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ON SOME PROPERTIES OF INTUITIONISTIC FUZZY EQUIVALENCE RELATIONS

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Abstract. Intuitionistic fuzzy sets, introduced by Atanassov, have gained successful applications in various fields. Then the intuitionistic fuzzy relations were developed. In this paper, firstly, the cover of an intuitionistic fuzzy sets and then the tolerance relations with their some property were studied. Finally some properties of intuitionistic fuzzy relations and intuitionistic fuzzy equivalence class w.r.t. IFRs were studied .

1. Introduction

The concept of fuzzy sets was introduced in Zadeh [5] as an extension of crisp sets, the usual two-valued sets in ordinary set theory, by expanding the truth value set to the real unit interval $[0, 1]$. In crisp set theory, if the membership degree of an element x is 1 then the nonmembership degree is 0. In fuzzy set theory, if he membership degree of an element x is $\mu(x)$ then the nonmembership degree is $1 - \mu(x)$ and thus it is fixed.

Intuitionistic fuzzy sets have been introduced by Atanassov in 1983 [1] and form an extension of fuzzy sets. While the nonmembership degree for each element of the universe is fix in fuzzy set theory, in intuitionistic fuzzy set theory, nonmembership degree is a more or less independet degree; the only condition that it is smaller then $1 - membership\ degree$. So, If X an universe then there exist two membersip and nonmembership degree for each $x \in X$, respectively $\mu_A(x)$ and $\nu_A(x)$ such that $0 \leq \mu_A(x) + \nu_A(x) \leq 1$.

Let X and Y be universes. The intuitionistic fuzzy relation between X and Y is defined an intuitionistic fuzzy set in $X \times Y$ in Atanassov [2, 3]. If R is an intuitionistic fuzzy relation between X and Y, $x \in X$, $y \in Y$ then the degree to which x is in relation R with y is denoted by $\mu_R(x, y)$.

Definition 1.1. Let L=[0,1] then

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$$L^* = \{(x_1, x_2) \in [0,1]^2 : x_1 + x_2 \leq 1\}$$
is a lattice with $(x_1, x_2) \leq (y_1, y_2) :\Longleftrightarrow$ "$x_1 \leq y_1$ *and* $x_2 \geq y_2$".
The units of this lattice are denoted by $0_{L^*} = (0,1)$ and $1_{L^*} = (1,0)$.

The lattice $(L^*, \leq)$ is a complete lattice: for each $A \subseteq L^*$ $supA = (sup\{x \in [0,1] : (y \in [0,1])((x,y) \in A)\}$,
$inf\{y \in [0,1] :(x \in [0,1])((x,y) \in A)\})$ and $infA = (inf\{x \in [0,1](y \in [0,1])((x,y) \in A)\}, sup\{y \in [0,1] \; (x \in [0,1])((x,y) \in A)\})$.

As is well known, every lattice $(L^*, \leq)$ has an equivalent definition as an algebraic structure $(L, \wedge, \vee)$ where the meet operator $\wedge$ and the join operator $\vee$ are linked the ordering " $\leq$ " by the following equivalence , for $a, b \in L$:

$$a \leq b \Longleftrightarrow a \vee b = b \Longleftrightarrow a \wedge b = a$$

The operators $\wedge$ and $\vee$ on $(L^*, \leq)$ are defined as follows ,for $(x_1, y_1), (x_2, y_2) \in L^*$:
$(x_1, y_1) \wedge (x_2, y_2) = (\min(x_1, x_2), \max(y_1, y_2))$
$(x_1, y_1) \vee (x_2, y_2) = (\max(x_1, x_2), \min(y_1, y_2))$

Definition 1.2. [1]An intuitionistic fuzzy set (shortly IFS) on a universe X is an object of the form

$$A = \{< x, \mu_A(x), \nu_A(x) >: x \in X\}$$

where $\mu_A(x)(\in [0,1])$ is called the"degree of membership of x in A ",$\nu_A(x)(\in [0,1])$is called the " degree of non- membership of x in A ",and where μ_A and ν_A satisfy the followıng condition:

$(x \in X)$ $(\mu_A(x) + \nu_A(x) \leq 1)$.

