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WC* PARTNER CURVES IN THE EUCLIDEAN 3−SPACE

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Abstract. In this article firstly, $\{T, N, B\}$ Frenet frame and $\{N, C, W\}$ alternative frame have been introduced. Later, a new type of special curve couple have defined as WC^*-partner curves and characterizations have given for these curves according to alternative moving frame. Also, some characterizations have obtained by using the distance function on the WC^*-partner curves.

1. Introduction

Numerous studies have been carried out on the differential geometry of curves in 3−dimensional Euclidean space. Especially between the Frenet roofs at the opposite points of the curves many new theories are obtained. One of them is a Bertrand curve. The Bertrand curves firstly described by Bertrand Russell in 1850. Two curves having a common principal normal vector are called Bertrand curves. The classical characterization for Bertrand curves is that a regular curve α in $\mathbb{E}^3$ is a Bertrand curve if and only if $a\kappa(s) + b\tau(s) = 1$ holds where κ and τ are curvature and torsion of curve and a, b are constant real numbers [9]. A lot of paper have been obtained on this curve [1, 2, 5, 7, 8, 9, 10]. The other famous curve pair is the Mannheim curve pair. This curve have been defined by Mannheim with the relation $\kappa^2 + \tau^2 = w^2$ =constant. Another definition can be made as two curves α and β in $\mathbb{E}^3$ are called Manneim partner curves if the principal normal vector fields of α coincide with the binormal vector fields of β at the corresponding points of curves [4, 6, 12].

The aim of this paper is to define a new kind of associated curve pairs and give characterizations for these curves. For this purpose, we use an alternative frame on original curve and define another curve by using this frame. This new curve pair is called WC^*-partner curves. Firstly, we give a brief summary of curve theory and alternative frame. In Section 3, the definition and main characterizations related to distance function and angle function of WC^*-partner curves have been introduced.

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2. Preliminaries

Let $\alpha = \alpha(s)$ be a regular unit speed curve in the Euclidean 3−space where s measures its arc length and $T = \alpha^{'}$ its unit tangent vector, $N = \dfrac{T^{'}}{\|T'\|}$ its principal normal vector and $B = T \times N$ its binormal vector. The triple $\{T, N, B\}$ be the Franet Frame of the curve $\alpha(s)$. Then the Frenet formula of the curve is given by

$$\begin{pmatrix} T^{'}(s) \\ N^{'}(s) \\ B^{'}(s) \end{pmatrix} = \begin{pmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{pmatrix} \begin{pmatrix} T(s) \\ N(s) \\ B(s) \end{pmatrix} \tag{2.1}$$

where $\kappa(s)$ and $\tau(s)$ are curvature and torsion of α, respectively. Moreover, the Frenet vectors satisfy

$$\sigma = \frac{\kappa^2}{(\kappa^2+\tau^2)^{3/2}}(\frac{\tau}{\kappa})^{'} = \frac{H^{'}}{\kappa(1+H^2)^{3/2}} = \text{constant},\ H = \frac{\tau}{\kappa}$$

From (2.1), the unit Darboux vector W of $\alpha(s)$ the equation

$$W = \frac{1}{\sqrt{\kappa^2+\tau^2}}(\tau T + \kappa B) \tag{2.2}$$

It is obvious that the Darboux vector is vertical to the principal normal vector field N from equation (2.2). With the help of the vector fields W and N, along $\alpha(s)$, $C = W \times N$ unit vector field is defined. This frame is designation by $\{N, C, W\}$ and alternative frame to curve rather than the Frenet frame $\{T, N, B\}$. The alternative frame and derivative formula of the alternative frame is given by

$$\begin{pmatrix} N \\ C \\ W \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ \frac{-\kappa}{\sqrt{\kappa^2+\tau^2}} & 0 & \frac{\tau}{\sqrt{\kappa^2+\tau^2}} \\ \frac{\tau}{\sqrt{\kappa^2+\tau^2}} & 0 & \frac{\kappa}{\sqrt{\kappa^2+\tau^2}} \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix} \tag{2.3}$$

and

$$\begin{pmatrix} N' \\ C' \\ W' \end{pmatrix} = \begin{pmatrix} 0 & \beta & 0 \\ -\beta & 0 & \gamma \\ 0 & -\gamma & 0 \end{pmatrix} \begin{pmatrix} N \\ C \\ W \end{pmatrix} \tag{2.4}$$

where

$$\beta = \kappa\sqrt{1+H^2},\ \gamma = \frac{H^{'}}{1+H^2},\ H = \frac{\tau}{\kappa}$$

