5TH IFSCOM2018 PROCEEDING BOOK
ISBN: 978-605-68670-0-2

IFSCOM2018
5TH IFS AND CONTEMPORARY MATHEMATICS CONFERENCE
SEPTEMBER, 05-09, 2018, KAHRAMANMARAS, TURKEY
pp: 1-7

# SPHERICAL PRODUCT SURFACE WITH HARMONIC GAUSS MAP IN GALILEAN 3-SPACE $\mathbb{G}_3$

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ABSTRACT. In this study, we handle spherical product surface having harmonic Gauss map in Galilean 3-space $\mathbb{G}_3$. We calculate the Laplacian operator of the Gauss map. Then, we give the necessary and sufficient conditions for spherical product surface to have harmonic Gauss map in $\mathbb{G}_3$.

## 1. INTRODUCTION

The term of finite type immersions is presented by Chen, and then the same author writes some papers related to this topic [15, 16]. If a submanifold $M$ is given in Euclidean $m-$space $\mathbb{E}^m$, and if an isometric immersion $x : M \rightarrow \mathbb{E}^m$, also known as the position vector field of $M$, is written as a finite sum of eigenvectors of the Laplacian $\Delta$ of $M$ for a constant map $x_0$, and non-constant maps $x_1, x_2, ..., x_k$, i.e.,

$$x = x_0 + \sum_{i=1}^{k} x_i,$$

then $x$ is called as a finite type. Here, $\Delta x = \lambda_i x_i$, $\lambda_i \in \mathbb{R}$, $1 \leq i \leq k$. If the numbers $\lambda_i$'s are different from each other, then the submanifold is called as $k-$type [14].

This term is extended to the Gauss map of $M$ as

$$\Delta G = a(G + C) \tag{1.1}$$

for a real number $a$ and a constant vector $C$ by Chen and Piccinni in [17]. In this respect, a submanifold satisfying (1.1) is said to have 1-type Gauss map $G$. Then many papers have been written about submanifolds having 1-type Gauss map $G$ [8, 9, 10].

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2010 *Mathematics Subject Classification.* Primary 53A05; Secondary 53A10.
*Key words and phrases.* Spherical product surface, harmonic Gauss map, Galilean 3-space.

Afterwards, in (1.1), the real number $a$ is replaced with a non-constant function $\lambda$. That is, the equation (1.1) becomes

$$\Delta G = \lambda(G + C). \tag{1.2}$$

A submanifold satisfying (1.2) is said to have pointwise 1-type Gauss map $G$. If the function $\lambda$ is non-constant, the pointwise 1-type Gauss map is called as proper. Also, if the vector $C$ is zero, the pointwise 1-type Gauss map is called as the first kind. Otherwise, second kind [18].

Surfaces satisfying (1.2) have been the subject of many studies such as [2, 3, 4, 19, 21, 24, 28]. In recent years, authors deal with the meridian surfaces with pointwise 1-type Gauss map in some spaces in [5, 6]. Also, authors study the tubular surfaces with pointwise 1-type Gauss map in Euclidean 4-space in [25, 26].

Recall that if $\Delta G = 0$, then $M$ is said to have harmonic Gauss map.

Kuiper defined the embeddings of product spaces and he give a new embedding in the $(m + n + d)$-dimensional Euclidean space $\mathbb{E}^{m+n+d}$ as here in below. Let $f : M \to \mathbb{E}^{m+d}$ be an embedding which is defined from an $m$-dimensional manifold $M$ into an $(m + d)$-dimensional Eulidean space $\mathbb{E}^{m+d}$, and $g : \mathbb{S}^n \to \mathbb{E}^{n+1}$ be a standard embedding defined from $n$-sphere into $\mathbb{E}^{n+1}$. In that case the new embedding is given as

$$\begin{aligned} X &= f \otimes g : M \times \mathbb{S}^n \to \mathbb{E}^{m+n+d} \\ (u, v) &\to X(u, v) = (f_1(u), f_2(u), ..., f_{m+d-1}(u), f_{m+d}(u)g(v)), \end{aligned} \tag{1.3}$$

where $u \in M$ and $v \in \mathbb{S}^n$. These types embeddings are taken from the embedding $f$ by rotating $\mathbb{E}^n$ about $\mathbb{E}^{m+d-1}$ in $\mathbb{E}^{m+n+d}$ [27].

