5th IFSCOM2018 PROCEEDING BOOK
ISBN: 978-605-68670-0-2

IFSCOM2018
5th Ifs And Contemporary Mathematics Conference
September, 05-09, 2018, Kahramanmaras, Turkey
pp:98-105

DETERMINANTS AND PERMANENTS OF HESSENBERG MATRICES WITH (s,t)-JACOBSTHAL AND (s,t)-JACOBSTHAL LUCAS SEQUENCES

YASEMIN TAŞYURDU

Abstract. In this study, we consider Hessenberg matrices with applications to (s,t)-Jacobsthal and (s,t)-Jacobsthal Lucas sequences. We define some Hessenberg matrices and obtain determinants and permanents of these Hessenberg matrices that give terms of (s,t)-Jacobsthal and (s,t)-Jacobsthal Lucas sequences. Also, we investigate the relationships between these sequences, and permanents and determinants of these matrices.

1. Introduction

There are so many article in the literature that concern about the number sequences. The well-known Fibonacci, Lucas, Pell, Jacobsthal number sequences contribute significantly to mathematics, especially to the field of number theory. In [7], Horadam defined the Jacobsthal and Jacobsthal Lucas sequences. Civciv and Türkmen study (s,t)-Fibonacci and (s,t)-Lucas number sequences ([4], [5]). Uygun defined (s,t)-Jacobsthal and (s,t)-Jacobsthal Lucas sequences [16].

For any real numbers s,t; the (s,t)-Jacobsthal $\{J_n(s,t)\}_n$ and (s,t)-Jacobsthal Lucas $\{j_n(s,t)\}_n$ number sequences are defined recurrently by

$$J_n(s,t) = sJ_{n-1}(s,t) + 2tJ_{n-2}(s,t), \quad J_0(s,t) = 0, J_1(s,t) = 1, \ n \geq 2 \tag{1.1}$$

and

$$j_n(s,t) = sj_{n-1}(s,t) + 2tj_{n-2}(s,t), \quad j_0(s,t) = 2, \quad j_1(s,t) = s, \ n \geq 2 \tag{1.2}$$

respectively, where $s > 0$, $t \neq 0$ and $s^2 + 8t > 0$ [16].

Throughout this paper, for convenience we will use the symbol J_n instead of $J_n(s,t)$ and the symbol j_n instead of $j_n(s,t)$. The first few values of these sequences are

$$\{J_n(s,t)\}_n = \left\{0,\ 1,\ s,\ s^2+2t,\ s^3+4st,\ s^4+6s^2t+4t^2, ...\right\}$$

2000 *Mathematics Subject Classification.* 11B37, 11B83, 11C20.

Key words and phrases. Jacobsthal Numbers, Jacobsthal Lucas Numbers, Hessenberg Matrix.

$$\{j_n(s,t)\}_n = \{2,\ s,\ s^2+4t,\ s^3+6st,\ s^4+8s^2t+8t^2,\ s^5+10s^3t+20st^2, ...\}.$$

Many properties of these numbers sequences are deduced directly from elemantary matrix algebra. In matrix algebra, determinant and permanent are two importance consepts. It is known that there are a lot of relationships between determinantal and permanental representations of matrices and well-known number sequences. Many researchers studied on determinantal and permanental representations of these number sequences ([12], [8], [17], [1]). Minc defined an $n \times n$ $(0,1)$-matrix $F(n,k)$ and showed that the permanents of $F(n,k)$ is equal to the generalized order-k Fibonacci numbers [10]. Kılıç and Taşçı studied permanents and determinants of Hessenberg matrices [9]. Gultekin and Tasyurdu obtain determinants and permanents of some Hessenberg matrices that give terms of polynomials $F_n(x,s,q)$ and $P_n(x,s,q)$ ([14], [15]). Gulec gave some determinantal and permanental representations of (s,t)-Pell numbers [6].

Let $M = [m_{ij}]$ be an $n \times n$ matrix and S_n is a symetric group of permutations over the set $\{1,2,...,n\}$. The determinant of M matrix defined by

$$\det M =_{\alpha \in S_n} sgn(\alpha)_{i=1}^{n} m_{i\alpha(i)}$$

where the sum ranges over all the permutations of the integers 1, 2, ..., n [13]. It can be denoted by $sgn(\alpha) = \pm 1$ the signature of α, equal to $+1$ if α is the product an even number of transposition, and -1 otherwise. The permanent of M matrix is defined by

$$perM =_{\alpha \in S_n} {}_{i=1}^{n} m_{i\alpha(i)}$$

where the summation extends over all permutations α of the symmetric group S_n [10].

