI AND M. ACHACHE

Abstract. This paper focuses on the numerical resolution of a Semi-definite least squares problems ($SDLS$) by an infeasible primal-dual type interior-point method based on the directions of Alizadeh-Haeberly-Overton (AHO) (Monteiro, 1997). Moreover, we also present some numerical experiments to illustrate the efficiency of this algorithm and a conclusion that ends the article is stated.

1. Introduction

Path-following interior-point methods of primal-dual type are the most attractive for solving linear optimization[1,7]. Their corresponding algorithms enjoy important theoretical and numerical properties such as the polynomial complexity and numerical efficiency. Thus motivated researches to extend it to more general optimization and mathematical problems, namely, complementarity problems, convex optimization, semidefinite optimization and convex quadratic semidefinite optimization. Interior-point methods are divided into two classes, namely, feasible and infeasible primal-dual interior point methods. Feasible primal-dual interior point algorithms require that the primal-dual starting point must be feasible i.e., it lies in the feasible set. This task is very hard to release in numerical practice. In order to eliminate this handicap, we shall use any starting point not necessarily lies in the feasible set of the considered problem. This type of methods is named as the infeasible interior-point methods.

We consider the following semidefinite least-squares problem ($SDLS$):

$$(SDLS\ )\begin{cases} \min f(x) = \frac{1}{2}\left\| X - C\right\|_F^2 \\ \langle A_i,\ X\rangle = b_i, i = 1,\dots,m \\ X \in \mathbb{S}_+^n \end{cases} \tag{1.1}$$

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and it's dual :

$$(1.2) \qquad (DSDLS\ ) \begin{cases} \max b^T y - \frac{1}{2}\|X\|_F^2 + \frac{1}{2}\|C\|_F^2 \\ \sum\limits_{i=1}^{m} y_i A_i - X + Z = -C \\ X, Z \in \mathbb{S}_+^n, y \in \mathbb{R}^m \end{cases}$$

where the vector $b \in \mathbb{R}^m$, the matrices $C \in \mathbb{R}^{n\times n}$ and A_i, $i = 1, \cdots, m$, are given and belong to the linear space of $n\times n$ symmetric matrices $\mathbb{S}^n$. I denotes the identity matrix in $\mathbb{R}^n$.The $\langle , \rangle$ operation is the inner product on $\mathbb{S}^n$ of two matrices X and Y, which is the trace of their product, i.e., $\langle X, Y\rangle = tr(XY) = \sum_{i,j} x_{ij} y_{ij}$. The inequality constraint $X \succeq 0$ indicates that the matrix X , belong to the cone of positive semidefinite matrices $\mathbb{S}_+^n$. We denote by $\mathbb{S}_{++}^n$ the cone of positive definite matrices of $\mathbb{S}^n$ and $\|\|_F$ denotes the Frobenius norm, i.e.,

$$\|X\|_F = \left(tr(X^T X)\right)^{\frac{1}{2}} = \left(\sum_{i,j=1}^{n} X_{ij}^2\right)^{\frac{1}{2}}, \ X = (x_{ij})$$

We denote by

$$\mathcal{F}_{(SDLS\ )} = \left\{X \in \mathbb{S}_+^n : \langle A_i,\ X\rangle = b_i, i = 1, \ldots, m\right\}$$

and

$$\mathcal{F}^0_{(SDLS\ )} = \left\{X \in \mathbb{S}_{++}^n : \langle A_i,\ X\rangle = b_i, i = 1, \ldots, m\right\}$$

the set of feasible and strictly feasible primal solutions for $(SDLS)$, respectively.

$$\mathcal{F}_{(DSDLS\ )} = \left\{(y, Z) \in \mathbb{R}^m \times \mathbb{S}_+^n : \sum_{i=1}^{m} y_i A_i - X + Z = -C\right\}$$

and

$$\mathcal{F}^0_{(DSDLS\ )} = \left\{(y, Z) \in \mathbb{R}^m \times \mathbb{S}_{++}^n : \sum_{i=1}^{m} y_i A_i - X + Z = -C\right\}$$

the sets of feasible and strictly feasible dual solutions for $(DSDLS)$, respectively.

The SDLSs problem have recently attracted considerable attention because it has a lot of applications in the domain of the applied mathematics and numerical linear algebra such as the nearest correlation matrix (NCM) and in preconditioning of linear system and error analysis of such iterative methods [12]. Many methods have been proposed to solve this problem. Alternating projections method is proposed by Higham in [11] to solve particular instances of semidefinite least-squares (and it could be generalized to any semidefinite least-squares). J. Malick propose a Lagrangian dualization of this least-squares problem, then he propose to solve the latter problem with a quasi-Newton algorithm [9]. Our aim is to propose an efficient primal dual interior point algorithm to solve the SDLSs problem.We are particularly interested by the infeasible primal dual path-following interior point algorithm. These methods enjoy best results such as polynomial complexity and numerical efficiency. This paper is organized as follows. In Section 2, we associate to the $(SDLS)$ problem the $(SDLS)_\mu$ perturbed problem by introducing the logarithmic barrier function followed by the optimal conditions for the $(SDLS)$, then we briefly present the primal-dual method of trajectory central. In section 3, we give a description of the interior-point algorithms obtained. In section 4, we present some numerical tests on several different examples to illustrate the effectiveness of this algorithm in solving the $(SDLS)$ problem.

2. Barrier penalization problems

In the rest of this document, we assume that the $(SDLS)$ problem meets the following conditions:

- Interior point Condition : (IPC) The set $\mathcal{F}^0_{(SDLS\ )} \times \mathcal{F}^0_{(DSDLS\ )}$ is non-empty.
- The matrix A_i $i = 1, \dots, m$ are linearly independent.

We associate to the $(SDLS)$ problem the following perturbed problem:

$$(SDLS)_\mu \begin{cases} \min f_\mu(X) \\ \langle A_i,\ X\rangle = b_i, i = 1, \dots, m \end{cases} \tag{2.1}$$

where

$$f_\mu(X) \ = \begin{cases} f(x) - \mu \ln \det X & \text{si} \quad X \succ 0 \\ +\infty & \text{si non} \end{cases}$$

The function $-\ln\det X$ is called the logarithmic barrier function associated with the cone $\mathbb{S}^n_+$ and $\mu > 0$,is the barrier parameter. It is shown if the IPC condition holds, that the problem (2.2) has a unique solution $(X(\mu), y(\mu))$, that the solution of (2.1) as a function of μ. The problem (2.1) is convex and differentiable. So the necessary and sufficient conditions for $X(\mu)$ to be an optimal for $(SDLS)_\mu$, is the existence of a vector $y(\mu)$ such as:

$$\begin{cases} \nabla f_\mu(X) - \sum\limits_{i=1}^{m} y_i(\mu) A_i = 0 \\ A_i \bullet X = b_i, \qquad i = 1, \dots, m \end{cases} \tag{2.2}$$

where

$$\nabla f_\mu(X) = X - C - \mu X^{-1}$$

Letting

$$Z(\mu) = \mu X^{-1}$$

then

$$X(\mu) Z(\mu) = \mu I,$$

and the system (2.2) can be rewritten as

$$\begin{cases} X(\mu) - Z(\mu) - \sum\limits_{i=1}^{m} y_i(\mu) A_i = C,\ X \succ 0,\ Z \succ 0 \\ A_i \bullet X(\mu) = b_i, \qquad i = 1, \dots, m \\ X(\mu) Z(\mu) = \mu I,\ \mu > 0. \end{cases} \tag{2.3}$$

We set $(X(\mu), y(\mu), Z(\mu))$as a solution of the system (2.3). The set

$$\mathcal{C} = \{(X(\mu), y(\mu), Z(\mu)) \ : \mu > 0\}$$

is called the central-path of the problem $(SDLS)$. If μ tends to zero then the limit of the system (2.2) exists and therefore it yields an optimal solution of (1.1) and (1.2). The infeasible primal-dual path-following interior point algorithms aim to trace approximately the central-path $\mathcal{C}$ by using at each iteration a damped Newton step while the initial starting point is not necessarily feasible i.e., it does not lie in $\mathcal{D}$ and get closer to the optimal solution of (1.1) as μ goes to zero.

Now we proceed to describe a damped Newton step produced by the algorithm for a given $\mu > 0$. Applying the Newton's method for (2.3) for a given infeasible point (X, y, Z) i.e., $X \succ 0, y \in \mathbb{R}^m$ and $Z \succ 0$ not necessarily in $\mathcal{F}_{(SDLS\ )}$.Then the Newton direction at this point is the unique solution of the following linear system:

$$
(2.4) \qquad \begin{cases} \Delta X - \Delta Z - \sum_{i=1}^{m} \Delta y_i A_i = C - (X - Z - \sum_{i=1}^{m} y_i A_i), \ X \succ 0, \ Z \succ 0 \\ A_i \bullet \Delta X = b_i - A_i \bullet X, \qquad i = 1, \ldots, m, \\ \Delta X Z + X \Delta Z = \sigma \mu I - XZ, \ \mu > 0, \end{cases}
$$

where $\sigma \in (0, 1)$ the centrality parameter.