In [11, 12], authors called these types embedding as rotational embeddings. Then they study special case of the rotational embeddings, spherical product surfaces, in Euclidean spaces by taking $m = 1$, $n = 1$ and $d = 1, 2$ in (1.3) respectively. Also, spherical product surface is studied in Galilean 3-space $\mathbb{G}_3$ in [7].

In the 19th century by C.F. Gauss, N.I. Lobachevsky and J. Bolyai, non-Euclidean geometries were set forth with the discovery of hyperbolic geometry, which accepts a new postulate (infinite number of parallels can be drawn to a line from a point outside the given line) instead of parallel postulate. G.F.B. Riemann laid the foundations of a new geometry called the elliptic geometry afterwards. Those geometries were generalized by F. Klein, and Euclid presented the existence of the nine geometries including the hyperbolic and elliptic ones [31]. Galilean geometry is a non-Euclidean geometry and associated with Galilei principle of relativity. This principle can be explained briefy as " in all inertial frames, all law of physics are the same." (Except for the Euclidean geometry in some cases), Galilean geometry is the easiest of all Klein geometries, and it is revelant to the theory of relativity of Galileo and Einstein. One can have a look at the studies of Yaglom [33] and Röschel [30] for Galilean geometry. Recently, many works related to Galilean geometry have been done by several authors in [13, 20, 22, 32].

## 2. Preliminaries

In this section, we give some basic properties of the Galilean space, curves and surfaces in Galilean space $\mathbf{G}_3$. Detailed informations can be found in [23]. The scalar product and cross product of two vectors $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$ in $\mathbb{G}_3$ are respectively defined as

$$\langle a, b\rangle = \begin{cases} a_1 b_1, & if a_1 \neq 0 \vee b_1 \neq 0 \\ a_2 b_2 + a_3 b_3 & if a_1 = 0 \wedge b_1 = 0, \end{cases} \tag{2.1}$$

$$a \times b = \begin{cases} \begin{vmatrix} 0 & e_2 & e_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}, & if a_1 \neq 0 \vee b_1 \neq 0 \\ \\ \begin{vmatrix} e_1 & e_2 & e_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}, & if a_1 = 0 \wedge b_1 = 0 \end{cases} \tag{2.2}$$

Also the length of the vector $a = (a_1, a_2, a_3)$ is given by

$$\|a\| = \begin{cases} |a_1|, & if a_1 \neq 0 \\ \sqrt{a_2^2 + a_3^2}, & if a_1 = 0 \end{cases} \tag{2.3}$$

[29].

A curve $\alpha : I \subset \mathbb{R} \to \mathbb{G}_3$ parametrized by Galilean invariant parameter (the arc-length on $\alpha$) is given as

$$\alpha(u) = (u, y(u), z(u)), \tag{2.4}$$

the curvature $\kappa(u)$ and the torsion $\tau(u)$ are defined by

$$\kappa(u) = \sqrt{y''^2(u) + z''^2(u)} \tag{2.5}$$

and

$$\tau(u) = \det \frac{(\alpha'(u), \alpha''(u), \alpha'''(u))}{\kappa^2(u)}. \tag{2.6}$$

The associated moving trihedron is given by

$$\begin{aligned} T(u) &= \alpha'(u) = (1, y'(u), z'(u)) \\ N(u) &= \frac{\alpha''(u)}{\kappa(u)} = \frac{1}{\kappa(u)}(0, y''(u), z''(u)) \\ B(u) &= (T \times N)(u) = \frac{1}{\kappa(u)}(0, -z''(u), y''(u)). \end{aligned} \tag{2.7}$$