Let $M = [m_{ij}]$ be an $m \times n$ matrix with row vectors $r_1, r_2, ..., r_m$. We call M is contractible on column k, if column k contains exactly two nonzero elements. Suppose that M is contractible on column k with $m_{ik} \neq 0 \neq m_{jk}$ and $i \neq j$. Then the $(m-1) \times (n-1)$ matrix $M_{ij:k}$ obtained from M replacing row i with $m_{jk} r_i + m_{ik} r_j$ and deleting row j and column k is called the contraction of M on column k relative to rows i and j. If M is contractible on row k with $m_{ki} \neq 0 \neq m_{kj}$ and $i \neq j$, then the matrix $M_{k:ij} = \left[M_{ij:k}^T\right]^T$ is called the contraction of M on row k relative to columns i and j.

Lemma 1.1. [2] *Let M be a nonnegative integral matrix of order n for $n > 1$ and let N be a contraction of M. Then*

$$perM = perN. \tag{1.3}$$

An $n \times n$ matrix $M_n = [m_{ij}]$ is called lower Hessenberg matrix if $m_{ij} = 0$ when $j - i > 1$, i.e.,

$$M_n = \begin{bmatrix} m_{11} & m_{12} & 0 & \cdots & 0 \\ m_{21} & m_{22} & m_{23} & \cdots & 0 \\ m_{31} & m_{32} & m_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ m_{n-1,1} & m_{n-1,2} & m_{n-1,3} & \cdots & m_{n-1,n} \\ m_{n,1} & m_{n,2} & m_{n,3} & \cdots & m_{n,n} \end{bmatrix}.$$

Theorem 1.2. [3] *Let M_n be an $n \times n$ lower Hessenberg matrix for all $n \geq 1$ and* $\det(M_0) = 1$. *Then*

$$\det(M_1) = m_{11}$$

and for $n \geq 2$

$$\det(M_n) = m_{n,n} \det(M_{n-1}) +_{r=1}^{n-1} \left[(-1)^{n-r} m_{n,r}{}_{j=r}^{n-1} m_{j,j+1} \det(M_{r-1}) \right]. \tag{1.4}$$

Theorem 1.3. [11] *Let M_n be an $n \times n$ lower Hessenberg matrix for all $n \geq 1$ and* $per(M_0) = 1$. *Then*

$$per(M_1) = m_{11}$$

and for $n \geq 2$

$$per(M_n) = m_{n,n} per(M_{n-1}) +_{r=1}^{n-1} \left[m_{n,r}{}_{j=r}^{n-1} m_{j,j+1} per(M_{r-1}) \right]. \tag{1.5}$$

In this paper, we define four type lower Hessenberg matrix and show that the determinant and permanent of these type matrices are (s,t)-Jacobsthal and (s,t)-Jacobsthal Lucas numbers.

2. Main Results and Proofs

2.1. The Determinantal Representations.

Definition 2.1. The $n-$square Hessenberg matrix $A_n(s,t) = (a_{ij})$ defined by

$$A_n(s,t) = \begin{bmatrix} s & -2t & 0 & 0 & 0 & \cdots & 0 \\ 1 & s & -2t & 0 & 0 & \cdots & 0 \\ 0 & 1 & s & -2t & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & 0 \\ \vdots & \vdots & \vdots & 0 & 1 & s & -2t \\ 0 & 0 & \cdots & 0 & 0 & 1 & s \end{bmatrix} \tag{2.1}$$

with $a_{i,i} = s$, $a_{i+1,i} = 1$, $a_{i,i+1} = -2t$ for $1 \leq i \leq n$ and 0 otherwise.

Theorem 2.2. *Let the matrix $A_n(s,t)$ be as in equation (2.1). Then for $n \geq 1$,*

$$\det A_n(s,t) = J_{n+1}$$

where J_n is the nth (s,t)-Jacobsthal number.