However, the resulting system may yields as a solution a search direction which is not symmetric. Since we want X and Y to be symmetric matrices, one must "symmetrizing"the perturbed complementary equation $XZ = \mu I$. Based on different symmetrization schemes, several search directions have been proposed in the literature of semidefinite optimization problems such as Kojima et all [19], Helmberg et all [21], Monteiro [20] and Nesterov and Todd (NT) [17],[18]. In this paper, we use the direction determined by the following system:

$$
(2.5) \qquad \begin{cases} X - Z - \sum_{i=1}^{m} y_i A_i = C, \ X \succ 0, \ Z \succ 0 \\ A_i \bullet X = b_i, \qquad i = 1, \ldots, m \\ \dfrac{XY + YX}{2} = \sigma \mu I, \ \mu > 0. \end{cases}
$$

This symmetrization is introduced by Alizadeh–Haeberly–Overton [23] and is called AHO-direction. Therefore the AHO direction is determined by the solution of the system:

$$
(2.6) \qquad \begin{cases} \Delta X - \Delta Z - \sum_{i=1}^{m} \Delta y_i A_i = C - (X - Z - \sum_{i=1}^{m} y_i A_i), \ X \succ 0, \ Z \succ 0 \\ A_i \bullet \Delta X = b_i - A_i \bullet X, \qquad i = 1, \ldots, m, \\ \Delta X Z + X \Delta Z + \Delta Z X + Z \Delta X = 2\sigma \mu I - (XZ + ZX), \ \mu > 0. \end{cases}
$$

The system (2.6) has a unique symmetric solution $(\Delta X,\ \Delta y,\ \Delta Y\ )$. We will refer to the assignment:

$$
X^+ = X + \alpha \Delta X, \quad y^+ = y + \alpha \Delta y, Z^+ = Z + \alpha \Delta Z
$$

as the damped Newton step with $\alpha > 0$, is the step-size.

3. An infeasible path-following algorithm for SDLS

We present an infeasible path-following interior-point algorithm for computing an optimal solution of $(SDLS)$ that uses the primal-dual interior-point framework proposed by many authors. In each iteration the algorithm starts with guesses (matrices) $X^0, Z^0 \succ 0$, $y^0 \in \mathbb{R}^m$, not necessarily feasible. We would like to update these matrices until we are within our desired tolerance of satisfying equations . We will stop our algorithm when the

$$
\max \left(\left\| C - (X - Z - \sum_{i=1}^{m} y_i A_i) \right\|_F, \|b - AX\|_F\,, \ \|XZ\|_F \right)
$$

is small enough. In order to implement our algorithm we need to compute a direction from and a suitable step size $\alpha > 0$ in each iteration such that $X + \alpha \Delta X \succ 0$, and $Z + \alpha \Delta Z \succ 0$.For computing the step-size $\alpha > 0$, so that $X = X + \alpha \Delta X \succ$

0, and $Z = Z + \alpha\Delta Z \succ 0$,we need to determine the maximum step-size $\alpha_{\max}$ so that if $0 < \alpha \leq \alpha_{\max}$ then $X = X + \alpha\Delta X \succ 0$ and $Z = Z + \alpha\Delta Z \succ 0$. Let α_X and α_Z be the maximum possible step-size on the direction ΔX and ΔZ, respectively. It is known that the condition $X + \alpha\Delta X \succ 0$ is equivalent to $I - \alpha X^{-1}\Delta X \succ 0$. In the other words, we must have $1 - \alpha\lambda_{\max}(X^{-1}\Delta X) > 0$ where $\lambda_{\max}(X^{-1}\Delta X)$ is the maximum eigenvalue of $X^{-1}\Delta X$. Thus

$$\alpha_X = \begin{cases} \dfrac{1}{\lambda_{\max}(X^{-1}\Delta X)} & \text{if } \lambda_{\max}(X^{-1}\Delta X) > 0 \\ \infty & \text{Otherwise.} \end{cases}$$

Similarly for $Z + \alpha\Delta Z$, we have

$$\alpha_Z = \begin{cases} \dfrac{1}{\lambda_{\max}(Z^{-1}\Delta Z)} & \text{if } \lambda_{\max}(Z^{-1}\Delta Z) > 0 \\ \infty & \text{Otherwise.} \end{cases}$$

Once these two allowed maximum step-sizes are determined, then the setep size α is taken as:

$$\alpha = \min(1, \rho \min(\alpha_X, \alpha_Y)) : \rho \in \,]0, 1[\, .$$

The outline of the generic infeasible primal-dual IP algorithm is presented in Figure 1.

4. The algorithm

Input

(1) An accuracy parameter $\epsilon > 0$;
(2) initial guesses $X^0,\ Z^0 \succ 0, y^0 \in \mathbb{R}^m$ and $\mu > 0$;
(3) matrices A_i, C, a vector b ; $i = 1, \dots, m$;

While $\max\left(\left\| C - (X - Z - \sum_{i=1}^{m} y_i A_i) \right\|_F, \|b - AX\|_F,\ \|XZ\|_F\right) > \epsilon$ **do**

begin

(1) Solve the system to obtain $(\Delta X,\ \Delta y,\ \Delta Z)$;
(2) Determine a step size $\alpha > 0$ s.t. $X + \alpha\Delta X \succ 0$ and $Z + \alpha\Delta Z \succ 0$;
(3) Update $X := X + \alpha\Delta X \succ 0$, $Z =: Z + \alpha\Delta Z \succ 0$;

end

Fig.1. Infeasible interior-point algorithms for solving ($SDLS$).

5. Computational experiments

We consider some problems of differents sizes, each problem is followed by a table containing the results obtained by our method. we implement the algorithm in Matlab (R2013b) and the experiments were conducted on a Pentium 4 3.0GHz PC with 2GB of RAM. In the implementation, we use $\varepsilon = 10^{-6}$, we start wilth a point (X^0, y^0, Z^0) not necessarily feasible. The number of iterations required and the time executed by the algorithm are denoted by "Iter" and "CPU" respectivly.

Exemple 1[5](Nearest correlation matrix problems (NCM)). This example of SDLS, is constructed from the following nearest correlation matrix problem:

$$\min_X \frac{1}{2} \|X - C\|_F^2 \ \textbf{s.t.}\ A_i \bullet X = b_i, i = 1, \dots, m,\ X \succeq 0,$$

with

$$C = \begin{bmatrix} 1 & 0.5 & 1 \\ 0.5 & 1 & 0.25 \\ 1 & 0.25 & 1 \end{bmatrix}, A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, A_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, b = e.$$

In this example the triplet starting point is taken as

$$X_0 = 2 * I,\ Z_0 = I,\ y_0 = [2,\ 2,\ 2]^T.$$

The obtained numerical results are summarized in Table 1.

μ	0.5	0.05	0.005
Iter	19	16	13
CPU	0.0374	0.0353	0.0338

Table 1. Numerical results for Example 1.

The obtained approximate primal-dual optimal solution is:

$$X^* = \begin{bmatrix} 1 & 0.4910 & 0.9684 \\ 0.4910 & 1 & 0.2582 \\ 0.9684 & 0.2582 & 1 \end{bmatrix}, Z^* = \begin{bmatrix} 0.0351 & -0.0090 & -0.0316 \\ -0.0090 & 0.0023 & 0.0082 \\ -0.0316 & 0.0082 & 0.0285 \end{bmatrix},$$

$$y^* = [-0.0350, -0.0023, -0.0285]^T.$$

The optimal values for both problems are $p^* = d^* = 0.0011$.

6. Conclusion

In this paper, we introduced a primal-dual infeasible interior point algorithm to solve a semidefinite least-squares problem using the directions of Alizadeh-Hueber-Overtion (AHO). The obtained algorithm gives a strictly feasible solution of ($SDLS$) and a primal-dual solution of ($SDLS$) and ($DSDLS$). Moreover, the numerical tests show that when the size becomes large the system becomes unstable and that is the disadvantage of this approach.

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SOME CHARACTERIZATIONS OF RULED SURFACES GENERATED BY S- CURVES

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Abstract. In this paper, we construct and study timelike special ruled surfaces, which are generated by S- Curves, in Minkoswki 3- Space. We investigate different properties of the constructed ruled surface.

1. Introduction

In differential geometry, surface theory is an important working area in differential geometry. It has been carefully studied by for researchers. A ruled surface is one of the special surfaces and it is thinkable as a geometric set of lines. [2, 4, 9, 10]. It is defined by the moving of a straightline (ruling) along a curve (base curve). This entrancing special surface is of great interest to many applications and has contribution in several areas, such as mathematical physics, kinematics and Computer Aided Geometric Design (CAGD) [11, 12].

Nowadays, a good deal of research on ruled surface theory has been conducted about ruled surfaces in Euclidean and Minkowski space [5, 15, 16]. In [3], the authors studied the ruled surface whose rulings are linear combinations of Frenet frame vectors of its base curve. They gave its position vector in the case of the base curve as general helix [1] and slant helix [7], respectively. Furthermore, in [13] the authors were interested in the study of ruled surface with alternative moving frame of its base curve. They investigated its most important properties and gave characterizations. Then, in [ ], the authors introduced ruled surfaces generated from any vector X,Bishop Darboux vector and Bishop vectors. Finally, in [8] they studied special ruled surfaces, whose rulings are linear combinations of Darboux frame vectors of its base curve relative to an arbitrary regular surface in Euclidean 3-space.

In [6], they construct a new coordinate system by rotating the axes of space system about the time one. We should care that the axis of time rotates differently than the axes of space. Therefore, the rotation of the Frenet frame of (t) which lies on a surface in $\mathbb{E}_1^3$ should be depend on the types of the curve, surface and axis of

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rotation. They called this frame "Shonoda-Saad frame" or simply S-frame. This frame gives us a relative differential geometry between Euclidean and Minkowski invariants of a chosen curve on a regular timelike surface . They investigated this special frame according to a regular spacelike curve $\alpha(t)$ on the surface in $\mathbb{E}_1^3$.

In this paper, we deal with timelike special ruled surfaces, which is generated by S- Curves, in Minkoswki 3- Space. We obtain some properties of these surfaces.

