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GENERALIZED METRIC SPACES AND FIXED POINT THEOREMS

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ABSTRACT. The present study aims to deal with famous fixed point theorems on generalized metric spaces. In this study we give definitions of some generalized metric spaces such as b-metric, rectangular, rectangular b-metric, $b_v(s)$ and so on and state some fixed point theorems proved on these spaces. Also, we introduce new generalized metric spaces and generalize some famous fixed point theorems to our generalized metric spaces.

1. INTRODUCTION AND PRELIMINARIES

Let (E, ρ) be a metric space and S a mapping on E. If there exists a point $u \in E$ such that $Su = u$, then point u is said to be a fixed point of S and the set of all fixed points of S is denoted by $F(S)$.

Fixed point theory is one of the most important and famous theory in mathematics since it has applications to very different type of problems arise in different branches. So, uniqueness and existence problems of fixed points are also important.

Some of the well known fixed point theorems are Banach, Kannan, Ciric and Reich fixed point theorems. Banach fixed point theorem proved by Stefan Banach in 1920 guarantees that a contractive mapping (a mapping S is called contractive if there exists a $c \in [0,1)$ such that $\rho(Su, Sw) \leq c\rho(u,w)$ for all $u, w \in E$) defined on a complete metric space has a unique fixed point, see [1].

In 1968, Kannan proved another fixed point theorem for mapping satisfying

$$\rho(Su, Sw) \leq \gamma(\rho(u, Su) + \rho(w, Sw)) \tag{1.1}$$

for all $u, w \in E$ and $\gamma \in \left[0, \frac{1}{2}\right)$, see [2]. Although contractivity condition implies the uniform continuity of S, Kannan type mappings (mappings which satisfy the inequality (1.1) need not to be continuous. Also both of these theorems characterize the completeness of metric spaces.

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In 1971, Reich [4] proved the following fixed point theorem on a complete metric space: Let E be a complete metric space and S a mapping on E, satisfy following condition:

$$\rho(Su, Sw) \leq \alpha\rho(u, w) + \beta\rho(u, Su) + \gamma\rho(w, Sw)$$

for all $u, w \in E$, where α, β, γ are nonnegative constants with $\alpha + \beta + \gamma < 1$. Then S has a unique fixed point.

In 1974, Ciric [3] proved the following fixed point theorem on a complete metric space, which generalizes the Banach contraction principle: Let E be a complete metric space and $S : E \to E$ be a quasi-contractive mapping; i.e., there exists a constant $q \in [0, 1)$ such that following inequality holds for all $u, w \in E$:

$$\rho(Su, Sw) \leq q \max\{\rho(u, w), \rho(u, Su), \rho(w, Sw), \rho(u, Sw), \rho(w, Su)\}.$$

Rhoades [14] proved that a mapping which satisfies for all $u, w \in E$

$$\rho(Su, Sw) \leq \rho(u, w) - \varphi(\rho(u, w)) \tag{1.2}$$

has a unique fixed point where (E, ρ) is a complete metric space and $\varphi : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}^+ \cup \{0\}$ is a nondecreasing and continuous function such that $\varphi(t) = 0$ iff $t = 0$. Here, $\mathbb{R}$ is the set of all real numbers. A mapping which satisfies inequality (1.2) is called weakly contractive. It is clear that weakly contractive mappings can be reduced to contractive mappings by taking $\varphi(t) = ct$ for $c \in (0, 1]$.

In 1989, Bakhtin introduced the notion of b-metric spaces by adding a multiplier to triangle ineuality. In 1994, Matthews [15] introduced the notion of partial metric spaces. In this kind of spaces, self-distance of any point need not to be zero. This space is used in the study of denotational semantics of dataflow network. In 2000, Branciari [16] introduced rectangular metric space by adding four points instead of three points in triangle inequality. These three spaces are the basis of other generalized metric spaces. After all these spaces, v-generalized metric space [16], rectangular b-metric spaces [17], $b_v(s)$ metric space [13], partial b-metric space [18] and partial rectangular b-metric space [19] were introduced in recent years. Below, we give definitions of some generalized metric spaces.

Definition 1.1. [20] Let E be a nonempty set and $\rho : E \times E \to [0, \infty)$ a function. (E, ρ) is called b-metric space if there exists a real number $s \geq 1$ such that following conditions hold for all $u, w, v \in E$:

(1) $\rho(u, w) = 0 \Leftrightarrow u = w$;
(2) $\rho(u, w) = \rho(w, u)$;
(3) $\rho(u, w) \leq s[\rho(u, v) + \rho(v, w)]$.

Clearly a b-metric space with $s = 1$ is exactly a usual metric space.

