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ON FILTER-TYPE SOFT SETS

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ABSTRACT. In this paper, we have established filter-type soft sets and studied its basic structural properties. Besides, we give a decision-making method using filter-type soft sets in a decision-making process in any topological universe.

1. INTRODUCTION

In science and technology, mankind has been searching for various ways to model the unknowns, ambiguities and fuzziness of the problems they have been experiencing for centuries. Of course, we use mathematical methods to solve all encountered problems. However, classical mathematical methods are not always useful. So, over time, many mathematicians have defined new ways and methods. In 1999, Molodtsov [1] defined the notion of soft set as a new very useful mathematical tool for dealing with uncertainties in any science. He showed that the soft set theory can be applied many directions such as analysis, probability theory, game theory etc. In [2, 3, 4], the set-theoretic operations such as subset, union, intersection, similarity etc. are given for soft sets. In operational research as a sub-field of applied mathematics, decision-making has an important place in solving daily life problems. Many decision-making techniques have been proposed until today. Soft set theory continues on its way to becoming a pioneering mathematical tool used in the decision-making process. In [8, 9, 10], some of decision-making methods are given using soft set theory. On the other hand, topology whose main objective is the study of the general abstract nature of continuity or closeness on spaces is one of the major area of mathematics. Besides, the notion of filter is defined by H. Cartan [5] in 1937 as alternative to similar to concept of a net in topology. We recommend [6] and [7] for detailed information on topological concepts and the concept of filter. In this paper, we define the concept of filter-type soft set over any topological universe and study some of basic properties. At the end of the paper,

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we propose a decision-making method using convergence of filters, and filter-type soft sets, and give some simple descriptive examples.

2. Preliminaries

In this section, we give the preliminary information necessary for the formation of this article. In 1999, Molodtsov [1] introduced the soft set theory which for deal with the uncertainties in real world phenomena, mathematically. Let U be an initial universe which involves the problems, E be a set of parameters associated with elements in U, and 2^U be a family of all subsets of U. The concept of a soft set is defined as follows;

Definition 2.1. [1] It is called that the pair (F, A) is a soft set over U such that F is a set-valued mapping from A to 2^U. In other words, the soft set over any nonempty set is a parameterization of some subsets of given set.

Let (F, A) be a soft set over U, then it is called that the family $F(A) = \{F(p)|p \in A\}$ is an *image family* of (F, A). For each $p \in A$, $F(p)$ is called *p-approximated set* in (F, A). We will denote the family of all soft sets over the universal set U with the $\mathcal{S}(U, E)$.

Set-theoretical operations among soft sets given by Maji et al. and Ali et al. in [2] and [3]. If (F, A) and (G, B) are two soft sets over the initial universal set U, then it is called that (F, A) is a *soft subset* of (G, B) such that $A \subseteq B$ and $F(p) \subseteq G(p)$ for each $p \in A$, and denoted by $(F, A)\widetilde{\subset}(G, B)$. If $(F, A)\widetilde{\subset}(G, B)$ and $(G, B)\widetilde{\subset}(F, A)$, then it is called that (F,A) is *soft equal* to (G,B). The *soft union* of (F, A) and (G, B) is the soft set (H, C) over U and denoted by $(H, C) = (F, A)\widetilde{\cup}(G, B)$ and defined that $C = A \cup B$ and

$$H(c) = \begin{cases} F(c) & \text{, if } c \in A - B \\ G(c) & \text{, if } c \in B - A \\ F(c) \cup G(c) & \text{, if } c \in A \cap B \end{cases}$$

for each $p \in C$. Similarly, the *soft intersection* of (F, A) and (G, B) is a soft set over U, denoted by $(H, C) = (F, A)\widetilde{\cap}(G, B)$ such that $C = A \cap B \neq \varnothing$ and $H(p) = F(p) \cap G(p)$ for each $p \in C$. We call that the soft set (F, A) is a *null soft set* over U if $F(p = \varnothing)$ for each $p \in A$ and (F, A) is called a *whole soft set* over U if $F(p) = U$ for each $p \in A$. The *soft complement* of (F, A) is a soft over U, denoted by $(F, A)^c = (F^c, A)$ and defined by the set valued mapping $F^c : A \to 2^U$ such that $F^c(p) = U - F(p)$ for each $p \in A$ [2, 3].

In [4], Min gave the concept of similarity of soft sets with each other. Let (F, A) and (G, B) be soft sets over U. Then it is called that (F, A) is *similar* to (G, B) if there exist a bijection $\varphi : A \to B$ such that $G \circ \varphi = F$.

