

ON THE SIGMA INDEX OF MONOGENIC SEMIGROUP GRAPHS

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ABSTRACT. In [9], Das et.al. considered the monogenic semi group S_M with zero having $\{0, x, x^2, \dots, x^n\}$. Also they defined undirected graph $\Gamma(S_M)$ associated with S_M whose vertices are the non-zero elements x, x^2, \dots, x^n and any two different vertices x^i and x^j are adjacent if $i + j > n$ (for $1 \leq i, j \leq n$).

In this paper we present sigma index monogenic semigroup graphs $\Gamma(S_M)$. Also we calculate forgotten (Zagreb) index of $\Gamma(S_M)$.

1. INTRODUCTION

Algebraic structures are very important for graph theory. Each commutative ring R can be described by means of a simple graph $\Gamma(R)$. There are many studies in the literature about zero-divisor graphs [3, 4, 5, 11, 12]. In a recently study Das, Akgunes and Cevik [9] the graph $\Gamma(S_M)$ is defined. The authors considered the finite multiplicative monogenic semigroup (with zero) $S_M = \{0, x, x^2, \dots, x^n\}$. The vertices of this graph obtained S_M monogenic semigroup are the non-zero elements $\{x, x^2, \dots, x^n\}$ and any two distinct vertices x^i and x^j are adjacent with the rule $x^i \cdot x^j = 0$ if and only if $i + j > n$ ($1 \leq i, j \leq n$).

In [9], later on giving the definition of $\Gamma(S_M)$, it has been presented some certain results for the diameter, girth, maximum and minimum degree, chromatic number, clique number, degree sequence, irregularity index and dominating number for it.

In [2], to indicate the importance of this graph and by considering *First Zagreb index*, *Second Zagreb index*, *Randić index*, *geometric-arithmetic index* and *Atom-bond Connectivity index*, we will express some other certain results over $\Gamma(S_M)$.

Topological indices are defined and used in many areas(physical, chemical, pharmacological, pharmaceutical, biological, etc.) to study several properties of different

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objects such as atoms and molecules. Several topological indices have been defined and studied by mathematicians and chemists [7, 10, 17, 18].

Two of the most important degree-based topological graph indices are the first and second Zagreb indices:

$$M_1(G) = \sum_{i \in V} d_i^2 \quad \text{and} \quad M_2(G) = \sum_{ij \in E(G)} d_i d_j$$

respectively. These were introduced in the 1970s [[14, 16]].

One additional Zagreb-type index is the forgotten index [[13]].

$$F = F(G) = \sum_{i \in V} d_i^3$$

Also, recently, Gutman et al. [[15]] defined a new graph invariant *Sigma Index* as

$$\sigma = \sigma(G) = \sum_{uv \in E(G)} (d_u - d_v)^2.$$

In this paper we calculate F and σ for $\Gamma(\mathcal{S}_M)$.

2. INDICES RESULTS OVER $\Gamma(\mathcal{S}_M)$

In this main section, we will present some theorems and some lemmas over $\Gamma(\mathcal{S}_M)$ in terms of special graph indices that studied previously for some other graphs.

Firstly we give an algorithm about all neighborhood of $\Gamma(\mathcal{S}_M)$.

Lemma 2.1. [2]

To make some simplification in our calculations and so results, let us present the following algorithm about adjacency on $\Gamma(\mathcal{S}_M)$. By the definition, the neighbors of the vertices can be defined as in the following steps:

- I_n : The vertex x^n is adjacent to the all vertices x^{i_1} ($1 \leq i_1 \leq n - 1$) except itself.
- I_{n-1} : The vertex x^{n-1} is adjacent to the all vertices x^{i_2} ($2 \leq i_2 \leq n - 2$) except itself and to the vertex x^n (by the first step I_n).
- I_{n-2} : The vertex x^{n-2} is adjacent to the all vertices x^{i_3} ($3 \leq i_3 \leq n - 3$), the vertex x^n (by the first step I_n) and the vertex x^{n-1} (by the second step I_{n-1}).

