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NUMERICAL SOLUTION FOR THE FRACTIONAL ORDER LINEAR DIFFERENTIAL EQUATIONS USING HERMITE COLLOCATION METHOD

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ABSTRACT. In this paper, the fractional order linear differential equations are solved by using the Hermite Collocation Method (HCM). To illustrate the accuracy of the method, two fractional order linear differential equations with variable and constant coefficients are studied. Obtained results are compared with some earlier works. It is seen that the method is very efficient and reliable due to obtained numerical results are satisfactorily.

1. INTRODUCTION

The fractional order differential equations can be used for modelling in many fields such as viscoelasticity, heat conduction, electrode-electrolyte polarization, electromagnetic waves, diffusion wave and control theory [1]. The general fractional order linear differential equation is given as the following form;

$$a_n(x)D_c^{\alpha_n}y(x) + a_{n-1}(x)D_c^{\alpha_{n-1}}y(x) + \ldots + a_0(x)D_c^{\alpha_0}y(x) = g(x) \tag{1.1}$$

where $D_c^{\alpha_n}$ is the derivative of y of order α_n in the sense of Caputo fractional differential operator, $y(x)$ is an unknown function of the independent variable x.

Many numerical methods have been developed to solve differential equations of fractional order which are usually difficult to solve analitically. The fractional order linear differential equations have been solved using various numerical methods in literature. Some of these methods are given as follows: fractional finite difference method [2], homotopy perturbation method [3], adomian decomposition method [4], homotpy analysis method [5], variational iteration method [6].

The main aim of this paper is to solve the fractional order linear differential equations numerically by using Hermite Collocation Method (HCM). The given method converts the mentioned equations to the linear algebraic systems of which unknowns are Hermite coefficients, by using collocation points. Since expressing

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this algebraic systems by matrices and using matrix algebra solution of the algebraic system can be obtained easily. Consequently, the numerical solutions of the fractional order linear equations are obtained in terms of truncated Hermite series.

2. Preliminaries

In this section, some basic subjects of the fractional calculus which are used throught this paper are given.

Definition 2.1. A generalized integral which is depended on parameter of p

$$\Gamma(p)=\int_0^\infty x^{p-1}e^{-x}dx,\ \Gamma:(0,\infty)\longrightarrow\mathbb{R} \tag{2.1}$$

is named as Gamma function or second type Euler integral. Gamma function have proporties that

(i) convergent for $0<p<\infty$
(ii) divergent for $p\leq 0$.

Definition 2.2. The function that defined by the integral

$$\beta(p,q)=\int_0^1 x^{p-1}(1-x)^{q-1}dx,\ \beta:(0,\infty)x(0,\infty)\longrightarrow\mathbb{R} \tag{2.2}$$

is named as Beta function or first type Euler integral. Beta function is

(i) convergent for $p>0$ and $q>0$
(ii) divergent for $p\leq 0$ and $q\leq 0$.

Definition 2.3. The Caputo fractional derivative from α-th order of function f is given as the following form;

$$D_c^\alpha f(x)=\frac{1}{\Gamma(m-\alpha)}\int_a^x (x-t)^{m-\alpha-1}f^{(m)}(t)dt \tag{2.3}$$

where f function can be continuously differentiable m times, α any positive integer and m is a positive integer such that $m\in\mathbb{N}$, $m-1<\alpha<m$.

Definition 2.4. One of the solutions of

$$y''-2xy'+2ny=0 \tag{2.4}$$

equation is $H_n(x)$ Hermite polynomials which are shown by

$$H_n(x)=\sum_{k=0}^{\lfloor n/2\rfloor}\frac{(-1)^k n!(2x)^{n-2k}}{k!(n-2k)!},\ n=0,1,\ldots \tag{2.5}$$

where $-\infty<x<\infty$. Hermite polynomials are orthogonal in terms of $w(x)=e^{-x^2}$ weight function between interval of $(-\infty,\infty)$ and satisfy the relation of

$$\int_{-\infty}^{\infty} e^{-x^2}H_n(x)H_m(x)dx=0 \tag{2.6}$$

for $m\neq n$ and $m,n\in\mathbb{N}_0$.