Proof. To prove $\det A_n(s,t) = J_{n+1}$, we use the mathematical induction on n. Then,

$$\begin{aligned} n &= 1, & \det A_1(s,t) &= s = J_2 \\ n &= 2, & \det A_2(s,t) &= \begin{vmatrix} s & -2t \\ 1 & s \end{vmatrix} = s^2 + 2t = J_3 \\ n &= 3, & \det A_3(s,t) &= \begin{vmatrix} s & -2t & 0 \\ 1 & s & -2t \\ 0 & 1 & s \end{vmatrix} = s^3 + 4st = J_4 \\ n &= 4, & \det A_4(s,t) &= \begin{vmatrix} s & -2t & 0 & 0 \\ 1 & s & -2t & 0 \\ 0 & 1 & s & -2t \\ 0 & 0 & 1 & s \end{vmatrix} = s^4 + 6s^2t + 4t^2 = J_5. \end{aligned}$$

Assume that it is true for n, namely

$$\det A_n(s,t) = J_{n+1},\ \ \det A_{n-1}(s,t) = J_n, ...$$

and we show that it is true for $n+1$. By our assumption and using equation (1.4), we have

$$\begin{aligned}
\det(A_{n+1}) &= a_{n+1,n+1}\det(A_n) +_{r=1}^{n} \left[(-1)^{n+1-r} a_{n+1,r}{}_{j=r}^{n} a_{j,j+1}\det(A_{r-1})\right] \\
&= s\det(A_n) +_{r=1}^{n-1} \left[(-1)^{n+1-r} a_{n+1,r}{}_{j=r}^{n} a_{j,j+1}\det(A_{r-1})\right] + (-1)a_{n+1,n}a_{n,n+1}\det(A_{n-1}) \\
&= s\det(A_n) + [(-1)(1)(-2t)\det(A_{n-1})] \\
&= s\det(A_n) + 2t\det(A_{n-1}) \\
&= sJ_{n+1} + 2tJ_n \\
&= J_{n+2}.
\end{aligned}$$

So the result holds for all integers n and the proof is complete. □

Definition 2.3. The $n-$square Hessenberg matrix $B_n(s,t) = (b_{ij})$ defined by

$$B_n(s,t) = \begin{bmatrix} s & -2t & 0 & 0 & 0 & \cdots & 0 \\ 2 & s & -2t & 0 & 0 & \cdots & 0 \\ 0 & 1 & s & -2t & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & 0 \\ \vdots & \vdots & \vdots & 0 & 1 & s & -2t \\ 0 & 0 & \cdots & 0 & 0 & 1 & s \end{bmatrix} \tag{2.2}$$

with $a_{i,i} = s$, $a_{i+1,i} = 1$, $a_{i,i+1} = -2t$ for $1 < i \leq n$, $a_{1,1} = s$, $a_{2,1} = 2$, $a_{1,2} = -2t$ and 0 otherwise.

Theorem 2.4. *Let the matrix $B_n(s,t)$ be as in equation (2.2). Then for $n > 1$,*

$$\det B_n(s,t) = j_n$$

where j_n is the nth (s,t)-Jacobsthal Lucas number.

Proof. o prove $\det B_n(s,t) = j_n$, we use the mathematical induction on n. Then,

$$\begin{aligned}
n &= 1, & \det B_1(s,t) &= s = j_1 \\
n &= 2, & \det B_2(s,t) &= \begin{vmatrix} s & -2t \\ 2 & s \end{vmatrix} = s^2 + 4t = j_2 \\
n &= 3, & \det B_3(s,t) &= \begin{vmatrix} s & -2t & 0 \\ 2 & s & -2t \\ 0 & 1 & s \end{vmatrix} = s^3 + 6st = j_3 \\
n &= 4, & \det B_4(s,t) &= \begin{vmatrix} s & -2t & 0 & 0 \\ 2 & s & -2t & 0 \\ 0 & 1 & s & -2t \\ 0 & 0 & 1 & s \end{vmatrix} = s^4 + 8s^2t + 8t^2 = j_4.
\end{aligned}$$

Assume that it is true for n, namely

$$\det B_n(s,t) = j_n,\ \ \det B_{n-1}(s,t) = j_{n-1}, ...$$

and we show that it is true for $n+1$. By our assumption and using equation (1.4), we have

$$\begin{aligned}
\det(B_{n+1}) &= b_{n+1,n+1}\det(B_n) +_{r=1}^{n}\left[(-1)^{n+1-r} b_{n+1,r}{}_{j=r}^{n} b_{j,j+1}\det(B_{r-1})\right] \\
&= s\det(B_n) +_{r=1}^{n-1}\left[(-1)^{n+1-r} b_{n+1,r}{}_{j=r}^{n} b_{j,j+1}\det(B_{r-1})\right] + (-1)b_{n+1,n}b_{n,n+1}\det(B_{n-1}) \\
&= s\det(B_n) + [(-1)(1)(-2t)\det(B_{n-1})] \\
&= s\det(B_n) + 2t\det(B_{n-1}) \\
&= sj_n + 2tj_{n-1} \\
&= j_{n+1}.
\end{aligned}$$