2. Some Properties of S-curves and Ruled Surfaces

Definition 2.1. Let $\varphi(u,v)$ be a timelike surface in $\mathbb{E}_1^3$ and $\alpha(s)$ be a regular spacelike curve lies on φ with timelike principal normal vector $\mathbf{U}$. The curve $\alpha(s)$ is called a spacelike S-curve of first or second type, if $\dfrac{\tau_g}{\kappa_g} = \tanh\Psi$ or $\dfrac{\kappa_g}{\tau_g} = \tanh\Psi$, $\Psi = \gamma$ or β, $\Psi \neq 0$, respectively. Then, we have

$$\begin{bmatrix} \mathbf{T}'_M \\ \mathbf{N}'_M \\ \mathbf{B}'_M \end{bmatrix} = \begin{bmatrix} 0 & \kappa_g^1 - \tau_g^2 & -\kappa_n \\ \tau_g^2 - \kappa_g^1 & 0 & \tau_g^1 - \kappa_g^2 \\ -\kappa_n & \tau_g^1 - \kappa_g^2 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T}_M \\ \mathbf{N}_M \\ \mathbf{B}_M \end{bmatrix}.$$

Here, $\kappa_g = \kappa \sinh\theta$, $\kappa_n = \kappa\cosh\theta$ and $\tau_g = \tau - \dfrac{d\theta}{ds}$, $\theta = \angle_H(\mathbf{N},\mathbf{G})$, $G = -\mathbf{T}\times_M \mathbf{N}_M$, $\kappa_g^1 = \kappa_g\cosh\gamma$, $\kappa_g^2 = \kappa_g \sinh\gamma$, $\tau_g^1 = \tau_g\cosh\gamma$, $\tau_g^2 = \tau_g\sinh\gamma$ and $-\kappa_n = \kappa_n - \dfrac{d\gamma}{ds}$, $\gamma = \angle_H(\mathbf{T},\mathbf{T}_M)$, [see 6].

The standard unit normal vector on a regular surface $\varphi(u,v)$ is identified by

$$U = \frac{\varphi_u \times \varphi_v}{\|\varphi_u \times \varphi_v\|}.$$

The Gauss curvature and mean curvature of the surface $\varphi(u,v)$ defined by

$$K = \frac{h_{11}h_2 - h_{12}^2}{EG-F^2},\ H = \frac{Eh_{22} - 2Fh_{12} + Gh_{11}}{EG-F^2},$$

respectively.

On the other hand, a ruled surface in $\mathbb{E}_1^3$ is generated by a one-parameter family of straight lines and has the parametric representation

$$\varphi : I\times\mathbb{R} \to \mathbb{E}_1^3,\ \varphi(u,v) = \alpha(u) + vX(u)$$

where I is an open interval of the real line $\mathbb{R}.\alpha(u)$ is called the base curve of the ruled surface and $X(u)$ are the unit vectors representing the direction of straight lines (rulings), [14].

3. Ruled Surfaces with S- Curves

In this section, we deal with ruled surfaces, which are generated by spacelike S-Curves with timelike unit normal vector field, in Minkoswki 3- Space.

Theorem 3.1. Let $\varphi(u,v) = \alpha(u) + v\mathbf{T}_M(u)$ be ruled surface, where $\alpha(u)$ is a unit spacelike S- Curves with timelike unit normal vector field in Minkoswki 3-

Space. Then, principal curvatures are

$$\begin{aligned}\kappa_{\max} &= H+\sqrt{H^2-K} \\ &= \frac{1}{2\left(\kappa_n^2+(\kappa_g^1-\tau_g^2)^2\right)^{3/2}}[v[\kappa_n(-(\kappa_g^1-\tau_g^2)^2+\kappa_n^2) \\ &\quad +(\kappa_g^1-\tau_g^2)(-\kappa_n+v[(\kappa_g^1-\tau_g^2)(\tau_g^1-\kappa_g^2)-\kappa_n'])+2\kappa_n(\kappa_g^1-\tau_g^2)]+\sqrt{A}\end{aligned}$$

$$\begin{aligned}\kappa_{\min} &= H-\sqrt{H^2-K} \\ &= \frac{1}{2\left(\kappa_n^2+(\kappa_g^1-\tau_g^2)^2\right)^{3/2}}[v[\kappa_n(-(\kappa_g^1-\tau_g^2)^2+\kappa_n^2) \\ &\quad +(\kappa_g^1-\tau_g^2)(-\kappa_n+v[(\kappa_g^1-\tau_g^2)(\tau_g^1-\kappa_g^2)-\kappa_n'])+2\kappa_n(\kappa_g^1-\tau_g^2)]-\sqrt{A}\end{aligned}$$

where

$$\begin{aligned}A &= v^2(\kappa_n^2(-(\kappa_g^1-\tau_g^2)^2+\kappa_n^2)^2+2(\kappa_n(-(\kappa_g^1-\tau_g^2)^2+\kappa_n^2)) \\ &\quad ((\kappa_g^1-\tau_g^2)(-\kappa_n+v[(\kappa_g^1-\tau_g^2)(\tau_g^1-\kappa_g^2)-\kappa_n'])+2\kappa_n(\kappa_g^1-\tau_g^2)) \\ &\quad +((\kappa_g^1-\tau_g^2)(-\kappa_n+v[(\kappa_g^1-\tau_g^2)(\tau_g^1-\kappa_g^2)-\kappa_n'])+2\kappa_n(\kappa_g^1-\tau_g^2))^2 \\ &\quad -\kappa_n^2(\kappa_g^1-\tau_g^2)^2\left(\kappa_n^2+(\kappa_g^1-\tau_g^2)^2\right)\end{aligned}$$

Proof. First derivatives of the surface $\varphi(u,v)$ are

$$\varphi_u(u,v)=\mathbf{T}_M+v[(\kappa_g^1-\tau_g^2)\mathbf{N}_M-\kappa_n\mathbf{B}_M],$$

$$\varphi_v(u,v)=\mathbf{T}_M.$$

The unit normal vector field of the surface is

$$U=-\frac{\kappa_n\mathbf{T}_M+(\kappa_g^1-\tau_g^2)\mathbf{B}_M}{\sqrt{\kappa_n^2+(\kappa_g^1-\tau_g^2)^2}}$$

Components of the First fundamental form are

$$E=1+v^2(-(\kappa_g^1-\tau_g^2)^2-\kappa_n^2),$$

$$F=G=1.$$

Second derivatives of the surface $\varphi(u,v)$ are

$$\begin{aligned}\varphi_{uu}(u,v) &= v(-(\kappa_g^1-\tau_g^2)^2+\kappa_n^2)\mathbf{T}_M \\ &\quad +[(\kappa_g^1-\tau_g^2)+v((\kappa_g^1-\tau_g^2)'-\kappa_n(\tau_g^1-\kappa_g^2))]\mathbf{N}_M \\ &\quad +(-\kappa_n+v[(\kappa_g^1-\tau_g^2)(\tau_g^1-\kappa_g^2)-\kappa_n'])\mathbf{B}_M,\end{aligned}$$

$$\varphi_{uv}(u,v)=(\kappa_g^1-\tau_g^2)\mathbf{N}_M-\kappa_n\mathbf{B}_M,$$

$$\varphi_{vv}(u,v)=0.$$

Second fundamental form of the surface $\varphi(u,v)$ are

$$\begin{aligned}h_{11} &= -\frac{v}{\sqrt{\kappa_n^2+(\kappa_g^1-\tau_g^2)^2}}[\kappa_n(-(\kappa_g^1-\tau_g^2)^2+\kappa_n^2) \\ &\quad +(\kappa_g^1-\tau_g^2)(-\kappa_n+v[(\kappa_g^1-\tau_g^2)(\tau_g^1-\kappa_g^2)-\kappa_n'])],\end{aligned}$$

$$h_{12} = -\frac{\kappa_n v(\kappa_g^1 - \tau_g^2)}{\sqrt{\kappa_n^2 + (\kappa_g^1 - \tau_g^2)^2}},$$

$$h_{22} = 0.$$

Then, the Gauss curvature of $\varphi(u, v)$ is

$$K = \frac{\kappa_n^2(\kappa_g^1 - \tau_g^2)^2}{\left(\kappa_n^2 + (\kappa_g^1 - \tau_g^2)^2\right)^2},$$

$$\begin{aligned} H &= \frac{v}{2\left(\kappa_n^2 + (\kappa_g^1 - \tau_g^2)^2\right)^{3/2}}[\kappa_n(-(\kappa_g^1 - \tau_g^2)^2 + \kappa_n^2) \\ &\quad +(\kappa_g^1 - \tau_g^2)(-\kappa_n + v[(\kappa_g^1 - \tau_g^2)(\tau_g^1 - \kappa_g^2) - \kappa_n']) + 2\kappa_n(\kappa_g^1 - \tau_g^2)]. \end{aligned}$$

Then, principal curvatures of this surface are

$$\begin{aligned} \kappa_{\max} &= H + \sqrt{H^2 - K} \\ &= \frac{1}{2\left(\kappa_n^2 + (\kappa_g^1 - \tau_g^2)^2\right)^{3/2}}[v[\kappa_n(-(\kappa_g^1 - \tau_g^2)^2 + \kappa_n^2) \\ &\quad +(\kappa_g^1 - \tau_g^2)(-\kappa_n + v[(\kappa_g^1 - \tau_g^2)(\tau_g^1 - \kappa_g^2) - \kappa_n']) + 2\kappa_n(\kappa_g^1 - \tau_g^2)] + \sqrt{A} \end{aligned}$$

$$\begin{aligned} \kappa_{\min} &= H - \sqrt{H^2 - K} \\ &= \frac{1}{2\left(\kappa_n^2 + (\kappa_g^1 - \tau_g^2)^2\right)^{3/2}}[v[\kappa_n(-(\kappa_g^1 - \tau_g^2)^2 + \kappa_n^2) \\ &\quad +(\kappa_g^1 - \tau_g^2)(-\kappa_n + v[(\kappa_g^1 - \tau_g^2)(\tau_g^1 - \kappa_g^2) - \kappa_n']) + 2\kappa_n(\kappa_g^1 - \tau_g^2)] - \sqrt{A} \end{aligned}$$

where

$$\begin{aligned} A &= v^2(\kappa_n^2(-(\kappa_g^1 - \tau_g^2)^2 + \kappa_n^2)^2 + 2(\kappa_n(-(\kappa_g^1 - \tau_g^2)^2 + \kappa_n^2)) \\ &\quad ((\kappa_g^1 - \tau_g^2)(-\kappa_n + v[(\kappa_g^1 - \tau_g^2)(\tau_g^1 - \kappa_g^2) - \kappa_n']) + 2\kappa_n(\kappa_g^1 - \tau_g^2)) \\ &\quad +((\kappa_g^1 - \tau_g^2)(-\kappa_n + v[(\kappa_g^1 - \tau_g^2)(\tau_g^1 - \kappa_g^2) - \kappa_n']) + 2\kappa_n(\kappa_g^1 - \tau_g^2))^2 \\ &\quad -\kappa_n^2(\kappa_g^1 - \tau_g^2)^2\left(\kappa_n^2 + (\kappa_g^1 - \tau_g^2)^2\right) \end{aligned}$$