By keep going these steps in this algorithm, we have two possibilities at the final two stages depending on the n is even or odd:

n is even:

- $I_{\frac{n}{2}+2}$: The vertex $x^{\frac{n}{2}+2}$ is adjacent not only to the vertices $x^{\frac{n}{2}-1}$, $x^{\frac{n}{2}}$ and $x^{\frac{n}{2}+1}$ but also adjacent to the vertices x^n (by I_n), x^{n-1} (by I_{n-1}), x^{n-2} (by I_{n-2}), ..., $x^{\frac{n}{2}+3}$ (by $I_{\frac{n}{2}+3}$).

$I_{\frac{n}{2}+1}$: The vertex $x^{\frac{n}{2}+1}$ is adjacent not only to the single vertex $x^{\frac{n}{2}}$ but also adjacent to the vertices x^n (by I_n), x^{n-1} (by I_{n-1}), x^{n-2} (by I_{n-2}), \dots , $x^{\frac{n}{2}+2}$ (by $I_{\frac{n}{2}+2}$).

n is odd:

$I_{\frac{n+1}{2}+2}$: The vertex $x^{\frac{n+1}{2}+2}$ is adjacent not only to the vertices $x^{\frac{n+1}{2}-2}$, $x^{\frac{n+1}{2}-1}$, $x^{\frac{n+1}{2}}$ and $x^{\frac{n+1}{2}+1}$ but also adjacent to the vertices x^n (by I_n), x^{n-1} (by I_{n-1}), x^{n-2} (by I_{n-2}), \dots , $x^{\frac{n+1}{2}+3}$ (by $I_{\frac{n+1}{2}+3}$).

$I_{\frac{n+1}{2}+1}$: The vertex $x^{\frac{n+1}{2}+1}$ is adjacent not only to the vertices $x^{\frac{n+1}{2}-1}$ and $x^{\frac{n+1}{2}}$ but also adjacent to the vertices x^n (by I_n), x^{n-1} (by I_{n-1}), x^{n-2} (by I_{n-2}), \dots , $x^{\frac{n+1}{2}+2}$ (by $I_{\frac{n+1}{2}+2}$).

In a usual way, let d_1, d_2, \dots, d_n denote the degrees of the vertices x, x^2, \dots, x^n in $\Gamma(\mathcal{S}_M)$, respectively. At this point we note that there exist so many studies regarding the degrees of vertices under different names such as *degree sequence* (we may refer [1, 9] and the references cited in them for a list of studies about this subject). In fact the following lemma is directly related with this subject which assures there exists an ordering $d_1 \leq d_2 \leq \dots \leq d_n$ among degrees. Although the proof of this lemma can be found in [9], one can also see easily by considering the above effective algorithm.

Lemma 2.2 ([2]).

$$(2.1) \quad \left. \begin{aligned} d_1 = 1, d_2 = 2, \dots, d_{\lfloor \frac{n}{2} \rfloor} = \lfloor \frac{n}{2} \rfloor, d_{\lfloor \frac{n}{2} \rfloor + 1} = \lfloor \frac{n}{2} \rfloor, \\ d_{\lfloor \frac{n}{2} \rfloor + 2} = \lfloor \frac{n}{2} \rfloor + 1, \dots, d_n = n - 1. \end{aligned} \right\}$$

Now we give a theorem and a lemma which are the playing an important role in our main results

Theorem 2.3 ([2]). *The second Zagreb index of the graph $\Gamma(\mathcal{S}_M)$ is*

$$M_2(\Gamma(\mathcal{S}_M)) = \begin{cases} \frac{5n^4 - 8n^3 + 10n^2 - 4n}{48} & ; \text{ if } n \text{ is even} \\ \frac{5n^4 - 8n^3 - 2n^2 + 8n - 3}{48} & ; \text{ if } n \text{ is odd} \end{cases}.$$

Lemma 2.4 ([15]). *For any simple connected graph G*

$$\sigma(G) = F(G) - 2M_2(G).$$

Now we will give first main theorem.