On the other hand, the concept of topological space arose the study of the real line and Euclidean space and the study of continuous functions on these spaces. Formal definition of a topology on given any universal set is a family of subsets of it which the countable intersection of elements of the family is member of the family and arbitrary union of elements of the family is member of it. In addition to this, the concept of filter on a set is a family of subsets of it which satisfy some specific properties. In topology, a filter is a generalization of a net. Formal definition of a filter on a topological space is given as follows:

Definition 2.2. [6, 7] A *filter* is a non-empty family $\mathcal{F}$ of subsets of a topological space U such that:

(1) $\varnothing \notin \mathcal{F}$,
(2) if $A \in \mathcal{F}$ and $A \subseteq B$, then $B \in \mathcal{F}$,
(3) if $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$.

Definition 2.3. [6, 7] Let $\mathcal{F}$ and $\mathcal{F}'$ be filters on U. We call that $\mathcal{F}$ is *coarser* than $\mathcal{F}'$ or $\mathcal{F}'$ is *finer* than $\mathcal{F}$ if $\mathcal{F} \subseteq \mathcal{F}'$.

Lemma 2.4. [6, 7] *Let $\{\mathcal{F}_i\}_{i\in I}$ be a non-empty family of filters on U. Then $\mathcal{F} = \bigcap_{i\in I} \mathcal{F}_i$ is a filter on U.*

Definition 2.5. [6, 7] Let $\mathcal{B}$ be a family of subsets of U. $\mathcal{B}$ is called a *base* of the filter it generates if it satisfies following axioms;

(1) $\mathcal{B} \neq \varnothing$ and $\varnothing \notin \mathcal{B}$,
(2) For each $A, B \in \mathcal{B}$, there exists $C \in \mathcal{B}$ such that $C \subseteq A \cap B$.

The filter is obtained by $\mathcal{F} = \{A \subseteq U \mid \exists B \in \mathcal{B}, B \subseteq A\}$.

Two filter base are said to be *equivalent* if they generate same filter.

Lemma 2.6. [6, 7] *Let $\mathcal{B}$ and $\mathcal{B}'$ be two filter bases on U. $\mathcal{B}$ is equivalent to $\mathcal{B}'$ if and only if every set of $\mathcal{B}$ contains a set of $\mathcal{B}'$ and every set of $\mathcal{B}'$ contains a set of $\mathcal{B}B$.*

There is a simple way to get a filter on U from any family $\mathcal{A} \subseteq 2^U$.

Definition 2.7. [6, 7] Let $\mathcal{A} \subseteq 2^U$. It is called that $\mathcal{A}$ has *the finite intersection property* if every finite subfamily of $\mathcal{A}$ has non-empty intersection.

Lemma 2.8. [6, 7]

(a) *If $\mathcal{A}$ has the finite intersection property, then*

$$\mathcal{F} = \{X \subseteq U \mid \textit{there are } n \in \mathbb{N} \textit{ and } X_1, \dots, X_n \in \mathcal{A} \textit{ such that } X_1 \cap \cdots \cap X_n \subseteq X\}$$

is the least filter on U including $\mathcal{A}$, it is called a filter generated *by the family $\mathcal{A}$ and $\mathcal{A}$ is called* subbasis *for $\mathcal{F}$.*

(b) *There is a filter including $\mathcal{A}$ if and only if $\mathcal{A}$ has the finite intersection property.*

Definition 2.9. [6, 7] Let $\mathcal{F}$ be a filter on U. We say that $\mathcal{F}$ *converges* to the point $x \in U$, and denoted by $\mathcal{F} \to x$, if every open neighborhood of x is in $\mathcal{F}$.

Theorem 2.10. [6, 7] *Let U be a topological space. U is Hausdorff iff every filter has at most one limit.*

Theorem 2.11. [6, 7] *Let U be a topological space. U is compact iff every filter can be extended to a convergent filter.*

3. Filter-Type Soft Sets

In this section, we define filter-type soft set over an initial topological universe and discuss some properties.

Throughout this paper U denotes the $(U, \mathcal{T})$ topological space.

Definition 3.1. A soft set (F, A) is called a *filter-type soft set* over U if the image family $F(A) = \{F(p) | p \in A\}$ of (F, A) is a filter on U for all $p \in A$.

The family of all filter-type soft set over U is denoted by $\mathfrak{F}(U, E)$.