Conclusion 3.2 Let $\Omega(u, v) = \alpha(u) + v\mathbf{N}_M(u)$ be ruled surface, where $\alpha(u)$ is a unit S- curve in Minkoswki 3- Space. Then Gauss curvature and mean curvature of the surface $\Omega(u, v)$ are

$$K = \frac{(\tau_g^1 - \kappa_g^2)^2}{\left((\tau_g^1 - \kappa_g^2)^2 + (-1 + v((\kappa_g^1 - \tau_g^2))^2\right)^2}[(\tau_g^2 - \kappa_g^1) - (1 + v(\tau_g^2 - \kappa_g^1))]^2,$$

$$\begin{aligned} H &= -\frac{1}{\left((\tau_g^1 - \kappa_g^2)^2 + (-1 + v((\kappa_g^1 - \tau_g^2))^2\right)^{3/2}}[v((\tau_g^1 - \kappa_g^2)(\tau_g^2 - \kappa_g^1)' - \kappa_n(\tau_g^1 - \kappa_g^2)) \\ &\quad -(1 + v(\tau_g^2 - \kappa_g^1))(-\kappa_n(1 + v(\tau_g^2 - \kappa_g^1)) + v(\tau_g^1 - \kappa_g^2)')]. \end{aligned}$$

Conclusion 3.3 Let $\Omega(u, v) = \alpha(u) + v\mathbf{N}_M(u)$ be ruled surface, where $\alpha(u)$ is a unit S- curve in Minkoswki 3- Space. Then principal curvatures of the surface

$\Omega(u,v)$ are

$$\begin{aligned}\kappa_{\max} &= -\frac{1}{\left((\tau_g^1-\kappa_g^2)^2+(-1+v((\kappa_g^1-\tau_g^2))^2\right)^{3/2}}([v((\tau_g^1-\kappa_g^2)(\tau_g^2-\kappa_g^1)'-\kappa_n(\tau_g^1-\kappa_g^2)) \\ &\quad -(1+v(\tau_g^2-\kappa_g^1))(-\kappa_n(1+v(\tau_g^2-\kappa_g^1))+v(\tau_g^1-\kappa_g^2)')]+\sqrt{A^*}, \\ \kappa_{\min} &= -\frac{1}{\left((\tau_g^1-\kappa_g^2)^2+(-1+v((\kappa_g^1-\tau_g^2))^2\right)^{3/2}}([v((\tau_g^1-\kappa_g^2)(\tau_g^2-\kappa_g^1)'-\kappa_n(\tau_g^1-\kappa_g^2)) \\ &\quad -(1+v(\tau_g^2-\kappa_g^1))(-\kappa_n(1+v(\tau_g^2-\kappa_g^1))+v(\tau_g^1-\kappa_g^2)')]-\sqrt{A^*}.\end{aligned}$$

where

$$\begin{aligned}A^* &= [v((\tau_g^1-\kappa_g^2)(\tau_g^2-\kappa_g^1)'-\kappa_n(\tau_g^1-\kappa_g^2))-(1+v(\tau_g^2-\kappa_g^1)) \\ &\quad (-\kappa_n(1+v(\tau_g^2-\kappa_g^1))+v(\tau_g^1-\kappa_g^2)')]^2+(\tau_g^1-\kappa_g^2)^2[(\tau_g^2-\kappa_g^1)-(1+v(\tau_g^2-\kappa_g^1))]^2)\end{aligned}$$

Conclusion 3.4 Let $\Gamma(u,v)=\alpha(u)+v\mathbf{B}_M(u)$ be ruled surface, where $\alpha(u)$ is a unit S- curve in Minkoswki 3- Space. Then Gauss curvature and mean curvature of the surface $\Gamma(u,v)$ are

$$K=0,$$

$$\begin{aligned}H &= \frac{1}{((1-v\kappa_n)-v^2(\tau_g^1-\kappa_g^2))^{3/2}}(v^2(\tau_g^1-\kappa_g^2)^2(\tau_g^2-\kappa_g^1) \\ &\quad +(1-v\kappa_n)^2(\tau_g^2-\kappa_g^1).\end{aligned}$$

Conclusion 3.5 Let $\Gamma(u,v)=\alpha(u)+v\mathbf{N}_M(u)$ be ruled surface, where $\alpha(u)$ is a unit S- curve in Minkoswki 3- Space. Then principal curvatures of the surface $\Gamma(u,v)$ are

$$\begin{aligned}\kappa_{\max} &= \frac{1}{((1-v\kappa_n)-v^2(\tau_g^1-\kappa_g^2))^{3/2}}(v^2(\tau_g^1-\kappa_g^2)^2(\tau_g^2-\kappa_g^1) \\ &\quad +(1-v\kappa_n)^2(\tau_g^2-\kappa_g^1),\end{aligned}$$

$$\kappa_{\min}=0.$$

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TOURISM MANAGEMENT APPLICATION IN PYTHAGOREAN FUZZY SETS WITH COPRAS METHOD

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ABSTRACT
Tourism has always been an important part of economic revenue for growing countries to balance their foreign trade. But with the pandemic, there has been an incredible decline in these incomes. Therefore, governments of growing countries have provided some advantages to tourists for enliven the economy. This situation has led tourists to consider more destinations. Multicriteria decision making methods (MCDM) are very practical tools to select best possible alternatives among many. In this paper, Pythagorean fuzzy COPRAS method is applied to tourism management problem and compared with some aggregation operators.

1. Introduction

For many years, classical sets have been used for real life problems and are still in use. But these problems also contain uncertain information which cannot be expressed by crisp numbers. Zadeh (1965) have paved the way for researchers by defining fuzzy sets which can use uncertain data while expressing the information. Zadeh expanded the characteristic function of classical set to interval of [0,1] and defined it as membership function to measure the belonging of an element in this set. This work has drawn great attention and fuzzy sets have started to be studying extensively. Atanassov (1986) defined intuitionistic fuzzy sets (IFSs) by adding a non-membership function to FSs and limited the sum of these two functions as 1. This approach was very important in terms of information expression since it was the first time that an information was presented as pairs. Yager (2013) extended the values of the membership and the non-membership functions of IFSs while the sum of them stayed in the interval [0,1] and called it Pythagorean fuzzy sets (PFSs). In PFSs sum of squares of membership functions is limited with 1 and thus, the pairs whose sums is greater than 1 can be processed in decision making problems. PFSs are in development (Biswas and Sarkar 2018; Garg 2016, 2017a, 2017b; Khan et al. 2019; Peng and Garg 2019; Peng and Selvachandran 2019; Rahman et al. 2017; Rani, Mishra, and Mardani 2020) and widely applied to real life problems with MCDM methods (Bolturk 2018; Liang et al. 2018; Pérez-Domínguez et al. 2018; Ren, Xu, and Gou 2016; Wu et al. 2019).

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Among MCDM methods, COPRAS (Complex Proportional Assessment) method proposed by Zavadskas et al. (1994) is known with its simplicity and practicability. It evaluates the alternatives considering the minimization and maximization of the criteria. The power of this method comes from the comparing of the alternatives with each other and reveal how good or bad they are from the other alternatives as a percentage. COPRAS method was applied to fuzzy sets by Zavadskas and Antucheviciene (2007) and then it was extended to intuitionistic fuzzy sets by Razavi Hajiagha, et al. (2013). With the definition of the PFS, COPRAS method has recently applied to PFSs (Buyukozkan and Gocer 2019; Dorfeshan and Meysam Mousavi 2019).

Tourism management has become a significant sector in economy due to pandemic conditions. Governments have granted some privileges to visitors that coming their countries. Turkey is one of the best tourism locations in the world and excuse tourists from restrictions such as lockdown and trip. These advantages make Turkey attractive for a travel location. In this work, PF-COPRAS method is simplified from hybrid models and applied to choose the best possible trip location for a tourism management problem which has not been studied before. Then, results are compared with basic aggregation operators to show the effect of PF-COPRAS method.

2. Pythagorean Fuzzy Sets

In this section, PFSs are presented which will be used on farther sections.

Definition 1. (Atanassov 1986) Let X be a non-empty set, then an IFS A in X is defined as

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in X\} \tag{1}$$

where $\mu_A, \nu_A : X \to [0,1]$ represents the degree of membership and the degree of non-membership of the element x such that for any $x \in X$,

$$0 \leq \mu_A(x) + \nu_A(x) \leq 1. \tag{2}$$

Here, $\pi_A = 1 - (\mu_A(x) + \nu_A(x))$ is called hesitancy degree of the element x in the set A. Moreover, the pair of $\left(\mu_A(x), \nu_A(x)\right)$ is called intuitionistic fuzzy number (IFS) and denoted as $a = (\mu_A, \nu_A)$.

Definition 2. (Yager 2013) Let X be a universe of discourse. A PFS P in X is given by

$$P = \{\langle x, \mu_P(x), \nu_P(x)\rangle | x \in X\} \tag{3}$$

where $\mu_P : X \to [0,1]$ represents the degree of membership and $\nu_P : X \to [0,1]$ the degree of non-membership of the element x such that for any $x \in X$,

$$0 \leq \left(\mu_P(x)\right)^2 + \left(\nu_P(x)\right)^2 \leq 1. \tag{4}$$

The degree of indeterminacy is $\pi_P = \sqrt{1 - \left(\left(\mu_P(x)\right)^2 + \left(\nu_P(x)\right)^2\right)}$. For convenience in decision making problems $\mu_P(x), \nu_P(x)$ a Pythagorean fuzzy number (PFN) denoted as $p = (\mu_P, \nu_P)$.