Since the principal normal vector N is common in both frames, from equations (2.1), (2.2) and (2.4) ,

$$\begin{aligned} C &= -\bar{\kappa}T + \bar{\tau}B \\ W &= \bar{\tau}T + \bar{\kappa}B \end{aligned} \tag{2.5}$$

or

$$\begin{aligned} T &= -\bar{\kappa}C + \bar{\tau}W \\ B &= \bar{\tau}C + \bar{\kappa}W \end{aligned} \tag{2.6}$$

where $\bar{\kappa} = \dfrac{\kappa}{\beta}$ and $\bar{\tau} = \dfrac{\tau}{\beta}$

A regular curve α is called a helix if the tangent lines of the curve make a constant angle with a fixed direction and a helix is characterized by the property that $\dfrac{\tau}{\kappa}$ is constant [14]. If the principal normal lines of the curve make a constant angle with a fixed direction, then the curve is called a slant helix and characterized by the equally

$$\frac{\gamma}{\beta}=\frac{H^{'}}{\kappa(1+H^2)^{3/2}}=\sigma=\text{constant}$$

[8]. Then the characterization of a slant helix according to alternative frame is given as follows.

Remark 1. *A regular curve α with alternative curvatures $\beta(s)$, $\gamma(s)$ is a slant helix if and only if $\dfrac{\gamma(s)}{\beta(s)}=$ constant.*

3. WC* PARTNER CURVES IN THE EUCLIDEAN 3-SPACE

Definition 1. *Let $\alpha=\alpha(s)$ and $\alpha^*=\alpha^*(s)$ be two regular space curves in the Euclidean 3−space with Frenet frame $\{T,N,B\}$ and $\{T^*,N^*,B^*\}$ curvatures κ,κ^* torsions τ,τ^* respectively, and let the alternative moving frames and alternative curvetures of curves be $\{N,C,W\}$, β,γ and $\{N^*,C^*,W^*\}$, β^*,γ^*, respectively. The curves α and α^* are called WC^*−partner curves or $\{\alpha,\alpha^*\}$ is caled α WC^*−curve pair if the vector fields W and C^* coincide i.e., $C^*=W$ holds at the corresponding points of the curves.*

By considering Definition 1, one can easily write the parametric representation of α^* as follow

$$\alpha^*(s)=\alpha(s)+\lambda(s)W(s) \tag{3.1}$$

where $\lambda=\lambda(s)$ is the distance function between corresponding points of the curves α and α^*. Because the vector fields W and C^* are the same, we are able to give the relationship between the alternative frames of α and α^* such as

$$\begin{aligned} N^* &= \sin\theta N+\cos\theta C \\ W^* &= \cos\theta N-\sin\theta C \end{aligned} \tag{3.2}$$

$$\begin{aligned} N &= -\sin\theta N^*+\cos\theta W^* \\ C &= \cos\theta N^*+\sin\theta W^* \end{aligned} \tag{3.3}$$

where $\theta=\theta(s)$ is the angle function between vector fields N and W^*. Now, we give some characterizations for WC^*−partner curves. Whenever we talk about the curves α and α^*, we will assume that the curves have frames and curvatures as given in Definition 1.