The class of IFSs on a universe X will be denoted $IFR(X)$.

An IFS A is said to be contained in an IFS B (notation $A \subseteq B$) if and only if, for all $x \in X : \mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$.

The intersection (resp.the union) of two IFSs A and B on X is definition as the IFSs

$$A \cap B = \{< x, \mu_A(x) \wedge \mu_B(x), \nu_A(x) \vee \nu_B(x) >: x \in X\}$$

(resp. $A \cup B = \{< x, \mu_A(x) \vee \mu_B(x), \nu_A(x) \wedge \nu_B(x) >: x \in X\}$)

2. Intuitionistic Fuzzy Tolerance Relation

Definition 2.1. Let X be a set $A \in IFS(X)$ and $R \in IFR(X)$. If the following conditions hold then R is called an intuitionistic fuzzy tolerance relation on A and writen as $R \in IFTR(A)$.

- TR1 For all $x, y \in X$, $\mu_R(x,y) = \mu_R(y,x)$, $\nu_R(x,y) = \nu_R(y,x)$
- TR2 For all $x, y \in X$, $\mu_R(x,y) \leq \mu_R(x,x) \wedge \mu_R(y,y)$, $\nu_R(x,y) \geq \nu_R(x,x) \vee \nu_R(y,y)$
- TR3 For all $x \in X$, $\mu_R(x,x) = \mu_A(x)$, $\nu_R(x,x) = \nu_A(x,x)$

If the condition $TR1$ holds, then R is called symmetric relation.

Definition 2.2. Let $A \in IFS(X)$ and $K = \{C : C \in IFS(X)\}$. If $A = \bigcup_{C \in K} C$ then K is called a intuitionistic fuzzy cover for A.

We write $IFC(A) = \{K : K \text{ is a cover for } A\}$

From the above definition, $\mu_A(x) = \sup_{C \in K} \mu_C(x)$ therefore $\mu_C(x) \leq \mu_A(x)$ and $\nu_A(x) = \inf_{C \in K} \nu_C(x)$ therefore $\nu_C(x) \geq \nu_A(x)$, for all $x \in X$. From the above discussion we have $\mu_C \subset \mu_A$ and $\nu_C \supset \nu_A$. Therefore, $C \subset A$ for all $C \in K$ and $0 \leq \sup_{C \in K} \mu_C(x) + \inf_{C \in K} \nu_C(x) \leq 1$.

Definition 2.3. Let X be a set, A$\in IFS(X)$, $K \in IFC(A)$. If we define

$$R_K = \{< (x, y), \sup_{C \in K} (\mu_C(x) \wedge \mu_C(x)), \inf_{C \in K} (\nu_C(x) \vee \nu_C(x)) >: x, y \in X\}$$

then it is clear that $R_K \in IFS(X \times X)$ from the inequality $0 \leq \sup_{C \in K} \mu_C(x) + \inf_{C \in K} \nu_C(x) \leq 1$.

Proposition 1. R_K is satisfied the intuitionistic fuzzy tolerance relation axioms TR1, TR2, TR3.

Proof. From the definition of R_K, TR1 and TR2 is easily satisfied.

For all $x, y \in X$,

$\mu_{R_C}(x, y) = \sup_{C \in K} (\mu_C(x) \wedge \mu_C(y)) \leq \mu_A(x) \wedge \mu_A(y)$

$= \sup_{C \in K} (\mu_C(x) \wedge \mu_C(x)) \wedge \sup_{C \in K} (\mu_C(y) \wedge \mu_C(y))$

$=\mu_{R_C}(x, x) \wedge \mu_{R_C}(y, y)$

$\nu_{R_C}(x, y) = \inf_{C \in K} (\nu_C(x) \vee \nu_C(y)) \geq \nu_A(x) \vee \nu_A(y)$

$= \sup_{C \in K} (\nu_C(x) \vee \nu_C(x)) \wedge \sup_{C \in K} (\nu_C(y) \vee \nu_C(y))$