3. Implementation of the HCM

In this section, the approximate solutions of the truncated Hermite series form can be obtained as follow:

$$y(x)=\sum_{n=0}^{N} a_n H_n(x^{\alpha}) \tag{3.1}$$

of the fractional order linear differential equation with variable or constant coefficients

$$\sum_{k=0}^{m} P_k(x) D_c^{k\alpha} y(x) = g(x) \tag{3.2}$$

with conditions for $\lfloor m.\alpha \rfloor = t \in \mathbb{N}_0,\ a \le x \le b$

$$\sum_{k=0}^{t-1} [a_{jk}(x) D_c^{k\alpha} y(a) + b_{jk}(x) D_c^{k\alpha} y(b)] = \lambda_j,\ j=0,1,2,\cdots,t-1. \tag{3.3}$$

Here a_n are the unknown Hermite coefficients, N can be chosen any positive integer such that $N \ge t$, $0 < \alpha \le 1$.

To find a solution in (3.1) of the problem (3.2) with the conditions (3.3), the collocation points can be used by

$$x_i = a + \left(\frac{b-a}{N}\right) i,\ i=0,1,\cdots,N,\ 0<a\le x\le b. \tag{3.4}$$

The matrix form of the approximate solution $y(x)$ given by (3.1) can be written as

$$y(x) = H(x^{\alpha})A \tag{3.5}$$

where $H(x^{\alpha}) = [H_0(x^{\alpha})\ H_1(x^{\alpha})\ \cdots\ H_N(x^{\alpha})]$ and $A=[a_0\ a_1\ \cdots\ a_N]^T$. Hermite polynomials given in (2.5) according to the odd and even values of N, x^{α} instead of x can be written as follows matrix form;
if N is odd

$$\underbrace{\begin{bmatrix} H_0(x^{\alpha}) \\ H_1(x^{\alpha}) \\ \vdots \\ H_{N-1}(x^{\alpha}) \\ H_N(x^{\alpha}) \end{bmatrix}}_{H^T(x^{\alpha})} = \underbrace{\begin{bmatrix} 2^0 & 0 & \cdots & 0 & 0 \\ 0 & 2^1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ (-1)^{\left(\frac{N-5}{2}\right)} \frac{2^0}{0!} \frac{(N-1)!}{\left(\frac{N-1}{2}\right)!} & 0 & \cdots & 2^{N-1} & 0 \\ 0 & (-1)^{\left(\frac{N-1}{2}\right)} \frac{2^1}{1!} \frac{N!}{\left(\frac{N-1}{2}\right)!} & \cdots & 0 & 2^N \end{bmatrix}}_{F} \underbrace{\begin{bmatrix} 1 \\ x^{\alpha} \\ \vdots \\ x^{\alpha(N-1)} \\ x^{\alpha N} \end{bmatrix}}_{X^T(x^{\alpha})},$$

if N is even

$$\underbrace{\begin{bmatrix} H_0(x^\alpha) \\ H_1(x^\alpha) \\ \vdots \\ H_{N-1}(x^\alpha) \\ H_N(x^\alpha) \end{bmatrix}}_{H^T(x^\alpha)} = \underbrace{\begin{bmatrix} 2^0 & 0 & \cdots & 0 & 0 \\ 0 & 2^1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & (-1)^{\left(\frac{N-2}{2}\right)} \frac{2^1}{1!} \frac{(N-1)!}{\left(\frac{N-2}{2}\right)!} & \cdots & 2^{N-1} & 0 \\ (-1)^{\left(\frac{N-4}{2}\right)} \frac{2^0}{0!} \frac{N!}{\left(\frac{N}{2}\right)!} & 0 & \cdots & 0 & 2^N \end{bmatrix}}_{F} \underbrace{\begin{bmatrix} 1 \\ x^\alpha \\ \vdots \\ x^{\alpha(N-1)} \\ x^{\alpha N} \end{bmatrix}}_{X^T(x^\alpha)}.$$