Theorem 1. *Let $\{\alpha,\alpha^*\}$ be WC^*−curve pair. γ is the torsion of $\{N,C,W\}$ and γ^* is the torsion $\{N^*,C^*,W^*\}$. Then the following relation exists.*

$$\frac{\gamma^*}{\gamma}=\sin\theta=\text{constant and }(\beta^*)^2+(\gamma^*)^2=\gamma^2$$

Proof. Since $W=C^*$, their derivatives are equal

$$\begin{aligned} (C^*)' &= -\beta^*N^*+\gamma^*W^* \\ W' &= -\gamma C \end{aligned}$$

from last equation and equation (3.3),

$$-\beta^*N^*+\gamma^*W^*=-\gamma(\cos\theta N^*-\sin\theta W^*)$$

$$\begin{aligned} \beta^* &= \gamma\cos\theta \\ \gamma^* &= \gamma\sin\theta \end{aligned}$$

So, we obtain that

$$\frac{\gamma^*}{\gamma} = \sin\theta$$

and

$$(\beta^*)^2 + (\gamma^*)^2 = \gamma^2$$

□

Theorem 2. *Let $\{\alpha, \alpha^*\}$ be WC^*-curve pair. $\theta = \theta(s)$ is the angle funtion between vector fields N and W^*. Then the following relation exists.*

$$\theta =_0^s \beta ds, s =_0^{s^*} \frac{\beta^*}{\lambda\cos\theta} ds^*$$

Proof. Using the equation (3.2) , we have

$$N^* = \sin\theta N + \cos\theta C$$

If we take the derivative of each side of the above equation accordin to s^*, we obtain

$$\frac{dN^*}{s^*}\frac{ds^*}{ds} = \cos\theta\frac{d\theta}{ds}N + \sin\theta N' - \sin\theta\frac{d\theta}{ds}C + \cos\theta C'$$

$$\beta^* C^* \frac{ds^*}{ds} = \cos\theta\frac{d\theta}{ds}N + \sin\theta(\beta C) - \sin\theta\frac{d\theta}{ds}C + \cos\theta(-\beta N + \gamma W)$$

from $W = C^*$, we have

$$\beta^* W\frac{ds^*}{ds} = (\cos\theta\frac{d\theta}{ds} - \beta\cos\theta)N + (\beta\sin\theta - \sin\theta\frac{d\theta}{ds})C + \gamma\cos\theta W$$

$$\beta^*\frac{ds^*}{ds} = \gamma\cos\theta \text{ and } s = s =_0^{s^*} \frac{\beta^*}{\lambda\cos\theta}ds^*$$

and

$$\cos\theta\frac{d\theta}{ds} - \beta\cos\theta = 0 \text{ and } \theta =_0^s \beta ds$$

□

Theorem 3. *Let $\{\alpha, \alpha^*\}$ be WC^*-curve pair. α^* is a slant helix if and only if*

$$\frac{\bar{\kappa} + \lambda\gamma}{\sin\theta(-\bar{\tau} - \lambda')}$$

is constant.

Proof. If we take the derivative of equation (3.1) according to s, we have

$$T^*\frac{ds^*}{ds} = T + \lambda' W + \lambda W'$$

and we use the alternative frame formulas,

$$(-\bar{\kappa}^* C^* + \bar{\tau}^* W^*)\frac{ds^*}{ds} = -\bar{\kappa}C + \bar{\tau}W + \lambda' W - \lambda(\gamma C)$$

from equation (3.2) and $W = C^*$,

$$[-\bar{\kappa}^* W + \bar{\tau}^*(\cos\theta N - \sin\theta C)]\frac{ds^*}{ds} = (-\bar{\kappa} - \lambda\gamma)C + (\bar{\tau} + \lambda')W$$

$$\begin{aligned}\bar{\kappa}^{*}\frac{ds^{*}}{ds} &= -\bar{\tau}-\lambda' \\ \bar{\tau}^{*}\frac{ds^{*}}{ds} &= \frac{\bar{\kappa}+\lambda\gamma}{\sin\theta}\end{aligned}$$

and

$$\frac{\bar{\tau}^{*}}{\bar{\kappa}^{*}} = \frac{\bar{\kappa}+\lambda\gamma}{\sin\theta(-\bar{\tau}-\lambda')}. \tag{3.4}$$