$=\nu_{R_C}(x, x) \vee \nu_{R_C}(y, y)$

therefore TR2 satisfied for R_C. □

Definition 2.4. Let $R \in IFR(X), c \in IFS(X)$. If

$$\begin{aligned} \mu_C(x) \wedge \mu_C(y) &\leq \mu_R(x, y) \\ \nu_C(x) \vee \nu_C(y) &\geq \nu_R(x, y) \end{aligned}$$

for all $x, y \in X$ then C is called an intuitionistic fuzzy pre-class of R. It is clear that the family of the intuitionistic fuzzy pre-class of R is partially ordered set with respect to including relation " $\subset$ ". The maximal intuitionistic fuzzy pre-class is called class of R. We write the family of the intuitionistic fuzzy pre-class of R with $K(R)$.

Proposition 2. Let X be a set, $A \in IFS(X)$, $\overline{K} = \{K : K \text{ is a cover for } A\}$. The relation $\varphi : \overline{K} \to IFR(X)$ defined by $K \mapsto R_K$ is a well-defined relation.

Theorem 2.5. *Let X be a set, $R \in IFR(X)$ is a intuitionistic fuzzy tolerance relation on X then the function φ defined at the above proposition is surjection.*

Proposition 3. Let $R \in IFTR_A(X)$ then there exist an intuitionistic fuzzy cover K of A such that $R = R_K$.

Proposition 4. If $K = K(R)$ is a family of intuitionistic fuzzy class of R then $R_{K(R)} = R$.

Example 2.6. Let $X = \{x_1, x_2, x_3, x_4\}$, $A = \{< x, 1, 0 >: x \in X\}$. If we define $K_1 = \{C_1, C_2, C_3, C_4\}$, $K_2 = \{C_1, C_2, C_3, C_5\}$ as the following

$$K_1 = \begin{array}{ccccc} & C_1 & C_2 & C_3 & C_4 \\ x_1 & (1.0, 0.0) & (0.1, 0.7) & (0.3, 0.4) & (0.6, 0.2) \\ x_2 & (0.1, 0.7) & (1.0, 0.0) & (0.5, 0.1) & (0.7, 0.2) \\ x_3 & (0.3, 0.4) & (0.5, 0.1) & (1.0, 0.0) & (0.3, 0.6) \\ x_4 & (0.6, 0.2) & (0.7, 0.2) & (0.3, 0.6) & (1.0, 0.0) \end{array}$$

$$K_1 = \begin{array}{ccccc} & C_1 & C_2 & C_3 & C_5 \\ x_1 & (1.0, 0.0) & (0.1, 0.7) & (0.3, 0.4) & (0.6, 0.2) \\ x_2 & (0.1, 0.7) & (1.0, 0.0) & (0.5, 0.1) & (0.7, 0.2) \\ x_3 & (0.3, 0.4) & (0.5, 0.1) & (1.0, 0.0) & (0.1, 0.8) \\ x_4 & (0.6, 0.2) & (0.7, 0.2) & (0.3, 0.6) & (1.0, 0.0) \end{array}$$

Then the result of the intuitionistic fuzzy relation is as follows;

$$R_{K_1} = R_{K_2} = \begin{array}{ccccc} & x_1 & x_2 & x_3 & x_4 \\ x_1 & (1.0, 0.0) & (0.6, 0.2) & (0.3, 0.4) & (0.6, 0.2) \\ x_2 & (0.6, 0.2) & (1.0, 0.0) & (0.5, 0.1) & (0.7, 0.2) \\ x_3 & (0.3, 0.4) & (0.5, 0.1) & (1.0, 0.0) & (0.5, 0.2) \\ x_4 & (0.6, 0.2) & (0.7, 0.2) & (0.5, 0.2) & (1.0, 0.0) \end{array}$$

in addition, if $K_3 = K_1 \cup K_1$ then the result is

$$K_3 = \begin{array}{cccccc} & C_1 & C_2 & C_3 & C_4 & C_5 \\ x_1 & (1.0, 0.0) & (0.1, 0.7) & (0.3, 0.4) & (0.6, 0.2) & (0.6, 0.2) \\ x_2 & (0.1, 0.7) & (1.0, 0.0) & (0.5, 0.1) & (0.7, 0.2) & (0.7, 0.2) \\ x_3 & (0.3, 0.4) & (0.5, 0.1) & (1.0, 0.0) & (0.3, 0.6) & (0.1, 0.8) \\ x_4 & (0.6, 0.2) & (0.7, 0.2) & (0.3, 0.6) & (1.0, 0.0) & (1.0, 0.0) \end{array}$$

and $R_{K_1} = R_{K_2} = R_{K_3}$ therefore, it can be said that the function φ should not be always need not to be a monomorphism.