Theorem 2.5. *The forgotten Zagreb index of the graph $\Gamma(\mathcal{S}_M)$ is*

$$F(\Gamma(\mathcal{S}_M)) = \begin{cases} \frac{1}{8}n^2(2n^2 - 3n + 2) & ; \text{ if } n \text{ is even} \\ \frac{1}{8}(n-1)^2(2n^2 + n - 1) & ; \text{ if } n \text{ is odd} \end{cases}.$$

Proof. In the proof, we will mainly consider (2.1). By the definition of forgotten Zagreb index,

$$\begin{aligned}
F(\Gamma(\mathcal{S}_M)) &= \sum_{x^i \in V} d_i^3 = 1^3 + 2^3 + \cdots + \underbrace{\left(\left\lfloor \frac{n}{2} \right\rfloor\right)^3 + \left(\left\lfloor \frac{n}{2} \right\rfloor\right)^3}_{\left(\left\lfloor \frac{n}{2} \right\rfloor\right)^3 + \left(\left\lfloor \frac{n}{2} \right\rfloor + 1\right)^3} + \cdots + (n-1)^3 \\
(2.2) \quad &1^3 + 2^3 + \cdots + \left(\left\lfloor \frac{n}{2} \right\rfloor\right)^3 + \left(\left\lfloor \frac{n}{2} \right\rfloor + 1\right)^3 + \cdots + (n-1)^3 + \left(\left\lfloor \frac{n}{2} \right\rfloor\right)^3 \\
&= \left(\frac{(n-1)n}{2}\right)^2 + \left(\left\lfloor \frac{n}{2} \right\rfloor\right)^3.
\end{aligned}$$

Now depends on the status of n in $\lfloor \frac{n}{2} \rfloor$, the result as desired. $\left(\text{Because } \lfloor \frac{n}{2} \rfloor = \begin{cases} \frac{n}{2} & ; \text{ if } n \text{ is even} \\ \frac{n-1}{2} & ; \text{ if } n \text{ is odd} \end{cases} \right)$
 \square

Now we will give second main theorem.

Theorem 2.6. *The sigma index of the graph $\Gamma(\mathcal{S}_M)$ is*

$$\sigma(\Gamma(\mathcal{S}_M)) = \begin{cases} = \frac{1}{24}n(n-1)(n-2)(n+2) & ; \text{ if } n \text{ is even} \\ = \frac{1}{24}n(n-1)^2(n+1) & ; \text{ if } n \text{ is odd} \end{cases}.$$

Proof. The main goal in here to characterize $\sigma = \sigma(G) = \sum_{uv \in E(G)} (d_u - d_v)^2$ depending on the total number of degrees. We will consider definition of sigma index, (2.3), (2.4) and (2.5) and we calculate the sigma index of the graph $\Gamma(\mathcal{S}_M)$.

Assume that n is even:

$$\begin{aligned}
\sigma(\Gamma(\mathcal{S}_M)) &= F(\Gamma(\mathcal{S}_M)) - 2M_2(\Gamma(\mathcal{S}_M)) \\
&= \frac{1}{8}n^2(2n^2 - 3n + 2) - 2\frac{5n^4 - 8n^3 + 10n^2 - 4n}{48} \\
&= \frac{1}{24}n(n-1)(n-2)(n+2).
\end{aligned}$$

Assume that n is odd:

Following same steps as in n is even case, we obtain

$$\begin{aligned}
\sigma(\Gamma(\mathcal{S}_M)) &= F(\Gamma(\mathcal{S}_M)) - 2M_2(\Gamma(\mathcal{S}_M)) \\
&= \frac{1}{8}(n-1)^2(2n^2 + n - 1) - 2\frac{5n^4 - 8n^3 - 2n^2 + 8n - 3}{48} \\
&= \frac{1}{24}n(n-1)^2(n+1)
\end{aligned}$$

Hence the result. \square

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