So the result holds for all integers n and the proof is complete. □

2.2. **The Permanental Representations.**

Definition 2.5. The $n-$square Hessenberg matrix $C_n(s,t) = (c_{ij})$ defined by

$$C_n(s,t) = \begin{bmatrix} s & 2t & 0 & 0 & 0 & \cdots & 0 \\ 1 & s & 2t & 0 & 0 & \cdots & 0 \\ 0 & 1 & s & 2t & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & 0 \\ \vdots & \vdots & \vdots & 0 & 1 & s & 2t \\ 0 & 0 & \cdots & 0 & 0 & 1 & s \end{bmatrix} \qquad (2.3)$$

with $c_{i,i} = s$, $c_{i+1,i} = 1$, $c_{i,i+1} = 2t$ for $1 \le i \le n$ and 0 otherwise.

Theorem 2.6. *Let the matrix $C_n(s,t)$ be as in equation (2.3). Then for $n \ge 1$,*

$$perC_n(s,t) = perC_n^{n-2}(s,t) = J_{n+1}$$

where J_n is the nth (s,t)-Jacobsthal number.

Proof. Let $C_n^r(s,t)$ be rth contraction of $C_n(s,t)$, $1 \le r \le n-2$. From Definition 2.5, the matrix $C_n(s,t)$ can be contracted on column 1,then we get the following $C_n^1(s,t)$

$$C_n^1(s,t) = \begin{bmatrix} s^2+2t & 2st & 0 & 0 & 0 & \cdots & 0 \\ 1 & s & 2t & 0 & 0 & \cdots & 0 \\ 0 & 1 & s & 2t & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & 0 \\ \vdots & \vdots & \vdots & 0 & 1 & s & 2t \\ 0 & 0 & \cdots & 0 & 0 & 1 & s \end{bmatrix} = \begin{bmatrix} J_3 & 2tJ_2 & 0 & 0 & 0 & \cdots & 0 \\ 1 & s & 2t & 0 & 0 & \cdots & 0 \\ 0 & 1 & s & 2t & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & 0 \\ \vdots & \vdots & \vdots & 0 & 1 & s & 2t \\ 0 & 0 & \cdots & 0 & 0 & 1 & s \end{bmatrix}$$

where $J_3 = s^2 + 2t$ and $J_2 = s$. Since $C_n^1(s,t)$ also can be contracted according to the first column,

$$C_n^2(s,t) = \begin{bmatrix} s^3+4st & 2s^2t+4t^2 & 0 & 0 & 0 & \cdots & 0 \\ 1 & s & 2t & 0 & 0 & \cdots & 0 \\ 0 & 1 & s & 2t & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & 0 \\ \vdots & \vdots & \vdots & 0 & 1 & s & 2t \\ 0 & 0 & \cdots & 0 & 0 & 1 & s \end{bmatrix} = \begin{bmatrix} J_4 & 2tJ_3 & 0 & 0 & 0 & \cdots & 0 \\ 1 & s & 2t & 0 & 0 & \cdots & 0 \\ 0 & 1 & s & 2t & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & 0 \\ \vdots & \vdots & \vdots & 0 & 1 & s & 2t \\ 0 & 0 & \cdots & 0 & 0 & 1 & s \end{bmatrix}$$

where $J_4 = s^3 + 4st$ and $J_3 = s^2 + 2t$. Continuing as similar, we obtain the rth contraction

$$C_n^r(s,t) = \begin{bmatrix} J_{r+2} & 2tJ_{r+1} & 0 & 0 & 0 & \cdots & 0 \\ 1 & s & 2t & 0 & 0 & \cdots & 0 \\ 0 & 1 & s & 2t & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & 0 \\ \vdots & \vdots & \vdots & 0 & 1 & s & 2t \\ 0 & 0 & \cdots & 0 & 0 & 1 & s \end{bmatrix}$$

for $3 \le r \le n-4$. Hence

$$C_n^{n-3}(s,t) = \begin{bmatrix} J_{n-1} & 2tJ_{n-2} & 0 \\ 1 & s & 2t \\ 0 & 1 & s \end{bmatrix}$$