The vectors $T, N$ and $B$ are called the vectors of the tangent, principal normal and the binormal line, respectively. Then, Frenet's formulas hold

$$\begin{aligned} T' &= \kappa N \\ N' &= \tau B \\ B' &= -\tau N \end{aligned} \tag{2.8}$$

[29]. If $\tau = 0$, $\alpha$ is called a plane curve in the Galilean space $\mathbb{G}_3$ [1].

Let $M$ be a surface parametrized by

$$X(u, v) = (x_1(u, v), x_2(u, v), x_3(u, v)) \tag{2.9}$$

in $\mathbb{G}_3$. If the surface $M$ does not have Euclidean tangent planes, then it is admissible. Therefore $M$ is admissible if and only if $\frac{\partial x_1(u,v)}{\partial u} \neq 0$ and $\frac{\partial x_1(u,v)}{\partial v} \neq 0$.

The first fundamental form of the surface $M$ is defined as

$$I = g_{11}d_u^2 + 2g_{12}d_u d_v + g_{22}d_v^2, \tag{2.10}$$

where $g_{11} = \langle X_u, X_u \rangle$ , $g_{12} = \langle X_u, X_v \rangle$ , and $g_{22} = \langle X_v, X_v \rangle$. For the local coordinate system $\{u_1 = u, u_2 = v\}$ of $M$, we denote the matrix (resp. the determinant) of the first fundamental form with $g = g_{ij}$ $(i, j = 1, 2)$, ( resp. $|g|$), and the inverse matrix with $g^{-1} = g^{ij}$. Then, the Laplasian operator $\Delta$ of the first fundamental form $I$ on $M$ is given by

$$\Delta = \frac{1}{\sqrt{|g|}} \sum_{i,j=1}^{2} \frac{\partial}{\partial u_i} \left( \sqrt{|g|} g^{ij} \frac{\partial}{\partial u_j} \right). \tag{2.11}$$

Now, we give the spherical product surface which is introduced in [7] in Galilean 3-space $\mathbb{G}_3$.
Let $\alpha(u) = (u, p(u))$ and $\beta(v) = (v, q(v))$ be unit speed plane curves in Galilean 3-space $\mathbb{G}_3$ for the smooth real-valued functions $p$ and $q$. Then the sperical product patch is given as

$$X = \alpha \otimes \beta : \mathbb{G}_2 \to \mathbb{G}_3,\ X(u, v) = (u, p(u)v, p(u)q(v)). \tag{2.12}$$

Here, the equation (2.12) indicates a surface which is called as spherical product surface in $\mathbb{G}_3$.

**Main Theorem.** *Let $M$ be a spherical product surface given with (2.12) in $\mathbb{G}_3$. $M$ has harmonic Gauss map if and only if $M$ has the following parametrization:*

$$X(u, v) = (u, p(u)v, p(u)(av + b)), \tag{2.13}$$

*where $a$ and $b$ are real constants.*

## 3. Proof of Theorem

Let $M$ be a spherical product surface given with (2.12) in $\mathbb{G}_3$. Tangent space of $M$ is spanned by the tangent vectors

$$\begin{aligned} X_u &= (1, p'(u)v, p'(u)q(v)), \\ X_v &= (0, p(u), p(u)q'(v)). \end{aligned} \tag{3.1}$$