The given matrix form is briefly expressed as

$$H^T(x^\alpha) = FX^T(x^\alpha) \tag{3.6}$$

or

$$H(x^\alpha) = X(x^\alpha)F^T. \tag{3.7}$$

Substitution of equation (3.7) into (3.5) yields

$$y(x) = X(x^\alpha)F^T A. \tag{3.8}$$

Now, the $k\alpha$-th order Caputo fractional derivative of equation (3.8) is written as

$$D_c^{k\alpha} y(x) = D_c^{k\alpha} X(x^\alpha) F^T A \tag{3.9}$$

or equivalently:

$$D_c^{k\alpha} X(x^\alpha) = X(x^\alpha)(B^T)^k \tag{3.10}$$

where the matrix B is defined as follow:

$$B = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \Gamma(\alpha+1) & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)} & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \\ 0 & 0 & 0 & \cdots & 0 & \frac{\Gamma(N\alpha+1)}{\Gamma((N-1)\alpha+1)} & 0 \end{bmatrix}.$$

If, we substitute equation (3.10) into equation (3.9) we have:

$$D_c^{k\alpha} y(x) = X(x^\alpha)(B^T)^k F^T A. \tag{3.11}$$

For the collocation points $x = x_i,\ i = 0, 1, 2, \cdots, N$, equation (3.2) is rewritten as follows

$$\sum_{k=0}^{m} P_k(x_i) D_c^{k\alpha} y(x_i) = g(x_i). \tag{3.12}$$

The matrix form of equation (3.12) is given as follows

$$\underbrace{\begin{bmatrix} P_k(x_0) & 0 & \cdots & 0 \\ 0 & P_k(x_1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & P_k(x_N) \end{bmatrix}}_{P_k} \underbrace{\begin{bmatrix} D_c^{k\alpha} y(x_0) \\ D_c^{k\alpha} y(x_1) \\ \vdots \\ D_c^{k\alpha} y(x_N) \end{bmatrix}}_{Y^{k\alpha}} = \underbrace{\begin{bmatrix} g(x_0) \\ g(x_1) \\ \vdots \\ g(x_N) \end{bmatrix}}_{G}.$$

Therefore, the matrix form is equivalent to

$$\sum_{k=0}^{m} P_k Y^{k\alpha} = G. \tag{3.13}$$

In equtaion (3.11), $x = x_i$ is written for obtaining of the matrix $Y^{k\alpha}$, equation (3.14) is found by

$$D_c^{k\alpha} y(x_i) = X(x_i^{\alpha})(B^T)^k F^T A. \tag{3.14}$$

The matrix form of this equation can be written as follows

$$\underbrace{\begin{bmatrix} D_c^{k\alpha} y(x_0) \\ D_c^{k\alpha} y(x_1) \\ \vdots \\ D_c^{k\alpha} y(x_N) \end{bmatrix}}_{Y^{k\alpha}} = \underbrace{\begin{bmatrix} X(x_0^{\alpha}) \\ X(x_2^{\alpha}) \\ \vdots \\ X(x_N^{\alpha}) \end{bmatrix}}_{X^{\alpha}} \left[(B^T)^k F^T A \right]. \tag{3.15}$$

Equation (3.15) is rewritten as

$$Y^{k\alpha} = X^{\alpha}(B^T)^k F^T A. \tag{3.16}$$

Then, by writing equation (3.16) in equation (3.13), the following equation is obtained

$$\sum_{k=0}^{m} P_k X^{\alpha}(B^T)^k F^T A = G. \tag{3.17}$$

In addition, denoting

$$W = [w_{pq}] = \sum_{k=0}^{m} P_k X^{\alpha}(B^T)^k F^T, \ p, q = 0, 1, 2, \cdots, N \tag{3.18}$$

equation (3.17) is briefly written by

$$WA = G \text{ or } [W; G] = A. \tag{3.19}$$

The augmented matrix of equation (3.19) is written as follows:

$$[W; G] = \begin{bmatrix} w_{00} & w_{01} & \cdots & w_{0N} & ; & g(x_0) \\ w_{10} & w_{11} & \cdots & w_{1N} & ; & g(x_1) \\ \vdots & \vdots & \vdots & \vdots & ; & \vdots \\ w_{(N-1)0} & w_{(N-1)1} & \cdots & w_{(N-1)N} & ; & g(x_{N-1}) \\ w_{N0} & w_{N1} & \cdots & w_{NN} & ; & g(x_N) \end{bmatrix}_{(N+1)x(N+1)}$$

Now, we have to establish the new form of equation (3.19) under the following initial conditions.

$$D_c^j y(a) = \lambda_j, \ j = 0, 1, 2, \cdots, t-1. \tag{3.20}$$

Using equation (3.14) in condition equation (3.20), the following equation is found

$$X^{\alpha}(a)(B^T)^j F^T A = \lambda_j. \tag{3.21}$$

The matrix form of equation (3.21) is written as

$$U_j A = \lambda_j \text{ or } [U_j; \lambda_j], \ j = 0, 1, 2, \cdots, t-1 \tag{3.22}$$

where

$$U_j = X^{\alpha}(a)(B^T)^j F^T \equiv [u_{j0}\ u_{j1}\ u_{j2}\ \cdots\ u_{jN}]\,. \tag{3.23}$$

After the initial conditions were added, the formed augmented matrix can be written as follows:

$$[\widetilde{W};\widetilde{G}] = \left[\begin{array}{ccccc} w_{00} & w_{01} & \cdots & w_{0N} & ; & g(x_0) \\ w_{10} & w_{11} & \cdots & w_{1N} & ; & g(x_1) \\ \vdots & \vdots & \vdots & \vdots & ; & \vdots \\ w_{(N-t-1)0} & w_{(N-t-1)1} & \cdots & w_{(N-t-1)N} & ; & g(x_{N-t-1}) \\ w_{(N-t)0} & w_{(N-t)1} & \cdots & w_{(N-t)N} & ; & g(x_{N-t}) \\ u_{00} & u_{01} & \cdots & u_{0N} & ; & \lambda_0 \\ u_{10} & u_{11} & \cdots & u_{1N} & ; & \lambda_1 \\ \vdots & \vdots & \vdots & \vdots & ; & \vdots \\ u_{(t-1)0} & u_{(t-1)1} & \cdots & u_{(t-1)N} & ; & \lambda_{t-1} \end{array}\right]$$

This augmented matrix is briefly written as

$$\widetilde{W}A = \widetilde{G} \tag{3.24}$$

If $det(\widetilde{W}) \neq 0$, A the matrix of Hermite coefficients which is solution of equation (3.24) is found by

$$A = (\widetilde{W})^{-1}\widetilde{G}. \tag{3.25}$$

Finally, substitution of these coefficients into the truncated Hermite series gives the desired solution of the form:

$$y(x) = \sum_{n=0}^{N} a_n H_n(x^{\alpha}). \tag{3.26}$$

4. Numerical Examples and Comparisons

In this section, the fractional order linear differential equation is solved for two test examples with constant and variable coefficient. Obtained solutions were compared with the exact solutions and some numerical results to show the accuracy and efficiency of the numerical method. All numerical computations were performed by using MatlabR2009b.

Example 1
As the first example, we consider the following the fractional order linear differential equation with constant coefficient:

$$D^2 y(x) + D_c^{\frac{3}{2}} y(x) + y(x) = x^2 + 2 + 4\sqrt{\frac{x}{\pi}},\ x \in [0,1] \tag{4.1}$$

and initial condition

$$y(0) = 0. \tag{4.2}$$

The exact solution for this example was given by $y(x) = x^2$. By applying the HCM for $N = 4$ and $\alpha = 0.5$, the approximate solution was obtained. In Table 1, the obtained results were compared with exact solutions and fractional finite difference method [2] solutions, also the absolute errors were calculated at the selected points of the given interval for Example 1.