3. Intuitionistic Fuzzy Equivalence Relation

Definition 3.1. Let X be a set, $R \in IFR(X)$.

ER1 For every $x \in X$,

$$\mu_R(x, x) = 1$$

$$\nu_R(x, x) = 0$$

then R is called an intuitionistic fuzzy reflexsive.

ER2 For every $x, y \in X$,

$$\mu_R(x, y) \leq \mu_R(y, x)$$

$$\nu_R(x, y) \geq \nu_R(y, x)$$

then R is called an intuitionistic fuzzy symmetric.

ER3 For every $x, y, z \in X$,

$$\mu_R(x, y) \wedge \mu_R(y, z) \leq \mu_R(x, z)$$

$$\nu_R(x, y) \vee \nu_R(y, z) \geq \nu_R(x, z)$$

then R is called an intuitionistic fuzzy transitive.

If an intuitionistic fuzzy relation satisfies the previous properties then it is called an intuitionistic fuzzy equivalence relation ($IFER(X)$).

Example 3.2. Let $X = \{x_1, x_2, x_3, x_4\}$, $A = \{< x, 1, 0 >: x \in X\}$. If we define the intuitionistic relation is defined as follows

$$R = \begin{array}{ccccc} & x_1 & x_2 & x_3 & x_4 \\ x_1 & (1.0, 0.0) & (0.6, 0.2) & (0.3, 0.4) & (0.6, 0.2) \\ x_2 & (0.6, 0.2) & (1.0, 0.0) & (0.3, 0.4) & (0.6, 0.2) \\ x_3 & (0.3, 0.4) & (0.3, 0.4) & (1.0, 0.0) & (0.3, 0.4) \\ x_4 & (0.6, 0.2) & (0.6, 0.2) & (0.3, 0.4) & (1.0, 0.0) \end{array}$$

then R$\in IFER(X)$.

Remark 3.3. If an intuitionistic fuzzy tolerance relation R is satisfied $ER3$ then $R \in IFER(X)$.

Theorem 3.4. *Let X be a set, $R \in IFR(X)$ and K be an intuitionistic fuzzy cover of A. $R_K \in IFER(X)$ if and only if for every $C_1, C_2 \in K$ there exists $C \in K$ such that*

$$h_1(C_1, C_2) \wedge \mu_{C_1}(x) \wedge \mu_{C_2}(y) \leq \mu_C(x) \wedge \mu_C(y)$$

$$h_2(C_1, C_2) \vee \nu_{C_1}(x) \vee \nu_{C_2}(y) \geq \nu_C(x) \vee \nu_C(y)$$

where $h_1(C_1, C_2) = \sup\limits_{x \in X} \mu_{C_1}(x) \wedge \mu_{C_2}(x)$ *and* $h_2(C_1, C_2) = \inf\limits_{x \in X} \nu_{C_1}(x) \vee \nu_{C_2}(x)$

Proof. Let K be a intuitionistic fuzzy cover of A that is satisfies the above properties. For every $x \in X$,

$\mu_{R_K}(x, x) = \sup\limits_{C \in K} \mu_C(x) \wedge \mu_C(x) = \sup\limits_{C \in K} \mu_C(x) \geq \sup\limits_{C \in K} \mu_C(x) \wedge \mu_C(y) = \mu_{R_K}(x, y)$

therefore $\mu_{R_K}(x, x) \geq \sup\limits_{y \in X} \mu_{R_K}(x, y)$.