Definition 1.2. [16] Let E be a nonempty set and let $\rho : E \times E \to [0, \infty)$ be a mapping. (E, ρ) is called a rectangular metric space if following conditions hold for all $u, w \in E$ and for all distinct points $c, d \in E \setminus \{u, w\}$:

(1) $\rho(u, w) = 0 \Leftrightarrow u = w$;
(2) $\rho(u, w) = \rho(w, u)$;
(3) $\rho(u, w) \leq \rho(u, c) + \rho(c, d) + \rho(d, w)$.

Definition 1.3. [17] Let E be a nonempty set and let $\rho : E \times E \to [0, \infty)$ be a mapping. (E, ρ) is called a rectangular b-metric space if there exists a real number

$s \geq 1$ such that ffollowing conditions hold for all $u, w \in E$ and for all distinct points $c, d \in E \setminus \{u, w\}$:

(1) $\rho(u, w) = 0 \Leftrightarrow u = w$;
(2) $\rho(u, w) = \rho(w, u)$;
(3) $\rho(u, w) \leq s\,[\rho(u, c) + \rho(c, d) + \rho(d, w)]$.

Definition 1.4. [16] Let E be a nonempty set and let $\rho : E \times E \to [0, \infty)$ be a mapping. (E, ρ) is called v-generalized metric space if following conditions hold for all $u, w \in E$ and for all distinct points $z_1, z_2, \ldots, z_v \in E \setminus \{u, w\}$:

(1) $\rho(u, w) = 0 \Leftrightarrow u = w$;
(2) $\rho(u, w) = \rho(w, u)$;
(3) $\rho(u, w) \leq \rho(u, z_1) + \rho(z_1, z_2) + \cdots + \rho\,(z_v, w)$.

Definition 1.5. [13] Let E be a nonempty set, $\rho : E \times E \to [0, \infty)$ a mapping and $v \in \mathbb{N}$. Then (E, ρ) is said to be a $b_v(s)$ metric space if there exists a real number $s \geq 1$ such that following conditions hold for all $u, w \in E$ and for all distinct points $z_1, z_2, \ldots, z_v \in E \setminus \{u, w\}$:

1. $\rho(u, w) = 0 \Leftrightarrow u = w$;
2. $\rho(u, w) = \rho(w, u)$;
3. $\rho(u, w) \leq s[\rho(u, z_1) + \rho(z_1, z_2) + \cdots + \rho\,(z_v, w)]$.

This metric space can be reduced to v-generalized metric space by taking $s = 1$, rectangular metric space by taking $v = 2$ and $s = 1$, rectangular b-metric space by taking $v = 2$, b-metric space by taking $v = 1$ and usual metric space by taking $v = s = 1$.

Definition 1.6. [15] Let E be a nonempty set and $\rho : E \times E \to [0, \infty)$ a mapping. (E, ρ) is called partial metric space if following conditions hold for all $u, w, v \in E$:

(1) $u = w \Leftrightarrow \rho(u, u) = \rho(u, w) = \rho(w, w)$;
(2) $\rho(u, u) \leq \rho(u, w)$;
(3) $\rho(u, w) = \rho(w, u)$;
(4) $\rho(u, w) \leq \rho(u, v) + \rho(v, w) - \rho(v, v)$.

It is clear that every metric space is also a partial metric spaces.

Definition 1.7. [18] Let E be a nonempty set and mapping $\rho : E \times E \to [0, \infty)$ a mapping. (E, ρ) is called partial b-metric space if there exists a real number $s \geq 1$ such that following conditions hold for all $u, w, v \in E$:

(1) $u = w \Leftrightarrow \rho(u, u) = \rho(u, w) = \rho(w, w)$;
(2) $\rho(u, u) \leq \rho(u, w)$;
(3) $\rho(u, w) = \rho(w, u)$;
(4) $\rho(u, w) \leq s[\rho(u, v) + \rho(v, w)] - \rho(v, v)$.

In 2017, Kamran et al. introduced following generalized metric space which they call extended b-metric space.

Definition 1.8. [21] Let E be a nonempty set and let $\theta : E \times E \to [1, \infty)$ be a function. A function $\rho_\theta : E \times E \to [0, \infty)$ is called an extended b-metric if for all $u, v, w \in E$ it satisfies:

(1) $\rho_\theta(u, w) = 0 \Leftrightarrow u = w$;
(2) $\rho_\theta(u, w) = \rho(w, u)$;
(3) $\rho_\theta(u, w) \leq \theta\,(u, w)\,[\rho_\theta(u, v) + \rho_\theta(v, w)]$.

The pair (E, ρ_θ) is called an extended b-metric space.It is clear that if $\theta(u, w) = s$ for all $u, w \in E$, then we obtain b-metric space.