Table 1: Comparison of HCM solutions with exact solutions for Example 1.

x	HCM	Exact Solution	Absolute Error	FFDM[2]
0.1	0.009999999999999	0.010000000000000	1.1588E-15	3.20056E-15
0.2	0.039999999999998	0.040000000000000	1.7208E-15	9.29812E-15
0.3	0.089999999999998	0.090000000000000	2.0539E-15	1.48076E-14
0.4	0.159999999999998	0.160000000000000	2.4980E-15	1.89848E-14
0.5	0.249999999999997	0.250000000000000	2.8866E-15	2.29816E-14
0.6	0.359999999999997	0.360000000000000	3.2196E-15	2.80331E-14
0.7	0.489999999999997	0.490000000000000	3.1641E-15	3.30291E-14
0.8	0.639999999999996	0.640000000000000	3.7748E-15	3.79696E-14
0.9	0.809999999999996	0.810000000000000	3.7748E-15	4.29656E-14
1	0.999999999999996	1.000000000000000	4.2188E-15	3.9968E-14

Example 2

As the second example, we consider the following the fractional order linear differential equation which variable coefficient:

$$D_c^{\frac{3}{2}} y(x) - x^{\frac{3}{2}} y(x) = 4\sqrt{\frac{x}{\pi}} - x^{\frac{7}{2}},\ x \in [0,1] \tag{4.3}$$

and initial condition

$$y(0) = 0. \tag{4.4}$$

The exact solution for this example was given by $y(x) = x^2$. By applying the HCM for $N = 4$ and $\alpha = 0.5$, the approximate solution was obtained. In Table 2, the obtained results were compared with exact solutions and fractional finite difference method [2] solutions, also the absolute errors were calculated at the selected points of the given interval for Example 2.

Table 2: Comparison of HCM solutions with exact solutions for Example 2.

x	HCM	Exact Solution	Absolute Error	FFDM[2]
0.1	0.010000000000002	0.010000000000000	2.1164E-15	2.08167E-17
0.2	0.040000000000002	0.040000000000000	2.3037E-15	2.01228E-16
0.3	0.090000000000002	0.090000000000000	2.4009E-15	5.96745E-16
0.4	0.160000000000002	0.160000000000000	2.0539E-15	9.99201E-16
0.5	0.250000000000002	0.250000000000000	1.6653E-15	1.9984E-15
0.6	0.360000000000001	0.360000000000000	1.1102E-15	2.9976E-15
0.7	0.490000000000001	0.490000000000000	9.4369E-16	2.9976E-15
0.8	0.640000000000000	0.640000000000000	0	3.9968E-15
0.9	0.809999999999999	0.810000000000000	6.6613E-16	6.10623E-15
1	0.999999999999999	1.000000000000000	1.1102E-15	6.99441E-15

References

[1] I. Podlubny, Fractional differntial equations, Academic Press, New York, (1999).

[2] R.B. Albadarneh , I.M. Batiha ,M. Zurigat, Numerical solutions for linear fractional differential equations of order $1 < \alpha < 2$ using finite difference method (FFDM), J. Math. Computer Sci., Vol.16, N.1, pp.103-111 (2016).

[3] O. Abdulaziz, I. Hashim, S. Momani, Solving system of fractional differential equations by homotopy perturbation method, Phys. Lett. A, Vol.372, N.4, pp.451-459 (2008).

[4] Daftardar-Gejji V., Jafari H., Solving a multi-order fractional differential equation using Adomian decomposition, Appl. Math. Comput., Vol.189, N.1, pp.541-548 (2007).
[5] Hashim I., Abdulaziz O., Momani S., Homotopy analysis method for fractional IVPs, Commun. Nonlinear Sci. Numer. Simul., Vol.14, N.3, pp.674-684, (2009).
[6] G. Wu, E.W.M. Lee, Fractional variational iteration method and its application, Phys. Lett., Vol.374, N.25, pp.2506-2509 (2010).

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