The effect of PFS comes from its corresponding constrain conditions. The information can be stated more in depth. In other words, all IFNs are PFNs but not all PFNs are the IFNs. The representation of the membership functions of IFS and PFS is given in Fig. 1.

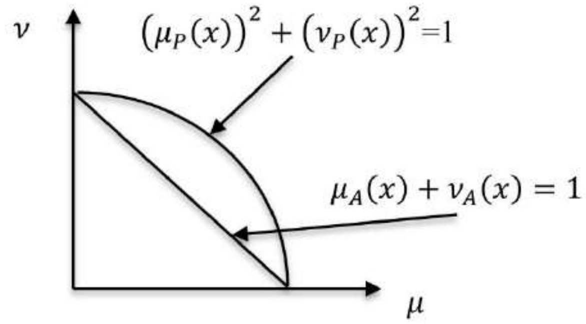

Fig. 1 Comparison of IFS and PFS membership functions.

Definition 3. (Zhang and Xu 2014) Let $p_1 = \left(\mu_{P_1}, \nu_{P_1}\right), p_2 = \left(\mu_{P_2}, \nu_{P_2}\right)$ and $p = (\mu_P, \nu_P)$ be three PFNs. Then, for $\lambda > 0$ the corresponding operations are defined as follows:

$$\begin{aligned}
&\textbf{1)}\quad p_1 \oplus p_2 = \left(\sqrt{\mu_{p_1}^2 + \mu_{p_2}^2 - \mu_{p_1}^2\mu_{p_2}^2}, \nu_{P_1}\nu_{P_2}\right)\\
&\textbf{2)}\quad p_1 \otimes p_2 = \left(\mu_{P_1}\mu_{P_2}, \sqrt{\nu_{p_1}^2 + \nu_{p_2}^2 - \nu_{p_1}^2\nu_{p_2}^2}\right)\\
&\textbf{3)}\quad \lambda p = \left(\sqrt{1 - \left(1 - \mu_p^2\right)^\lambda}, \nu_p^\lambda\right)\\
&\textbf{4)}\quad p^\lambda = \left(\mu_p^\lambda, \sqrt{1 - \left(1 - \nu_p^2\right)^\lambda}\right)
\end{aligned} \tag{5}$$

Definition 4. (Zhang and Xu 2014) For any PFN $p = (\mu_P, \nu_P)$, the score and the accuracy functions of p is defined as

$$s(p) = (\mu_P)^2 - (\nu_P)^2 \text{ and } a(p) = (\mu_P)^2 + (\nu_P)^2 \tag{6}$$

where $s(p) \in [-1,1]$ and $a(p) \in [0,1]$. For any PFNs p_1 and p_2

1. If $s(p_1) > s(p_2)$, then $p_1 > p_2$.
2. If $s(p_1) = s(p_2)$, then
   i. If $a(p_1) > a(p_2) \Rightarrow p_1 > p_2$
   ii. If $a(p_1) = a(p_2)$, then $p_1 \approx p_2$

Definition 5. (Zhang 2016) Let $p_i(i = 1,2, \dots, n)$ be a collection of PFNs, a Pythagorean fuzzy weighted averaging (PFWA) operatoris defined as follows:

$$\text{PFWA}(p_1, p_2, \dots, p_n) = \oplus_{i=1}^{n} (w_i p_i)$$

which can be described as

$$\text{PFWA}(p_1, p_2, \dots, p_n) = \left(\sqrt{1 - \prod_{i=1}^{n} (1 - (\mu_i)^2)^{w_i}}, \prod_{i=1}^{n} (\nu_i)^{w_i} \right) \quad (7)$$

where $w = (w_1, w_2, \dots w_n)^\top$ is the weight vector of p_i with $w_i \in [0,1]$ and $\sum_{i=1}^{n} w_i = 1$.

Definition 6. (Zhang 2016) Let $p_i(i = 1,2, \dots, n)$ be a collection of PFNs, a Pythagorean fuzzy weighted geometric (PFWG) operatoris defined as follows:

$$\text{PFWG}(p_1, p_2, \dots, p_n) = \otimes_{i=1}^{n} ({p_i}^{w_i})$$

which can be written as

$$\text{PFWG}(p_1, p_2, \dots, p_n) = \left(\prod_{i=1}^{n} (\mu_i)^{w_i}, \sqrt{1 - \prod_{i=1}^{n} (1 - (\nu_i)^2)^{w_i}} \right) \quad (8)$$

where $w = (w_1, w_2, \dots w_n)$ is the weight vector of p_i with $w_i \in [0,1]$ and $\sum_{i=1}^{n} w_i = 1$.

Definition 7. (Zhang and Xu 2014) Let A and B be two PFSs, then the distance between these two sets defined as:

$$d(A,B) = \frac{1}{2n}\sum_{i=1}^{n}\left(\left|\mu_A^2(x_i) - \mu_B^2(x_i)\right| + \left|v_A^2(x_i) - v_B^2(x_i)\right| + \left|\pi_A^2(x_i) - \pi_B^2(x_i)\right|\right) \tag{9}$$

3. Pythagorean Fuzzy COPRAS Method

The proposed method is adapted from AHP integrated COPRAS method which was proposed in (Buyukozkan and Gocer 2019).

Let $A = \{A_1, A_2, \ldots, A_m\}$ be a set of alternatives, $C = \{C_1, C_2, \ldots, C_n\}$ be a set of criteria and $w = [w_1, w_2, \ldots, w_n]$ be a weight vector with respect to criteria where $\sum_{j=1}^{n} w_j = 1$ and $w_j \geq 0$. Then, the steps of PFS-based COPRAS method are given as follows:

Step 1. Construct decision making matrix $A = \left(a_{ij}\right)_{m\times n} = \left(\langle \mu_{ij}, v_{ij}\rangle\right)_{m\times n}$:

$$A = \begin{array}{c} \\ A_1 \\ A_2 \\ \ldots \\ A_m \end{array} \begin{array}{c} \begin{array}{cccc} C_1 & C_2 & \ldots & C_n \end{array} \\ \begin{bmatrix} \langle \mu_{11}, v_{11}\rangle & \langle \mu_{12}, v_{12}\rangle & \cdots & \langle \mu_{1n}, v_{1n}\rangle \\ \langle \mu_{21}, v_{21}\rangle & \langle \mu_{22}, v_{22}\rangle & \cdots & \langle \mu_{2n}, v_{2n}\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle \mu_{m1}, v_{m1}\rangle & \langle \mu_{m2}, v_{m2}\rangle & \cdots & \langle \mu_{mn}, v_{mn}\rangle \end{bmatrix} \end{array} \tag{10}$$

Step 2. Obtain weighted normalized decision matrix D:

$$D_{mn} = A_{mn} \otimes w_n = \begin{bmatrix} a_{11}w_1 & \cdots & aw_n \\ \vdots & \ddots & \vdots \\ a_{m1}w_1 & \cdots & a_{mn}w_n \end{bmatrix} \tag{11}$$

where $w_i = (w_1, w_2, \ldots w_n)^{\mathrm{T}}$ is the weight vector and $\sum_{i=1}^{n} w_i = 1$.

Step 3. Determine the sum of criteria values for benefit and cost:

Let $J_1 = \{1,2, \ldots, l\}$ be a benefit criterion set and $J_2 = \{l+1, l+2, \ldots, n\}$ be a cost criterion set, then S_i^+ is the sum of benefit criteria values and S_i^- is the sum of cost criteria values which are formulated as

$$S_i^+ = \oplus_{j=1}^{l} d_{ij}$$
$$S_i^- = \oplus_{j=l+1}^{n} d_{ij} \quad (12)$$

Step 4. Defuzzify S_i^+ and S_i^- using the following equation which is suggested by (Kahraman et al. 2018):

$$Deff(\mu_i, \nu_i) = \frac{\sqrt{\mu_i} - (\nu_i)^2}{2} \quad (13)$$

Step 5. Calculate the degree of relative importance:

$$Q_i = S_i^+ + \frac{S_{-min} \times \sum_{i=1}^{m} \quad S_i^-}{S_i^- \times \sum_{i=1}^{m} \frac{S_{-min}}{S_i^-}}, i = 1,2, \dots, m \quad (14)$$

where S_{min}^- is the minimum value of S_i^-. The bigger Q_i is, the better alternative is.

Step 6. Rank the alternatives with performance index:

$$P_i = \left[\frac{Q_i}{Q_{max}}\right] \times 100\% \quad (15)$$

After the performance of each alternative is determined, alternatives are ranked according to descending order of P_i.

4. Numerical Application and Results

Tourism has become an important part of economic income for countries during the pandemic. Especially, the growing countries need that revenues for keep their foreign trade in balance. Therefore, governments provide convenience to tourists and ease the restrictions for travels. For example, tourists coming to Turkey are exempt from the lockdown. Recently, Turkish government has announced that the CPR test will not be requested from the tourists coming to Turkey. These eases make Turkey attractive for tourists. In the following, we develop a numerical example of tourism management adapted from (Merigó et al. 2012) using PFNs.

Assume that a group of tourists are planning a travel to Turkey. Since Turkey can be considered as tourism attraction centre, there are lots of options to decide for a trip. After a general evaluation for different alternatives, 7 trip locations are chosen for alternatives.

A_1: Ayvalık

A_2: Kuşadası

A_3: Didim

A_4: Bodrum

A_5: Marmaris

A_6: Datça

A_7: Alanya

Each of them suggests criteria to evaluate these alternatives and choose the best option. After each one's wishes is considered thoroughly, 8 criteria are selected for assessment.