In 1993, Czervik [22] proved Banach fixed theorem for contraction mappings in b-metric spaces. In 2016, Shukla proved [7] generalized Banach fixed point theorem. In 2001, Rhoades [14] proved some theorems on weakly contractive maps. For further studies please see [5, 6, 7, 8, 9, 10, 11, 12].

2. Main Results

In this section we give generalizations of some famous fixed point theorems in complete $b_v(s)$ metric space. These theorems generalize and extend many results in literature.

Theorem 2.1. *Let E be a complete $b_v(s)$ metric space and S a weakly contractive mapping on E. Then S has a unique fixed point.*

Following is Kannan fixed point theorem in $b_v(s)$ metric spaces.

Theorem 2.2. *Let E be a complete $b_v(s)$ metric space and S a Kannan type mapping on E such that $s\gamma \leq 1$. Then S has a unique fixed point.*

Now, we give Ciric fixed point theorem in $b_v(s)$ metric space. But, to prove this theorem we need to prove following definition and lemma.

Definition 2.3. Let E be a $b_v(s)$ metric space. A sequence $\{u_n\}$ is said to converge to u in the strong sense if and only if $\{u_n\}$ is Cauchy and converges to $u \in E$.

Lemma 2.4. *Let E be a $b_v(s)$ metric space. Let $\{u_n\}$ and $\{w_n\}$ be sequences in E converging to u and w in the strong sense, respectively. Then*

$$\rho(u, w) \leq s \liminf_{n\to\infty} \rho(u_n, w_n)$$

holds.

Theorem 2.5. *Let E be a complete $b_v(s)$ metric space and let S be a mapping on E such that there exists $r \in [0, 1)$ satisfying*

$$d(Su, Sw) \leq r \max\{\rho(u, w), \rho(u, Su), \rho(w, Sw), \rho(u, Sw), \rho(w, Su)\}$$

for any $u, w \in E$. If $rs < 1$, then S has a unique fixed point.

Now, motivated and inspired by generalized metric spaces, we introduce partial $b_v(s)$ metric space.

Definition 2.6. Let E be a nonempty set and $\rho : E \times E \to [0, \infty)$ be a mapping and $v \in \mathbb{N}$. Then (E, ρ) is said to be a partial $b_v(s)$ metric space if there exists a real number $s \geq 1$ such that following conditions hold for all $u, w, z_1, z_2, \ldots, z_v \in E$:

(1) $u = w \Leftrightarrow \rho(u, u) = \rho(u, w) = \rho(w, w)$;
(2) $\rho(u, u) \leq \rho(u, w)$;
(3) $\rho(u, w) = \rho(w, u)$;
(4) $\rho(u, w) \leq s[\rho(u, z_1) + \rho(z_1, z_2) + \ldots + \rho(z_{v-1}, z_v) + \rho(z_v, y)] - \sum_{i=1}^{v} \rho(z_i, z_i)$.

It is easy to see that every $b_v(s)$ metric space is a partial $b_v(s)$ metric space. However, the converse is not true in general.

Remark 2.7. In Definition 2.6;

(1) if we take $v = 2$, then we derive partial rectangular b-metric space.

(2) if we take $v = 1$, then we derive partial b-metric space.
(3) if we take $v = s = 1$, then we derive partial metric space.

Now we give an analogue of Banach contraction principle. Our proof is very different from the original proof of Banach contraction principle in usual metric space.

Theorem 2.8. *Let (E, ρ) be a complete partial $b_v\,(s)$ metric space and $S : E \to E$ be a contraction mapping, i.e., S satisfies*

$$\rho(Su, Sw) \le \lambda \rho(u, w)$$

for all $u, w \in E$, where $\lambda \in [0, 1)$. Then S has a unique fixed point $b \in S$ and $\rho(b, b) = 0$.

Now, we give an analogue of Kannan fixed point theorem.

Theorem 2.9. *Let (E, ρ) be a complete partial $b_v\,(s)$ metric space and $S : E \to E$ a mapping satisfying the following condition:*

$$\rho(Su, Sy) \le \lambda \left[\rho(u, Su) + \rho(w, Sw)\right]$$

for all $u, w \in E$, where $\lambda \in [0, \frac{1}{2})$, $\lambda \neq \frac{1}{s}$. Then S has a unique fixed point $b \in E$ and $\rho(b, b) = 0$.

Theorem 2.10. *Let (E, ρ) be a complete partial $b_v\,(s)$ metric space and $S : E \to E$ a mapping satisfying:*

$$\rho(Su, Sw) \le \lambda \max\left\{\rho(u, w), \rho(u, Su), \rho(w, Sw)\right\}$$

for all $u, w \in E$ and $\lambda \in \left[0, \frac{1}{s}\right)$. Then, S has a unique fixed point $b \in E$ and $\rho(b, b) = 0$.