Example 3.2. Let $E = \{1, 2, 3, 4, 5\}$ be the parameter set, $U = \{a, b, c\}$ be the initial universe and $\mathcal{T} = \{\varnothing, U, \{a, b\}, \{b, c\}, \{b\}\}$ be a topology on U. For given the soft set $(F, A) = \{1 = \{a, b\}, 3 = \{b, c\}, 5 = \{b\}\}$ over the U where $A = \{1, 3, 5\} \subset E$, clearly (F, A) is a filter-type soft set over U.

Example 3.3. Let $\mathbb{R}$ be a real number set, define $F : \mathbb{R} \to 2^{\mathbb{R}}$ such that $F(x) = (x, \infty)$. Then (F, A) is a filter-type soft set over $\mathbb{R}$.

Example 3.4. Let (F_x, A) be a soft set over U such that $F_x(p)$ is a neighborhood of $x \in U$. So (F_x, A) is a filter-type soft set over U. It is called that this filter-type soft set is *vicinitic soft set* for $x \in U$ with respect to A. In any topological space, there are as many vicinitic soft set as the number of points in it.

Note that, a whole soft set over U is a filter-type soft set.

Example 3.5. From Example 3.2, if we tkae the soft subset $(G, B) = \{1 = \{b\}\}$ of (F, A), then it is not a filter-type soft set over U.

Obviously, we have following theorems.

Theorem 3.6. *Let (F, A) and (G, B) be two filter-type soft sets. Then, $(F, A)\widetilde{\subset}(G, B)$ iff $A \subseteq B$, $F(p) = G(p)$ for each $p \in A$ and $F(A) \subseteq G(B)$.*

Theorem 3.7. *If $A \subseteq B$ and $F(A) \subseteq G(B)$, then $(F, A)\widetilde{\subset}(G, B)$.*

We can give following definition using Definition 2.3.

Definition 3.8. Let (F, A) and (G, B) be two filter-type soft sets. It is called that (F, A) is *coarser* than (G, B) or (G, B) is *finer* than (F, A) if $F(A) \subseteq G(B)$, and denoted by $(F, A)\widetilde{\preccurlyeq}(G, B)$.

Example 3.9. From Example 3.2, let $(F, A) = \{1 = \{a, b\}, 3 = \{b, c\}, 5 = \{b\}\}$ and $(G, B) = \{2 = \{a, b\}, 3 = \{b, c\}, 4 = \{b\}, 5 = U\}$ be filter-type soft sets over U. Then we have that $(F, A)\widetilde{\preccurlyeq}(G, B)$.

Note that, if $(F, A)\widetilde{\preccurlyeq}(G, B)$, then it does not have to be $(F, A)\widetilde{\subset}(G, B)$.

Theorem 3.10. *The relation $\widetilde{\preccurlyeq}$ on the family of filter-type soft sets over U is a partial order relation, i.e. the pair $(\mathfrak{F}(U, E), \widetilde{\preccurlyeq})$ is a poset.*

Proof. The proof is straightforward. □

Definition 3.11. The maximal elements in $(\mathfrak{F}(U, E), \widetilde{\preccurlyeq})$ are called *ultrafilter-type soft sets*.

Example 3.12. Consider the universes given in Example 3.2. Define the soft set $(F, A) = \{1 = \{a\}, 2 = \{a, b\}, 3 = \{a, c\}, 4 = U\}$. Then we obtain that $F(A)$ is an ultrafilter on U. So (F, A) is an ultrafilter-type soft set over U.

Theorem 3.13. *Soft intersection of two filter-type soft sets is a filter-type soft set.*

Proof. From definition of soft intersection and Lemma 2.4, it is obvious. □

We can see from following example that soft union of two filter-type soft sets is not always filter-type.

Example 3.14. Let (F, A) be a filter-type soft sets in Example 3.2 and we take the filter-type soft set $(G, B) = \{2 = \{a, c\}, 4 = \{b, c\}, 5 = \{c\}\}$ on U. Then their soft union $(H, C) = \{1 = \{a, b\}, 2 = \{a, c\}, 3 = \{b, c\}, 4 = \{b, c\}, 5 = \{b, c\}\}$ is not a filter-type soft set over U.