$\nu_{R_K}(x, x) = \inf\limits_{C \in K} \nu_C(x) \vee \nu_C(x) = \inf\limits_{C \in K} \nu_C(x) \leq \inf\limits_{C \in K} \nu_C(x) \vee \nu_C(x) = \nu_{R_K}(x, y)$

therefore $\nu_{R_K}(x, x) \leq \nu_{R_K}(x, y)$.

thus R is an intuitionistic fuzzy reflexive relation.

For $x, y, z \in X$

$\mu_{R_K}(x, y) \wedge \mu_{R_K}(y, z) = \sup\limits_{C \in K} \mu_C(x) \wedge \mu_C(y) \wedge \sup\limits_{C \in K} \mu_C(y) \wedge \mu_C(z)$

$= \sup\limits_{C_1, C_2 \in K} (\mu_{C_1}(x) \wedge \mu_{C1}(y) \wedge \mu_{C_2}(y) \wedge \mu_{C_2}(z))$

$= \sup\limits_{C_1, C_2 \in K} (\mu_{C1}(y) \wedge \mu_{C_2}(y) \wedge \mu_{C_1}(x) \wedge \mu_{C_2}(z))$

$= \sup\limits_{C_1, C_2 \in K} ((\sup\limits_{C_1, C_2 \in K} \mu_{C1}(y) \wedge \mu_{C_2}(y)) \wedge \mu_{C_1}(x) \wedge \mu_{C_2}(z))$

$= \sup\limits_{C_1, C_2 \in K} (h_1(C_1, C_2) \wedge \mu_{C_1}(x) \wedge \mu_{C_2}(z))$

$\leq \sup\limits_{C \in K} \mu_C(x) \wedge \mu_C(z) = \mu_{R_K}(x, z)$

i.e. $\mu_{R_K}(x, y) \wedge \mu_{R_K}(y, z) \leq \mu_{R_K}(x, z)$ for every $x, y, z \in X$.

and

$\nu_{R_K}(x, y) \vee \nu_{R_K}(y, z) = \inf\limits_{C \in K} \nu_C(x) \vee \nu_C(y) \vee \inf\limits_{C \in K} \nu_C(y) \vee \nu_C(z)$

$= \inf\limits_{C_1, C_2 \in K} (\nu_{C_1}(x) \vee \nu_{C1}(y) \vee \nu_{C_2}(y) \vee \nu_{C_2}(z))$

$= \inf\limits_{C_1, C_2 \in K} (\nu_{C1}(y) \vee \nu_{C_2}(y) \vee \nu_{C_1}(x) \vee \nu_{C_2}(z))$

$= \inf\limits_{C_1, C_2 \in K} ((\inf\limits_{C_1, C_2 \in K} \nu_{C1}(y) \vee \nu_{C_2}(y)) \vee \nu_{C_1}(x) \vee \nu_{C_2}(z))$

$= \inf\limits_{C_1, C_2 \in K} (h_1(C_1, C_2) \vee \nu_{C_1}(x) \vee \nu_{C_2}(z))$

$\geq \inf\limits_{C \in K} \nu_C(x) \vee \nu_C(z) = \mu_{R_K}(x, z)$

i.e.$\nu_{R_K}(x,y) \vee \nu_{R_K}(y,z) \geq \mu_{R_K}(x,z)$ for every $x,y,z \in X$.
Therefore $R_K \in IFER(X)$
Conversely,
let $R_K \in IFER(X)$ then $\mu_{R_K}(x,y) \wedge \mu_{R_K}(y,z) \leq \mu_{R_K}(x,z)$ and $\nu_{R_K}(x,y) \vee \nu_{R_K}(y,z) \geq \mu_{R_K}(x,z)$ for every $x,y,z \in X$. Hence

$$\sup_{C_1,C_2\in K}(\mu_{C_1}(x) \wedge \mu_{C1}(y) \wedge \mu_{C_2}(y) \wedge \mu_{C_2}(z)) \leq \sup_{C\in K}(\mu_C(x) \wedge \mu_C(z))$$

therefore there exist $C \in K$ for $C_1, C_2 \in K$ such that $\mu_{C_1}(x) \wedge \mu_{C1}(y) \wedge \mu_{C_2}(y) \wedge \mu_{C_2}(z) \leq \sup_{C\in K}(\mu_C(x) \wedge \mu_C(z))$ then we get