which by contraction of $C_n^{n-3}(s,t)$ on first column, we get

$$C_n^{n-2}(s,t) = \begin{bmatrix} sJ_{n-1} + 2tJ_{n-2} & 2tJ_{n-1} \\ 1 & s \end{bmatrix} = \begin{bmatrix} J_n & 2tJ_{n-1} \\ 1 & s \end{bmatrix}$$

by using equation (1.1). From the equation (1.3), we obtain

$$perC_n(s,t) = perC_n^{n-2}(s,t) = sJ_n + 2tJ_{n-1} = J_{n+1}.$$

So the proof is complete. □

Definition 2.7. The $n-$square Hessenberg matrix $D_n(s,t) = (d_{ij})$ defined by

$$D_n(s,t) = \begin{bmatrix} s & 2t & 0 & 0 & 0 & \cdots & 0 \\ 2 & s & 2t & 0 & 0 & \cdots & 0 \\ 0 & 1 & s & 2t & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & 0 \\ \vdots & \vdots & \vdots & 0 & 1 & s & 2t \\ 0 & 0 & \cdots & 0 & 0 & 1 & s \end{bmatrix} \tag{2.4}$$

with $d_{i,i} = s$, $d_{i+1,i} = 1$, $d_{i,i+1} = 2t$ for $1 < i \le n$, $d_{1,1} = s$, $d_{2,1} = 2$, $d_{1,2} = 2t$ and 0 otherwise.

Theorem 2.8. *Let the matrix $D_n(s,t)$ be as in equation (2.4). Then for $n \ge 1$,*

$$perD_n(s,t) = perD_n^{n-2}(s,t) = j_n$$

where j_n is the nth (s,t)-Jacobsthal Lucas number.

Proof. Let $D(s,t)$ be rth contraction of $D_n(s,t)$, $1 \le r \le n-2$. From Definition 2.7, the matrix $D_n(s,t)$ can be contracted on column 1,then we get the following $D_n^1(s,t)$

$$D_n^1(s,t) = \begin{bmatrix} s^2+4t & 2st & 0 & 0 & 0 & \cdots & 0 \\ 1 & s & 2t & 0 & 0 & \cdots & 0 \\ 0 & 1 & s & 2t & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & 0 \\ \vdots & \vdots & \vdots & 0 & 1 & s & 2t \\ 0 & 0 & \cdots & 0 & 0 & 1 & s \end{bmatrix} = \begin{bmatrix} j_2 & 2tj_1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & s & 2t & 0 & 0 & \cdots & 0 \\ 0 & 1 & s & 2t & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & 0 \\ \vdots & \vdots & \vdots & 0 & 1 & s & 2t \\ 0 & 0 & \cdots & 0 & 0 & 1 & s \end{bmatrix}$$

where $j_2 = s^2 + 4t$ and $j_1 = s$. Since $D_n^1(s,t)$ also can be contracted according to the first column,

$$D_n^2(s,t) = \begin{bmatrix} s^3+6st & 2s^2t+8t^2 & 0 & 0 & 0 & \cdots & 0 \\ 1 & s & 2t & 0 & 0 & \cdots & 0 \\ 0 & 1 & s & 2t & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & 0 \\ \vdots & \vdots & \vdots & 0 & 1 & s & 2t \\ 0 & 0 & \cdots & 0 & 0 & 1 & s \end{bmatrix} = \begin{bmatrix} j_3 & 2tj_2 & 0 & 0 & 0 & \cdots & 0 \\ 1 & s & 2t & 0 & 0 & \cdots & 0 \\ 0 & 1 & s & 2t & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & 0 \\ \vdots & \vdots & \vdots & 0 & 1 & s & 2t \\ 0 & 0 & \cdots & 0 & 0 & 1 & s \end{bmatrix}$$

where $j_3 = s^3 + 6st$ and $j_2 = s^2 + 4t$. Continuing as similar, we obtain the rth contraction

$$D_n^r(s,t) = \begin{bmatrix} j_{r+1} & 2tj_r & 0 & 0 & 0 & \cdots & 0 \\ 1 & s & 2t & 0 & 0 & \cdots & 0 \\ 0 & 1 & s & 2t & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & 0 \\ \vdots & \vdots & \vdots & 0 & 1 & s & 2t \\ 0 & 0 & \cdots & 0 & 0 & 1 & s \end{bmatrix}$$