Then from (3.1), the coefficients of the first fundamental form and the Gauss map $G$ of $M$ are as follow:

$$\begin{aligned} g_{11} &= \langle X_u, X_u \rangle = 1, \\ g_{12} &= \langle X_u, X_v \rangle = 0, \\ g_{22} &= \langle X_v, X_v \rangle = p^2 \left(1 + (q')^2\right), \end{aligned} \tag{3.2}$$

and

$$G = \frac{(0, -q', 1)}{\sqrt{1 + (q')^2}} \tag{3.3}$$

[7]. Using (3.2), we get the following matrix (resp. inverse matrix) of the first fundamental form

$$g = g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & p^2\left(1 + (q')^2\right) \end{pmatrix}, \quad g^{-1} = g^{ij} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{p^2\left(1+(q')^2\right)} \end{pmatrix}, \tag{3.4}$$

where $|g| = p^2\left(1+(q')^2\right)$. From (2.11) and (3.4), we calculate the Laplacian operator $\Delta$ of $M$ as

$$\Delta = -\frac{p'}{|p|}\frac{\partial}{\partial u} - \frac{\partial^2}{\partial u^2} + \frac{qq'}{p^2\left(1+(q')^2\right)^2}\frac{\partial}{\partial v} - \frac{1}{p^2\left(1+(q')^2\right)}\frac{\partial^2}{\partial v^2}. \tag{3.5}$$

Thus, the Laplacian of the Gauss map is written as

$$\Delta G = \frac{qq'}{p^2\left(1+(q')^2\right)^2}\frac{\partial}{\partial v}\frac{(0,-q',1)}{\sqrt{1+(q')^2}} - \frac{1}{p^2\left(1+(q')^2\right)}\frac{\partial^2}{\partial v^2}\frac{(0,-q',1)}{\sqrt{1+(q')^2}}. \tag{3.6}$$

Differentiating the Gauss map with respect to $v$ two times and and substituting them in (3.6), we obtain the Laplacian of the Gauss map is

(3.7)

$$\Delta G = \frac{1}{p^2\left(1+(q')^2\right)^{\frac{7}{2}}}\left(0, -4q'\left(q''\right)^2 + q''' + \left(q'\right)^2 q''', -3\left(q'q''\right)^2 + \left(q''\right)^2 + q'q''' + \left(q'\right)^3 q'''\right).$$

Suppose that $M$ has harmonic Gauss map. Then, the vector given with (3.7) is zero. Thus, we have two equalities

$$\begin{aligned} -4q'\left(q''\right)^2 + q''' + \left(q'\right)^2 q''' &= 0, \\ -3\left(q'q''\right)^2 + \left(q''\right)^2 + q'q''' + \left(q'\right)^3 q''' &= 0. \end{aligned} \tag{3.8}$$

From the first equation of (3.8), we have

$$4q'\left(q''\right)^2 = q'''\left(1+(q')^2\right). \tag{3.9}$$

Susbtituting (3.9) in the second equation of (3.8), we get

$$\left(q''\right)^2\left(1+(q')^2\right) = 0,$$

which means $q'' = 0$, i.e. $q$ is a linear function as $q(v) = av + b$ for the real numbers $a$ and $b$.

**Example.** *Let us consider the spherical product surface having harmonic Gauss map given with (2.13) in Galilean 3-space $\mathbb{G}_3$. Taking the functions $p(u) = cosu$ and $q(v) = 3v + 4$, we plot the graph of the surface*

$$X(u,v) = (u, (cosu)v, cosu\,(3v+4)) \tag{3.10}$$

*via the maple plotting command plot3d($[x_1, x_2, x_3], u = a..b, v = c..d.$)*

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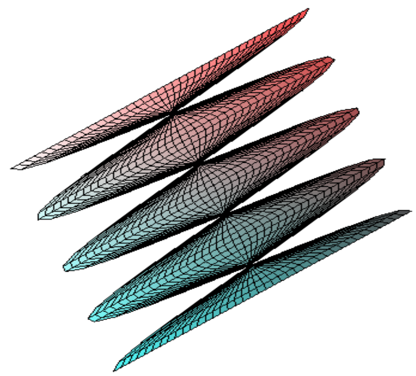

Figure 1. Spherical product surface having harmonic Gauss map with $p(u) = cosu$.


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