$$h_1(C_1,C_2)\wedge\mu_{C_1}(x)\wedge\mu_{C_2}(z) = \sup_{y\in X}(\mu_{C_1}(x)\wedge\mu_{C1}(y)\wedge\mu_{C_2}(y)\wedge\mu_{C_2}(z)) \leq \mu_C(x)\wedge\mu_C(z)$$

and

$$\inf_{C_1,C_2\in K}(\nu_{C_1}(x) \vee \nu_{C_1}(y) \vee \nu_{C_2}(y) \vee \nu_{C_2}(z)) \geq \inf_{C\in K}(\nu_C(x) \vee \nu_C(z))$$

therefore there exist $C \in K$ for $C_1, C_2 \in K$ such that $\nu_{C_1}(x) \vee \nu_{C1}(y) \vee \nu_{C_2}(y) \vee \nu_{C_2}(z) \geq \inf_{C\in K}(\nu_C(x) \vee \nu_C(z))$, then we get

$$h_2(C_1,C_2)\vee\nu_{C_1}(x)\vee\nu_{C_2}(z) = \inf_{y\in X}(\nu_{C_1}(x)\vee\nu_{C1}(y)\vee\nu_{C_2}(y)\vee\nu_{C_2}(z)) \geq \nu_C(x)\vee\nu_C(z)$$

□

Definition 3.5. Let X be a set, $A \in IFS(X)$, $R \in IFER(X)$ and $a \in X$.

$$[a]_R = \{< x, \mu_{[a]_R}, \nu_{[a]_R} >: x \in X\}$$

where $\mu_{[a]_R} = \mu_R(a,x)$, $\nu_{[a]_R} = \nu_R(a,x)$ is called an intuitionistic fuzzy equivalence class of a w.r.t R.

From the definition of $[a]_R$ it is clear that $0 \leq \mu_{[a]_R} + \nu_{[a]_R} \leq 1$ i.e. $[a]_R \in IFR(X)$.

Example 3.6. If the intuitionistic fuzzy relation will be used like the the previous example the result of the intuitionistic fuzzy equivalence classes will be as follows,

$[x_1]_R = \{(x_1,1,0),(x_2,0.6,0.2),(x_3,0.3,0.4),(x_4,0.6,0.2)\}$
$[x_2]_R = \{(x_1,0.6,0.2),(x_2,1,0),(x_3,0.3,0.4),(x_4,0.6,0.2)\}$
$[x_3]_R = \{(x_1,0.3,0.4),(x_2,0.3,0.4),(x_3,1,0),(x_4,0.3,0.4)\}$
$[x_4]_R = \{(x_1,0.6,0.2),(x_2,0.6,0.2),(x_3,0.3,0.4),(x_4,1,0)\}$

Theorem 3.7. *Let X be a set, $R \in IFR(X)$. C is an intuitionistic fuzzy equivalence class of R if and only if C is an intuitionistic fuzzy class of R.*

Proof. First we show that an intuitionistic fuzzy equivalence class of R is an intuitionistic fuzzy pre-class of R. Let $x,y \in X$,
$\mu_{[a]_R}(x) \wedge \mu_{[a]_R}(y) = \mu_R(a,x) \wedge \mu_R(a,y) = \mu_R(x,a) \wedge \mu_R(a,y) \leq \mu_R(x,y)$
and
$\nu_{[a]_R}(x) \vee \nu_{[a]_R}(y) = \nu_R(a,x) \vee \nu_R(a,y) = \nu_R(x,a) \vee \nu_R(a,y) \geq \nu_R(x,y)$
therefore $[a]_R$ is an intuitionistic fuzzy pre-class of R.
Let C be an intuitionistic fuzzy classs of R then there exist an $a \in X$ such that $\mu_C(x) \leq \mu_C(a)$ and $\nu_C(x) \geq \nu_C(a)$ therefore $\mu_C(x) = \mu_C(a) \wedge \mu_C(x) \leq \mu_R(a,x) = \mu_{[a]_R}(x)$ and $\nu_C(x) = \nu_C(a) \vee \nu_C(x) \geq \nu_R(a,x) = \nu_{[a]_R}(x)$