C_1: Price of trip

C_2: Tourist activities

C_3: Beaches

C_4: Road length to destination

C_5: Bars and restaurants close to the destination

C_6: Peace and stability.

C_7: Shopping opportunities

C_8: Night entertainments

The weights of criteria are determined by the tourist group as $w = [0.13\ 0.10\ 0.12\ 0.11\ 0.16\ 0.17\ 0.10\ 0.11\ ]$. The decision matrix A is constructed as Table 1 according to their preferences with respect to criteria. The steps of Pythagorean fuzzy COPRAS Method are performed in MATLAB.

Step 1. Decision making matrix $A = \left(a_{ij}\right)_{7\times 8}$ is constructed according to given information by tourists which is given by Table 1:

Table 1. Decision Matrix A

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
A_1	$\langle 0.7, 0.3\rangle$	$\langle 0.3, 0.5\rangle$	$\langle 0.1, 0.9\rangle$	$\langle 0.7, 0.3\rangle$	$\langle 0.6, 0.1\rangle$	$\langle 0.6, 0.6\rangle$	$\langle 0.7, 0.4\rangle$	$\langle 0.1, 0.5\rangle$
A_2	$\langle 0.1, 0.8\rangle$	$\langle 0.6, 0.3\rangle$	$\langle 0.3, 0.8\rangle$	$\langle 0.8, 0.2\rangle$	$\langle 0.6, 0.5\rangle$	$\langle 0.8, 0.3\rangle$	$\langle 0.1, 0.5\rangle$	$\langle 0.7, 0.2\rangle$
A_3	$\langle 0.5, 0.7\rangle$	$\langle 0.9, 0.4\rangle$	$\langle 0.8, 0.3\rangle$	$\langle 0.8, 0.3\rangle$	$\langle 0.3, 0.1\rangle$	$\langle 0.4, 0.5\rangle$	$\langle 0.4, 0.3\rangle$	$\langle 0.9, 0.4\rangle$
A_4	$\langle 0.5, 0.6\rangle$	$\langle 0.5, 0.3\rangle$	$\langle 0.9, 0.1\rangle$	$\langle 0.4, 0.8\rangle$	$\langle 0.7, 0.2\rangle$	$\langle 0.3, 0.6\rangle$	$\langle 0.5, 0.5\rangle$	$\langle 0.4, 0.1\rangle$
A_5	$\langle 0.8, 0.2\rangle$	$\langle 0.4, 0.4\rangle$	$\langle 0.9, 0.1\rangle$	$\langle 0.9, 0.3\rangle$	$\langle 0.3, 0.5\rangle$	$\langle 0.8, 0.3\rangle$	$\langle 0.6, 0.4\rangle$	$\langle 0.2, 0.1\rangle$
A_6	$\langle 0.2, 0.1\rangle$	$\langle 0.6, 0.7\rangle$	$\langle 0.6, 0.1\rangle$	$\langle 0.9, 0.2\rangle$	$\langle 0.4, 0.3\rangle$	$\langle 0.6, 0.7\rangle$	$\langle 0.4, 0.8\rangle$	$\langle 0.4, 0.1\rangle$
A_7	$\langle 0.2, 0.2\rangle$	$\langle 0.2, 0.3\rangle$	$\langle 0.1, 0.7\rangle$	$\langle 0.8, 0.6\rangle$	$\langle 0.6, 0.4\rangle$	$\langle 0.2, 0.7\rangle$	$\langle 0.6, 0.8\rangle$	$\langle 0.2, 0.4\rangle$

Step 2. Weighted normalized decision matrix D is obtained using Eq. (11):

$$D = \begin{bmatrix} \langle 0.29, 0.88\rangle & \langle 0.10, 0.93\rangle & \langle 0.03, 0.99\rangle & \langle 0.27, 0.88\rangle & \langle 0.26, 0.69\rangle & \langle 0.27, 0.92\rangle & \langle 0.26, 0.91\rangle & \langle 0.03, 0.93\rangle \\ \langle 0.04, 0.97\rangle & \langle 0.21, 0.89\rangle & \langle 0.12, 0.97\rangle & \langle 0.33, 0.84\rangle & \langle 0.26, 0.90\rangle & \langle 0.40, 0.81\rangle & \langle 0.03, 0.93\rangle & \langle 0.27, 0.84\rangle \\ \langle 0.19, 0.95\rangle & \langle 0.39, 0.91\rangle & \langle 0.34, 0.87\rangle & \langle 0.33, 0.88\rangle & \langle 0.12, 0.69\rangle & \langle 0.17, 0.89\rangle & \langle 0.13, 0.89\rangle & \langle 0.41, 0.90\rangle \\ \langle 0.19, 0.94\rangle & \langle 0.17, 0.89\rangle & \langle 0.43, 0.76\rangle & \langle 0.14, 0.98\rangle & \langle 0.32, 0.77\rangle & \langle 0.13, 0.92\rangle & \langle 0.17, 0.93\rangle & \langle 0.14, 0.78\rangle \\ \langle 0.35, 0.81\rangle & \langle 0.13, 0.91\rangle & \langle 0.43, 0.76\rangle & \langle 0.41, 0.88\rangle & \langle 0.12, 0.90\rangle & \langle 0.40, 0.81\rangle & \langle 0.21, 0.91\rangle & \langle 0.07, 0.78\rangle \\ \langle 0.07, 0.74\rangle & \langle 0.21, 0.97\rangle & \langle 0.23, 0.76\rangle & \langle 0.41, 0.84\rangle & \langle 0.17, 0.82\rangle & \langle 0.27, 0.94\rangle & \langle 0.13, 0.98\rangle & \langle 0.14, 0.78\rangle \\ \langle 0.07, 0.81\rangle & \langle 0.06, 0.89\rangle & \langle 0.03, 0.96\rangle & \langle 0.33, 0.95\rangle & \langle 0.26, 0.86\rangle & \langle 0.08, 0.94\rangle & \langle 0.21, 0.98\rangle & \langle 0.07, 0.90\rangle \end{bmatrix}$$

Step 3. Sum of criteria values for benefit and cost are determined using Eq. (12):

$$S_i^+ = \begin{bmatrix} \langle 0.4501, 0.4941\rangle \\ \langle 0.5599, 0.4921\rangle \\ \langle 0.6430, 0.3892\rangle \\ \langle 0.5740, 0.3452\rangle \\ \langle 0.6044, 0.3576\rangle \\ \langle 0.4607, 0.4313\rangle \\ \langle 0.3526, 0.6105\rangle \end{bmatrix} \text{ and } S_i^- = \begin{bmatrix} \langle 0.3863, 0.7490\rangle \\ \langle 0.3278, 0.8138\rangle \\ \langle 0.3730, 0.8363\rangle \\ \langle 0.2345, 0.9131\rangle \\ \langle 0.5202, 0.7106\rangle \\ \langle 0.4140, 0.6210\rangle \\ \langle 0.3332, 0.7669\rangle \end{bmatrix}$$

Here, C_1 and C_4 are the cost criteria, others are benefit criteria.

Step 4. S_i^+ and S_i^- are defuzzified using Eq. (13):

$$S_i^+ = [0.2134 \quad 0.2531 \quad 0.3252 \quad 0.3192 \quad 0.3248 \quad 0.2463 \quad 0.1106]$$

$$S_i^- = [0.0302 \quad -0.0449 \quad -0.0443 \quad -0.1747 \quad 0.1081 \quad 0.1289 \quad -0.0054]$$

Step 5. The degree of relative importance is calculated using Eq. (14):

$$Q = [0.2138 \quad 0.2528 \quad 0.3249 \quad 0.3192 \quad 0.3249 \quad 0.2464 \quad 0.1085]$$

Step 6. The alternatives are ranked with performance index using Eq. (15):

$$P = [65.7855 \quad 77.8025 \quad 100.0000 \quad 98.2321 \quad 99.9890 \quad 75.8411 \quad 33.3926]$$

According to the performance index, the alternatives are ordered as

$$A_3 > A_5 > A_4 > A_2 > A_6 > A_1 > A_7.$$

Then, A_3 (Didim) is the best location to travel.

Comparison of results

In order to compare this result with the existing aggregation operators given in Eq. (7) and Eq. (8), an analysis is conducted to calculate results with score function given in Eq. (6). Using tourists' preferences from Table 1, first PFWA operator is used to aggregate the decision matrix and then score function is used to rank the alternatives. Same operations are conducted for PFWG operator. The results are given as follow:

i. If PFWA operator is applied to decision matrix A, aggregated values are evaluated as:

$$PFWA(\mathrm{A}) = \begin{bmatrix} \langle 0.5671, 0.3701 \rangle \\ \langle 0.6223, 0.4005 \rangle \\ \langle 0.7036, 0.3255 \rangle \\ \langle 0.6053, 0.3152 \rangle \\ \langle 0.7328, 0.2541 \rangle \\ \langle 0.5893, 0.2679 \rangle \\ \langle 0.4707, 0.4682 \rangle \end{bmatrix}$$

Then, the score values are obtained as:

$s(PFWA(A)) = [0.1846 \quad 0.2269 \quad 0.3891 \quad 0.2671 \quad 0.4725 \quad 0.2755 \quad 0.0024]$

According to score values, ranking of the alternatives are ordered as

$A_5 > A_3 > A_6 > A_4 > A_2 > A_1 > A_7$

Then, A_5 (Marmaris) is the best location to travel.

ii. If PFWG operator is applied to decision matrix A, aggregated values are evaluated as:

$$PFWG(\mathrm{A}) = \begin{bmatrix} \langle 0.3907, 0.5606 \rangle \\ \langle 0.4031, 0.5528 \rangle \\ \langle 0.5468, 0.4313 \rangle \\ \langle 0.4943, 0.5042 \rangle \\ \langle 0.5469, 0.3295 \rangle \\ \langle 0.4681, 0.5143 \rangle \\ \langle 0.2852, 0.5790 \rangle \end{bmatrix}$$

Then, the score values are obtained as:

$$s(PFWG(A)) = [-0.1616 \quad -0.1431 \quad 0.1130 \quad -0.0099 \quad 0.1905 \quad -0.0454 \quad -0.2539]$$

According to score values, ranking of the alternatives are ordered as

$A_5 > A_3 > A_4 > A_6 > A_2 > A_1 > A_7$

Then, A_5 (Marmaris) is the best location to travel.