Theorem 3.15. *Let (F,A) and (G,B) be filter-type soft sets over U. $(F,A)\widetilde{\cup}(G,B)$ is a filter-type soft set iff either $(F,A)\widetilde{\subset}(G,B)$ or $(G,B)\widetilde{\subset}(F,A)$.*

Proof. Suppose that both $(F,A)\widetilde{\subset}(G,B)$ and $(G,B)\widetilde{\subset}(F,A)$ are not provided. Say $(F,A)\widetilde{\cup}(G,B)=(H,C)$, so we have $C=A\cup B$ and

$$H(c)=\begin{cases} F(c) & \text{, if } c\in A-B \\ G(c) & \text{, if } c\in B-A \\ F(c)\cup G(c) & \text{, if } c\in A\cap B \end{cases}$$

for each $p\in C$ from definition of soft union. Since $H(C)$ is a filter on U, we have $F(p)\in F(A)$ and $G(p)\in G(B)$ such that $F(p)\notin G(B)$ and $G(p)\notin F(A)$ from hypothesis and Theorem 3.6 for $p\in A\cap B$. Now $F(p)$ and $G(p)$ are in $H(C)$, so their intersection $F(p)\cap G(p)$ is in $H(C)$ because $H(C)$ is a filter. But then $F(p)\cap G(p)$ must be in either $F(A)$ or in $G(B)$. If $F(p)\cap G(p)\in F(A)$ and $F(p)\cap G(p)\subseteq G(p)$, then $G(p)$ would be in $F(A)$. This is a contradiction.

Conversely, suppose that either $(F,A)\widetilde{\subset}(G,B)$ or $(G,B)\widetilde{\subset}(F,A)$. Then we have either $(F,A)\widetilde{\cup}(G,B)=(F,A)$ or $(F,A)\widetilde{\cup}(G,B)=(G,B)$, respectively. So this is the desired result. □

From the notion of similarity of soft sets we have following theorem.

Theorem 3.16. *If (F,A) similar to (G,B) and (F,A) is a filter-type soft set over U, then (G,B) is also filter-type soft set.*

Proof. Suppose that (F,A) similar to (G,B), we have a bijection φ from A to B such that $F=G\circ\varphi$. Since φ is a bijection then we have $\varphi(A)=B$. We need to show that $G(B)$ is a filter on U. So we will get

$$G(B)=G(\varphi(A))=(G\circ\varphi)(A)=F(A)$$

from the similarity of soft sets. Since $F(A)$ is a filter on U, then $G(B)$ is filter on U. Thus (G,B) is a filter-type soft set over U. □

Definition 3.17. Let (F,A) be a soft set over U. Then it is called that (F,A) is a *basic filter-type soft set* if $F(A)$ is a filter base on U.

Note that, every filter-type soft set is a basic filter-type soft set. If we have a basic filter-type soft set then we can obtain a filter-type soft set which is generated by it using Definition 2.5. Namely, let (F,A) be a basic filter-type soft set, then if we define the soft set (F^*,A) as for each $p\in A$, there exists $F(p)\in F(A)$ such that $F(p)\subseteq F^*(p)$, the soft set (F^*,A) is a filter-type soft set.

Moreover, let (F,A) be any soft set over U. We know that $F(A)$ is a subset of 2^U. Let $F(e_i)\in F(A)$ for each $i\in\{1,2,\dots,n\}$ and $\bigcap_{i=1}^{n}F(e_i)\neq\varnothing$. Then there is a filter $\mathcal{F}$ which is contain $F(A)$ from Lemma 2.8. Therefore $F(A)$ is a subbasis of $\mathcal{F}$, and (F,A) is called a *subbasic filter-type soft set* over U. Hence every soft set over the initial universe U whose p-approximated sets have finite intersection property is a subbasic filter-type soft set.

Example 3.18. Let $\mathbb{N}$ be the set of natural numbers and define the soft set $(F,\mathbb{N})$ over $\mathbb{N}$ such that $F(n)=\{n,n+1,\dots\}$ where $n\in\mathbb{N}$. Then $(F,\mathbb{N})$ is a basic filter-type soft set. This soft set is referred to as *Fréchet*.

Using Lemma 2.6, we can obtain a relation among basic filter-type soft sets. Let (F,A) and (G,B) be two basic filter-type soft sets, then define the relation $\sim$

among basic filter-type soft sets such that $(F, A) \sim (G, B)$ iff the filter base $F(A)$ is equivalent to the filter base $G(B)$. Obviously, the relation $\sim$ is an equivalence relation among basic filter-type soft sets.

Definition 3.19. If $(F, A) \sim (G, B)$ then the parameters sets A and B are called *equivalent* and denoted by $A \approx B$.