Conversely,

let $[a]_R$ is an intuitionistic fuzzy equivalence class of R and let $[a]_R \subset C$ then

$\mu_C(a) \geq \mu_{[a]_R}(a) = \mu_R(a,a) \geq \sup_{x\in X} \mu_R(a,x)$ and $\nu_C(a) \leq \nu_{[a]_R}(x) = \nu_R(a,a) \leq \inf_{x\in X} \nu_R(a,x)$

if we use this result,

$\mu_{[a]_R}(x) \leq \mu_C(x) = \mu_C(x) \wedge \mu_C(a) \leq \mu_R(a,x) = \mu_{[a]_R}(x)$

and

$\nu_{[a]_R}(x) \geq \nu_C(x) = \nu_C(x) \vee \nu_C(a) \geq \nu_R(a,x) = \nu_{[a]_R}(x)$ therefore we get $[a]_R = C$. □

Proposition 5. Let X be a set, $R \in IFER(X)$ and let $a, b \in X$, $[a]_R = [b]_R$ if and only if $\mu_R(a,b) = \mu_R(a,a)$ and $\nu_R(a,b) = \nu_R(a,a)$.

Proof. Let $[a]_R = [a]_R$ then

$\mu_R(a,b) = \mu_{[a]_R}(b) = \mu_{[b]_R}(b) = \mu_R(b,b) = \mu_R(a,a)$ and

$\nu_R(a,b) = \nu_{[a]_R}(b) = \nu_{[b]_R}(b) = \nu_R(b,b) = \nu_R(a,a)$

Conversely,

let $\mu_R(a,b) = \mu_R(a,a)$ and $\nu_R(a,b) = \nu_R(a,a)$ then

$\mu_{[a]_R}(x) = \mu_R(a,x) \geq \mu_R(a,b) \wedge \mu_R(b,x) = \mu_R(b,x) = \mu_{[b]_R}(x)$

therefore $\mu_{[a]_R}(x) \geq \mu_{[b]_R}(x)$.

From the same way we get $\mu_{[b]_R}(x) \geq \mu_{[a]_R}(x)$ then $\mu_{[b]_R}(x) = \mu_{[a]_R}(x)$

and

$\nu_{[a]_R}(x) = \nu_R(a,x) \leq \nu_R(a,b) \wedge \nu_R(b,x) = \nu_R(b,x) = \nu_{[b]_R}(x)$ therefore $\nu_{[a]_R}(x) \leq \nu_{[b]_R}(x)$. From the same way we get $\nu_{[b]_R}(x) \leq \nu_{[a]_R}(x)$ then $\nu_{[b]_R}(x) = \nu_{[a]_R}(x)$ □

Proposition 6. Let X be a set, $R \in IFER(X)$ and let $a, b \in X$, $[a]_R \neq [a]_R$ then $h_1([a]_R, [b]_R) < \mu_R(x,x)$ and $h_2([a]_R, [b]_R) > \nu_R(x,x)$,for every $x \in X$.

Proof. We assume that $h_1([a]_R, [b]_R) = \mu_R(x,x)$ and $h_2([a]_R, [b]_R) = \nu_R(x,x)$,for every $x \in X$. Then $\sup_{x\in X}(\mu_{[a]_R}(x) \wedge \mu_{[b]_R}(x)) = \mu_R(x,x)$ therefore $\sup_{x\in X}(\mu_R(a,x) \wedge \mu_R(x,b)) \leq \sup_{x\in X} \mu_R(a,b) = \mu_R(a,b)$ then $\mu_R(x,x) \leq \mu_R(a,b)$

Ffrom the same way we get $\mu_R(a,b) \leq \mu_R(x,x)$ hence $\mu_R(x,x) = \mu_R(a,b)$ i.e. $\mu_{[a]_R} = \mu_{[b]_R}$

If we use the above method we get $\nu_{[a]_R} = \nu_{[b]_R}$. This is conradict to aur assumption. Therefore $h_1([a]_R, [b]_R) < \mu_R(x,x)$ and $h_2([a]_R, [b]_R) > \nu_R(x,x)$, for every $x \in X$. □

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