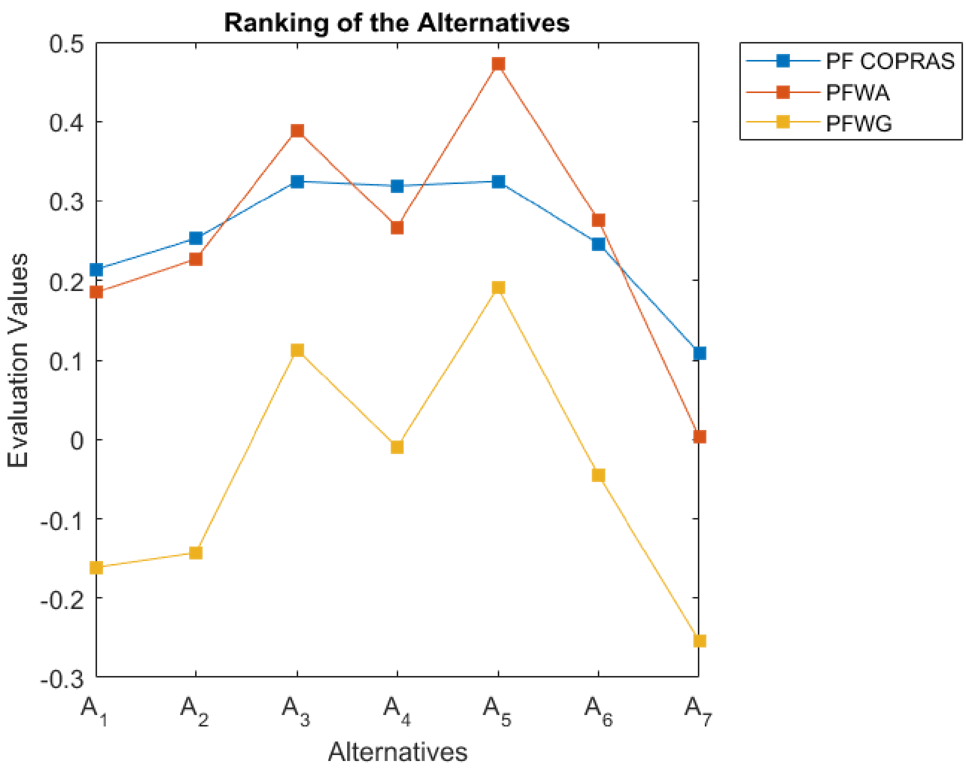

Figure 1. Ranking comparison of the alternatives with COPRAS, PFWA and PFWG.

Table 2. Ranking comparison

Methods	**Rankings**
PF COPRAS	$A_3 > A_5 > A_4 > A_2 > A_6 > A_1 > A_7$
PFWA Operator	$A_5 > A_3 > A_6 > A_4 > A_2 > A_1 > A_7$
PFWG Operator	$A_5 > A_3 > A_4 > A_6 > A_2 > A_1 > A_7$

As can be seen in the Figure 1 and Table 2, PF COPRAS method affect the results. Although A_5 is the best option for aggregation operators, A_3 is the best option for PF COPRAS method. MCDM method have a great effect on selecting alternatives when compared to aggregation operators.

5. Conclusion

In this study, Pythagorean fuzzy COPRAS method is applied to a decision-making problem considering all countries tourism incomes worldwide. The tourism management area is picked for numerical example and the selection of the best trip option problem is examined. Seven travel locations are selected with respect to eight criteria. First, a decision matrix is created in accordance with a group of tourists' requests and the importance of each criteria is identified. Then, the best possible trip location is acquired with PF COPRAS method. To compare results, same decision matrix is aggregated with PFWA and PFWG operators. Then alternatives are ranked with score function. The results obtained with aggregation operators are quite similar while the best option changes with PF COPRAS method. These results show the power of the MCDM methods.

In future studies, this problem can be adapted to other MCDM methods such as TOPSIS, TODIM, etc. Furthermore, picture fuzzy set, spherical fuzzy set and neutrosophic set extensions of these MCDM methods can also be applied to this kind of problems.

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ON PARTIAL DERIVATIVES OF SPLIT TRIPLET FUNCTIONS

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Abstract. Quaternions and split quaternions are not commutative by multiplication. A split triplet is obtained when the coefficient of one element of the vector part of split quaternions is zero. In some special cases, triplets are commutative. In this study, partial derivatives of split triplet functions are obtained.

1. Introduction

A real quaternion Q is defined by

$$Q = a + bi + cj + dk$$

where w, x, y, z are reel numbers and

$$i^2 = j^2 = k^2 = ijk = -1$$

$$\begin{aligned} ij &= k, \ jk = i, \ ki = j, \\ ji &= -k, \ kj = -i, \ ik = -j. \end{aligned}$$

The conjugate of a real quaternion Q

$$\bar{Q} = a - bi - cj - dk$$

and the norm of Q is

$$\begin{aligned} |Q| &= \sqrt{|Q|^2} \\ &= \sqrt{Q\overline{Q}} \\ &= \sqrt{a^2 + b^2 + c^2 + d^2}. \end{aligned}$$

The set of quaternions is denoted by H [1].

If one of the coefficients of i, j or k is zero, then quaternion Q is defined as a triplet. The triplets are components of three-dimensional space. They can be

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obtained from quaternions which are four-dimensional space components [5].

A split quaternion q is defined by

$$q = t + xi + yj + zk$$

where t, x, y, z are reel numbers and

$$i^2 = -1,\ j^2 = k^2 = ijk = 1$$
$$ij = k,\ jk = -i,\ ki = j.$$

The set of split quaternions is denoted by $\hat{H}$ [2]. Similarly, the norm of q is

$$\begin{aligned} |q| &= \sqrt{|q|^2} \\ &= \sqrt{q\overline{q}} \\ &= \sqrt{w^2 + x^2 - y^2 - z^2}. \end{aligned}$$

If one of the coefficients of i, j or k is zero, then quaternion q is defined as a split triplet. If the coefficient of k is zero than $q = t + xi + yj + 0.k$ is a triplet. The split triplets are components of three-dimensional Lorentzian space. They can be obtained from split quaternions which are four-dimensional Lorentzian space components.

2. Preliminaries

Consider the split quaternionic function $f = f_1 + if_2 + jf_3 + kf_4$, whose components are real valued functions. If one of the coefficients of i, j or k is zero, then quaternion f is defined as a triplet function. Let's coefficient of k is zero. Then, $f = f_1 + if_2 + jf_3 + k.0$ is a triplet function.

We can give the definition of derivative that

$$f'(q) = \frac{df}{dq} = \lim_{\Delta q \to 0} [f(q + \Delta q) - f(q)](\Delta q)^{(-1)}$$

where $q = t + xi + yj + 0k$ is a triplet. Then, $f(q) = f_1(q) + if_2(q) + jf_3(q) + k.0$.

In complex numbers algebra,

$$df/dz = \begin{bmatrix} \partial f_1/\partial x & \partial f_2/\partial x \\ \partial f_1/\partial y & \partial f_2/\partial y \end{bmatrix}$$

where $z = x + iy$ complex number and $f = f_1 + if_2$ complex function. So, $f(z) = f_1(z) + if_2(z)$ is wrtten.

$$\begin{aligned} \frac{\partial f}{\partial x} &= \frac{\partial f}{\partial z}\frac{\partial z}{\partial x} = \frac{\partial f}{\partial z}1 = f\ '(z) \\ &\Longrightarrow\ f\ '(z) = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial x}i \end{aligned}$$

$$\begin{aligned} \frac{\partial f}{\partial y} &= \frac{\partial f}{\partial z}\frac{\partial z}{\partial y} = \frac{\partial f}{\partial z}i = f\ '(z)i \\ &\Longrightarrow\ f\ '(z) = -\frac{\partial f}{\partial y}i = \frac{\partial f_2}{\partial y} - \frac{\partial f_1}{\partial y}i \end{aligned}$$

Real parts and the coefficient of i are equal. Also,

$$T_z = \begin{bmatrix} x & y \\ -y & x \end{bmatrix}$$

is the matrix representation of z complex number and

$$\frac{\partial f}{\partial x} = \frac{\partial f_1}{\partial x} + i\frac{\partial f_2}{\partial x}$$

is complex derivative. We can write that

$$\begin{aligned} T_{z'} &= \begin{bmatrix} \partial f_1/\partial x & \partial f_2/\partial x \\ -\partial f_2/\partial x & \partial f_1/\partial x \end{bmatrix} \\ &= \begin{bmatrix} \partial f_1/\partial x & \partial f_2/\partial x \\ \partial f_1/\partial y & \partial f_2/\partial y \end{bmatrix} \end{aligned}$$

by considering the matrix representation of z. Here,

$$\partial f_1/\partial x = \partial f_2/\partial y,\ \partial f_2/\partial x = -\partial f_1/\partial y$$

are Cauchy-Riemann terms [4].

3. Partial Derivatives of Split Triplet Functions

We can give the following theorem similar to the case with complex numbers and by considering the theorem given in [6].

Theorem 3.1. *We can write that*

$$\frac{df}{dq} = \begin{bmatrix} \partial f_1/\partial t & \partial f_2/\partial t & \partial f_3/\partial t & 0 \\ \partial f_1/\partial x & \partial f_2/\partial x & 0 & 0 \\ \partial f_1/\partial y & 0 & \partial f_3/\partial y & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$f = f_1 + if_2 + jf_3 + k.0$ is a split triplet function whose components are real valued functions and $q = t + ix + jy + k.0$ is a split triplet.