We know that equivalent filter bases generate same filter. Therefore, if we have two equivalent basic filter-type soft sets (F, A) and (G, B) then we have $F^*(A) = G^*(B)$.

Since the soft set theory is easy to understand, it is very useful in solving daily life problems. The theory of soft sets is a pioneering mathematical tool often used in decision-making theory. Many decision-making methods have been developed using soft sets in recent years. Here, we will propose a new decision-making method different from others using topology and filters.

Let U be a problem space which is a topological universe, E be a set of parameters which is related to elements in U. Let (F, A) be a filter-type soft set over U which is constructed according to the problem. The decision maker chooses the element x in U if $F(A)$ converges to x. If $F(A)$ does not converges to any element of U, then the decision-maker said to be *instable*, otherwise the decision-maker said to be *stable*.

Example 3.20. A soft set (F, E) describes attractiveness of the houses which Mrs. Kandemir is going to buy. Let $U = \{h_1, h_2, h_3, h_4\}$ be a set of four houses under consideration. $E = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8\}$ be a set of parameters such that e_1 denotes the expensiveness of houses, e_2 denotes the beauty of houses, e_3 denotes that the house is made of wood, e_4 denotes the cheapness of the houses, e_5 denotes that the house is in the green surroundings, e_6 denotes the modernity of the houses, e_7 denotes that the house is in good repair, e_8 denotes that the house is in bad repair. The topology on U is given as $\mathcal{T} = \{\varnothing, U, \{h_1\}, \{h_1, h_2\}, \{h_1, h_3\}, \{h_1, h_2, h_3\}\}$. Let us give the filter-type soft set (F, E) over U as

$$\begin{aligned}(F, E) = \{&e_1 = \{h_1, h_2\}, e_2 = \{h_2\}, e_3 = \{h_2, h_3\}, e_4 = \{h_2, h_4\},\\ &e_5 = \{h_1, h_2, h_3\}, e_6 = \{h_1, h_2, h_4\}, e_7 = \{h_2, h_3, h_4\}, e_8 = U\}.\end{aligned}$$

Obviously, $F(E)$ is a filter on U and we obtain that $F(E) \to h_2$. Hence, Mrs. Kandemir will choose the house h_2.

Note that, a filter may converge to more than one element.

Theorem 3.21. *Let (F, A) be a filter-type soft set over U. If $F(A)$ converges to x and U is Hausdorff, then x is unique, i.e. the decision-maker certainly only selects x.*

Proof. It is straightforward from Theorem 2.10. □

In a decision-making process, the decision-maker may be undecided. We have defined this as above if we use the filter-type soft set, i.e. when we have a filter-type soft set over a topological problem universe, if the filter induced by a filter-type soft set does not converges to any element of the universe, then we have called that the decision-maker is instable. We can stabilize the decision-maker according to the properties of the topological space. If the universe is compact space, then we obtain following theorem.

Theorem 3.22. *Let (F, A) be a filter-type soft set over U. If U is compact, then $F(A)$ can be extended to a converge filter, i.e. the decision-maker can be stable.*

Proof. The proof is straightforward from Theorem 2.11. □

We have just mentioned above that if we have a soft set over a topological universe whose p -approximated sets have finite intersection property, then it is a subbasic filter-type soft set. Therefore, it can be extended to a filter-type soft set. If the topological universe U is a compact space, then we can expand it to be convergent filter from Theorem 3.22. We can interpret this in the way the decision-maker's instability can be stabilized in the decision-making process on a compact space.

When the problem space of daily life problem is finite, all topologies defined on it are compact. Thus, the decision-makers instability in finite space can be become absolutely stable.

4. Conclusion

In this paper, we have defined the concept of a filter-type soft set over any initial topological universe. We have also investigated some basic set-theoretical properties of filter-type soft sets. In recent years, soft set theory has been studied by many scientists since the applicability of daily life is easy. Many scientists have used soft sets in decision making and have proposed various decision algorithms using soft set technique. Since the speed of the decision in a decision-making process is very important, the mathematical algorithm used in the process must not be aggravation. In this paper, we have also given a decision method using the concepts of filter and soft set in any topological space. In this way, we propose that there is direct daily application of topology, which is a very abstract field of mathematic. If the topological space is compact, we have stated that the decision maker certainly has a decision. Therefore we can decide directly on a topological problem space free of heavy algorithms using filter-type soft sets. So this article can be a helpful resource for those who work on this field. The author hope that this article shed light on the scientist which is working in this area.

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