If we take $s = 1$ in the definition of partial $b_v\,(s)$ metric space, then we introduce following partial v-generalized metric space.

Definition 2.11. Let E be a nonempty set and $\rho : E \times E \to [0, \infty)$ be a mapping and $v \in \mathbb{N}$. Then (E, ρ) is said to be a partial v-generalized metric space if following conditions hold for all $u, w, z_1, z_2, \ldots, z_v \in E$:

(1) $u = w \Leftrightarrow \rho(u, u) = \rho(u, w) = \rho(w, w)$;
(2) $\rho(u, u) \le \rho(u, w)$;
(3) $\rho(u, w) = \rho(w, u)$;
(4) $\rho(u, w) \le \rho(u, z_1) + \rho(z_1, z_2) + \ldots + \rho(z_{v-1}, z_v) + \rho(z_v, y) - \sum_{i=1}^{v} \rho(z_i, z_i)$.

Theorems which we prove in complete partial $b_v\,(s)$ metric space are valid in partial v-generalized metric space.

Now, we introduce $b_v(\theta)$ metric space motivated by extended b-metric space defined by Kamran et al. in 2017.

Definition 2.12. Let E be a nonempty set, $\theta : E \times E \to [1, \infty)$ a function and $v \in \mathbb{N}$. Then $\rho_\theta : E \times E \to [0, \infty)$ is called $b_v(\theta)$ metric if for all $u, z_1, z_2, ..., z_v, w \in E$, each of them different from each other, it satisfies

(1) $\rho_\theta(u, w) = 0 \Leftrightarrow u = w$;
(2) $\rho_\theta(u, w) = \rho_\theta(w, u)$;
(3) $\rho_\theta(u, w) \le \theta\,(u, w) \left[\rho_\theta(u, z_1) + \rho_\theta(z_1, z_2) + \cdots + \rho_\theta\,(z_v, w)\right]$.

The pair (E, ρ_θ) is called $b_v(\theta)$ metric space.

Now we can give Banach fixed point theorem in complete $b_v(\theta)$ metric space.

Theorem 2.13. *Let (E, ρ_θ) be a complete $b_v(\theta)$ metric space with a bounded function θ and $S : E \to E$ a contraction mapping, i.e., there exists a constant $c \in [0, 1)$ such that*

$$\rho_\theta(Su, Sw) \leq c\rho_\theta(u, w)$$

for all $u, w \in E$. Then S has a unique fixed point.

Now, we generalize Banach fixed point theorem for weakly contractive mappings in $b_v(\theta)$ metric space.

Theorem 2.14. *Let E be a complete $b_v(\theta)$ metric space and S a weakly contractive mapping on E. Then S has a unique fixed point.*

Now, we give Reich fixed point theorem in complete $b_v(\theta)$ metric space.

Theorem 2.15. *Let (E, ρ_θ) be a complete $b_v(\theta)$ metric space with a bounded function θ and $S : E \to E$ a mapping satisfying:*

$$\rho_\theta(Su, Sw) \leq \alpha\rho_\theta(u, w) + \beta\rho_\theta(u, Su) + \gamma\rho_\theta(w, Sw)$$

for all $u, w \in E$ where α, β, γ are nonnegative constants with $\alpha + \beta + \gamma < 1$ and $\Gamma_1 < \frac{1}{\Gamma_2}$ where $\Gamma_1 = \min\{\beta, \gamma\}$ and $\Gamma_2 = \max\{\theta(u, Su), \theta(Su, u)\}$. Then S has a unique fixed point. Moreover, sequence $\{u_n\}$ defined by $u_n = Su_{n-1}$ converges strongly to the unique fixed point of S.

In Reich fixed point theorem, if we get $\alpha = 0$, then we obtain following generalized Kannan fixed point theorem in $b_v(\theta)$ metric spaces.

Theorem 2.16. *Let E be a complete $b_v(\theta)$ metric space and S a mapping on E satisfying:*

$$\rho_\theta(Su, Sw) \leq \beta\rho_\theta(u, Su) + \gamma\rho_\theta(w, Sw)$$

for all $u, w \in E$ where β and γ are nonnegative constants with $\beta + \gamma < 1$ and $\Gamma_1 < \frac{1}{\Gamma_2}$ where $\Gamma_1 = \min\{\beta, \gamma\}$ and $\Gamma_2 = \max\{\theta(u, Su), \theta(Su, u)\}$. Then S has a unique fixed point.

2.1. References.

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