for $3 \leq r \leq n-4$. Hence

$$D_n^{n-3}(s,t) = \begin{bmatrix} j_{n-2} & 2tj_{n-3} & 0 \\ 1 & s & 2t \\ 0 & 1 & s \end{bmatrix}$$

which by contraction of $D_n^{n-3}(s,t)$ on first column, we get

$$D_n^{n-2}(s,t) = \begin{bmatrix} sj_{n-2}+2tj_{n-3} & 2tj_{n-2} \\ 1 & s \end{bmatrix} = \begin{bmatrix} j_{n-1} & 2tj_{n-2} \\ 1 & s \end{bmatrix}$$

by using equation (1.2). From the equation (1.3), we obtain

$$perD_n(s,t) = perD_n^{n-2}(s,t) = sj_{n-1} + 2tj_{n-2} = j_n.$$

So the proof is complete. □

As the other way, equation (1.5) can be used for proofs of Theorem 2.6 and Theorem 2.8 too.

The authors declare that there is no conflict of interest regarding the publication of this paper.

The authors express their sincere thanks to the referee for his/her careful reading and suggestions that helped to improve this paper.

References

[1] I. Aktaş, H. Köse, Hessenberg matrices and the Pell-Lucas and Jacobsthal numbers, International Journal of Pure and Applied Mathematics, 101(3), (2015), 425-432.

[2] R. Brualdi, P. Gibson, Convex polyhedra of doubly stochastic matrices I: applications of the permanent function, J. Comb. Theory 22(2), (1977), 194–230.

[3] N. D. Cahill, J.R. D'Errico, D. A. Narayan and J. Y. Narayan, Fibonacci determinants, College Math. J. 33(3), (2002), 221-225.

[4] H. Civciv, R Turkmen, On the (s,t)-Fibonacci and Fibonacci Matrix Sequences, Ars. Comb. 87, (2008), 161-173.

[5] H. Civciv, R Turkmen, Notes on the (s,t)-Lucas and Lucas Matrix Sequences, Ars. Comb. 89, (2008), 271-285.

[6] H. H. Güleç, Permanents and Determinants of Tridiagonal Matrices with (s,t)-Pell Lucas Numbers, International Journal of Mathematical Analysis, 11(23), (2017), 1117 - 1122.
[7] A. F. Horadam, Jacobsthal Representation Numbers, The Fibonacci Quarterly, 34(1), (1996), 40-54.
[8] K. Kaygısız, A. Şahin, Determinant and permanent of Hessenberg matrices and Fibonacci type numbers, Gen. Math. Notes 9(2), (2012), 32-41.
[9] E. Kılıç, D. Taşçı, On the generalized Fibonacci and Pell sequences by Hessenberg matrices, Ars Combin. 94 (2010) 161-174.
[10] H. Minc, Permanents, Encyclopedia Math. Appl., vol. 6, Addison–Wesley Publishing Company, London, (1978).
[11] A. A. Ocal, N. Tuglu, E. Altinisik, On the representation of k-generalized Fibonacci and Lucas numbers, Appl. Math. Comput. 170(1), (2005), 584-596.
[12] J. L. Ramîrez, Hessenberg Matrices and the Generalized Fibonacci-Narayana Sequence, Filomat 29(7), (2015), 1557-1563.
[13] D. Serre, Matrices: Theory and Applications, Springer Verlag, New York, (2002).
[14] Y. Taşyurdu, İ. Gültekin, Determinantal and permanental representation of q-Fibonacci polynomials, Qscience Connect, 25, (2014), 1-5.
[15] Y. Taşyurdu, İ. Gültekin, On The Pell Polynomials $P_n(x,s,q)$ and Tridiogonal Matrix, Research & Reviews: Discrete Mathematical Structures, 1(1), (2014), 24-27.
[16] Ş. Uygun, The (s,t)-Jacobsthal and (s,t)-Jacobsthal Lucas Sequences, Applied Mathematical Sciences, 9(70), (2015), 3467-3476.
[17] F. Yilmaz and D. Bozkurt, Hessenberg matrices and the Pell and Perrin numbers, J. Number Theory 131(8), (2011), 1390-1396.

Department of Mathematics, Faculty of Sciences and Art, The University of Erzincan Binali Yildirim, Erzincan 24000, Turkey.

Current address: Department of Mathematics, Faculty of Sciences and Art, The University of Erzincan Binali Yıldırım, Erzincan 24000, Turkey.

E-mail address: `ytasyurdu@erzincan.edu.tr`