Proof. We can write

$$\begin{aligned} \frac{\partial f}{\partial t} &= \frac{\partial f}{\partial q}\frac{\partial q}{\partial t} = \frac{\partial f}{\partial q}1 = f'(q) \\ &\Longrightarrow f'(q) = \frac{\partial f_1}{\partial t} + i\frac{\partial f_2}{\partial t} + j\frac{\partial f_3}{\partial t} + k.0 \end{aligned}$$

$$\begin{aligned} \frac{\partial f}{\partial x} &= \frac{\partial f}{\partial q}\frac{\partial q}{\partial x} = \frac{\partial f}{\partial q}i = f'(q)i \\ &\Longrightarrow f'(q) = \frac{-\partial f}{\partial x}i = \frac{\partial f_2}{\partial x} - i\frac{\partial f_1}{\partial x} - j.0 + k\frac{\partial f_3}{\partial x} \end{aligned}$$

$$\begin{aligned} \frac{\partial f}{\partial y} &= \frac{\partial f}{\partial q}\frac{\partial q}{\partial y} = \frac{\partial f}{\partial q}j = f'(q)j \\ &\Longrightarrow f'(q) = \frac{\partial f}{\partial y}j = \frac{\partial f_3}{\partial y} + i.0 + j\frac{\partial f_1}{\partial y} + k\frac{\partial f_2}{\partial y} \end{aligned}$$

equations. Here, coefficients are equal. Also,

$$T_q = \begin{bmatrix} t & x & y & 0 \\ -x & t & 0 & -y \\ y & 0 & t & x \\ 0 & -y & -x & t \end{bmatrix}$$

is the matrix representation of split triplet and

$$\frac{\partial f}{\partial t} = \frac{\partial f_1}{\partial t} + i\frac{\partial f_2}{\partial t} + j\frac{\partial f_3}{\partial t} + k.0$$

is split triplet derivative. We can write that

$$\begin{aligned} T_{f\ '} &= \begin{bmatrix} \partial f_1/\partial t & \partial f_2/\partial t & \partial f_3/\partial t & 0 \\ -\partial f_2/\partial t & \partial f_1/\partial t & 0 & -\partial f_3/\partial t \\ \partial f_3/\partial t & 0 & \partial f_1/\partial t & \partial f_2/\partial t \\ 0 & -\partial f_3/\partial t & -\partial f_2/\partial t & \partial f_1/\partial t \end{bmatrix} \\ &= \begin{bmatrix} \partial f_1/\partial t & \partial f_2/\partial t & \partial f_3/\partial t & 0 \\ \partial f_1/\partial x & \partial f_2/\partial x & 0 & 0 \\ \partial f_1/\partial y & 0 & \partial f_3/\partial y & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

(See [3] for similar operations). Thus, proof is complete. □

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ON SPLIT TRIPLET AND GRADIENT

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Abstract. A triplet is the special case of a quaternion. Likewise, a split triplet is the special case of a split quaternion. In general, they are not commutative according to the multiplication process. In this paper, the gradient of split triplet functions are obtained.

1. Introduction

A real quaternion Q is defined by

$$Q = a + bi + cj + dk$$

where w, x, y, z are reel numbers and

$$i^2 = j^2 = k^2 = ijk = -1$$

$$\begin{aligned} ij &= k,\ jk = i,\ ki = j, \\ ji &= -k,\ kj = -i,\ ik = -j. \end{aligned}$$

The norm of a real quaternion q is

$$|Q|^2 = Q\overline{Q} = a^2 + b^2 + c^2 + d^2.$$

The set of quaternions is denoted by H [1].

If one of the coefficients of i, j or k is zero, then quaternion Q is defined as a triplet. The triplets are components of three-dimensional space. They can be obtained from quaternions which are four-dimensional space components [5].

A split quaternion q is defined by

$$q = t + xi + yj + zk$$

where t, x, y and z are reel numbers and

$$i^2 = -1,\ j^2 = k^2 = ijk = 1$$

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$$ij = k, \; jk = -i, \; ki = j.$$

The set of split quaternions is denoted by $\hat{H}$ [2].

If one of the coefficients of i, j or k is zero, then quaternion q is defined as a split triplet. If the coefficient of k is zero than $q = t + xi + yj + 0.k$ is a triplet. The split triplets are components of three-dimensional Lorentzian space. They can be obtained from split quaternions which are four-dimensional Lorentzian space components.

2. Preliminaries

For split quaternions q_1 and q_2

$$q_1 = \mu q_2 \mu^{-1}$$

considering that the split quaternions q_1 and q_2 are similar if there is at least one μ split quaternion satisfying the equation. We can apply this feature for split triplet, which is the special case of split quaternion. Similar calculates are in [3] for split quaternion. Hence,

$$\begin{aligned}
q^i &= -iqi = -i(t + ix + jy + 0z)i \\
&= -i(ti - x - ky + 0z) \\
&= t + ix - jy \\
q^j &= -jqj = -j(t + ix + jy + 0z)j \\
&= -j(tj + kx + y + 0z) \\
&= -t + ix - jy \\
q^k &= -kqk = -k(t + ix + jy + 0z)k \\
&= -k(tk - jx - iy + 0z) \\
&= -t + ix + jy
\end{aligned}$$

involutions are obtained. Then, it is written

$$\begin{aligned}
q &= t + ix + jy \\
q^i &= t + ix - jy \\
q^j &= -t + ix - jy \\
q^k &= -t + ix + jy
\end{aligned}$$

equation system. So,

$$\begin{aligned}
t &= \frac{1}{4}(q + q^i - q^j - q^k) \\
x &= \frac{1}{4i}(q + q^i + q^j + q^k) \\
y &= \frac{1}{4j}(q - q^i - q^j + q^k)
\end{aligned}$$

are obtained. Hence,

$$\begin{aligned} dt &= \frac{1}{4}(dq + dq^i - dq^j - dq^k) \\ dx &= \frac{-i}{4}(dq + dq^i + dq^j + dq^k) \\ dy &= \frac{j}{4}(dq - dq^i - dq^j + dq^k) \end{aligned}$$

are written.

3. The Gradient of Split Triplet Functions

Now let's replace these values in partial derivatives of the function f. Using these values in the partial derivatives of the function f,

$$\begin{aligned} \frac{df}{dq} &= \frac{\partial f}{\partial t}\frac{\partial t}{\partial q} + \frac{\partial f}{\partial x}\frac{\partial x}{\partial q} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial q} \\ &= \frac{\partial f}{\partial t}\frac{1}{4} + \frac{\partial f}{\partial x}\frac{(-i)}{4} + \frac{\partial f}{\partial y}\frac{j}{4} \\ &= \frac{1}{4}(\frac{\partial f}{\partial t} - i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y}) \\ \frac{df}{dq^i} &= \frac{\partial f}{\partial t}\frac{\partial t}{\partial q^i} + \frac{\partial f}{\partial x}\frac{\partial x}{\partial q^i} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial q^i} \\ &= \frac{\partial f}{\partial t}\frac{1}{4} + \frac{\partial f}{\partial x}\frac{1}{4i} + \frac{\partial f}{\partial y}\frac{(-1)}{4j} \\ &= \frac{1}{4}(\frac{\partial f}{\partial t} - i\frac{\partial f}{\partial x} - j\frac{\partial f}{\partial y}) \\ \frac{df}{dq^j} &= \frac{\partial f}{\partial t}\frac{\partial t}{\partial q^j} + \frac{\partial f}{\partial x}\frac{\partial x}{\partial q^j} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial q^j} \\ &= \frac{\partial f}{\partial t}\frac{1}{4} + \frac{\partial f}{\partial x}\frac{1}{4i} + \frac{\partial f}{\partial y}\frac{(-1)}{4j} \\ &= \frac{1}{4}(-\frac{\partial f}{\partial t} - i\frac{\partial f}{\partial x} - j\frac{\partial f}{\partial y}) \\ \frac{df}{dq^k} &= \frac{\partial f}{\partial t}\frac{\partial t}{\partial q^k} + \frac{\partial f}{\partial x}\frac{\partial x}{\partial q^k} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial q^k} \\ &= \frac{\partial f}{\partial t}\frac{(-1)}{4} + \frac{\partial f}{\partial x}\frac{1}{4i} + \frac{\partial f}{\partial y}\frac{1}{4j} \\ &= \frac{1}{4}(-\frac{\partial f}{\partial t} - i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y}) \end{aligned}$$

equations can be written.It is obtained that

$$\begin{bmatrix} \frac{\partial f(q,q^i,q^j,q^k)}{\partial q} \\ \frac{\partial f(q,q^i,q^j,q^k)}{\partial q^i} \\ \frac{\partial f(q,q^i,q^j,q^k)}{\partial q^j} \\ \frac{\partial f(q,q^i,q^j,q^k)}{\partial q^k} \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 1 & -i & j \\ 1 & -i & -j \\ -1 & -i & -j \\ -1 & -i & j \end{bmatrix}\begin{bmatrix} \frac{df}{dt} \\ \frac{df}{dx} \\ \frac{df}{dy} \end{bmatrix}$$

in matrix form. It can be written that

$$\begin{bmatrix} \frac{\partial f(q,q^i,q^j,q^k)}{\partial q^*} \\ \frac{\partial f(q,q^i,q^j,q^k)}{\partial q^{i^*}} \\ \frac{\partial f(q,q^i,q^j,q^k)}{\partial q^{j^*}} \\ \frac{\partial f(q,q^i,q^j,q^k)}{\partial q^{k^*}} \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & i & -j \\ 1 & i & j \\ -1 & i & j \\ -1 & i & -j \end{bmatrix} \begin{bmatrix} \frac{df}{dt} \\ \frac{df}{dx} \\ \frac{df}{dy} \end{bmatrix}$$

for conjugate. Here,

$$\nabla f = \begin{bmatrix} \frac{df}{dt} \\ \frac{df}{dx} \\ \frac{df}{dy} \end{bmatrix}$$

is the gradient of f.

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