Because $\bar{\tau}^{*} = \dfrac{\tau^{*}}{\beta^{*}}$ and $\bar{\kappa}^{*} = \dfrac{\kappa^{*}}{\beta^{*}}$, $\dfrac{\bar{\tau}^{*}}{\bar{\kappa}^{*}} = \dfrac{\tau^{*}}{\kappa^{*}}$ is obtained. Then from equation (3.4), α^{*} is a slant helix if and only if

$$\frac{\bar{\kappa}+\lambda\gamma}{\sin\theta(-\bar{\tau}-\lambda')}$$

is constant. □

Theorem 4. *Let $\{\alpha,\alpha^{*}\}$ be $WC^{*}-$curve pair. α^{*} is a slant helix if and only if*

$$\frac{\gamma^{*}}{\beta^{*}} = constant$$

Proof. We know

$$(N^{*})' = \beta^{*}C^{*}$$

and

$$(W^{*})' = -\gamma^{*}C^{*}$$

using above two equations, we obtain that

$$\frac{\gamma^{*}}{\beta^{*}} = -\frac{(W^{*})'}{(N^{*})'}$$

Also if we take the derivative of equation (3.2), then we have

$$\begin{aligned}(N^{*})' &= \cos\theta\frac{d\theta}{ds}N+\sin\theta N'-\sin\theta\frac{d\theta}{ds}C+\cos\theta C' \\ &= \cos\theta\frac{d\theta}{ds}N+\sin\theta(\beta C)-\sin\theta\frac{d\theta}{ds}C+\cos\theta(-\beta N+\gamma W) \\ &= \left(\cos\theta\frac{d\theta}{ds}-\cos\theta\beta\right)N+\left(\sin\theta\beta-\sin\theta\frac{d\theta}{ds}\right)C+\gamma\cos\theta W\end{aligned}$$

and from the Theorem 3, we know that

$$\beta = \frac{d\theta}{ds}$$

so we obtain that

$$(N^{*})' = \gamma\cos\theta W \tag{3.5}$$

Similarly if we take the derivative of equation (3.2)

$$\begin{aligned}(W^{*})' &= -\sin\theta\frac{d\theta}{ds}N+\cos\theta N'-\cos\theta\frac{d\theta}{ds}C-\sin\theta C' \\ &= -\sin\theta\frac{d\theta}{ds}N+\cos\theta(\beta C)-\cos\theta\frac{d\theta}{ds}C-\sin\theta(-\beta N+\gamma W) \\ &= \left(-\sin\theta\frac{d\theta}{ds}+\sin\theta\beta\right)N+\left(\cos\theta\beta-\cos\theta\frac{d\theta}{ds}\right)C-\gamma\sin\theta W\end{aligned}$$

$$(W^{*})' = -\gamma\sin\theta W \tag{3.6}$$

By proportioning the equations (3.5) and (3.6) , we get

$$\frac{\gamma^*}{\beta^*} = -\frac{-\gamma \sin\theta W}{\gamma\cos\theta W} = -\tan\theta = \text{constant}$$

□

Theorem 5. *Let $\{\alpha, \alpha^*\}$ be WC^*-curve pair. Then the following relation exists.*

$$\frac{\gamma}{\beta} = -\frac{\beta^*}{\beta\cos\theta}$$

Proof. Using alternative frame $\{N, C, W\}$, we have

$$N' = \beta C \text{ and } W' = -\gamma C$$

If we rate these two equations, we obtain,

$$\frac{\gamma}{\beta} = -\frac{W'}{N'}$$

and we know that

$$(C^*)' = -\beta^* N^* + \gamma^* W^*$$

so from the following equations

$$W = C^*$$

$$W' = -\beta^* N^* + \gamma^* W^*$$

and from equation (3.2) , we get

$$\begin{aligned} W' &= -\beta^*(\sin\theta N + \cos\theta C) + \gamma^*(\cos\theta N - \sin\theta C) \\ &= (-\beta^* \sin\theta + \gamma^* \cos\theta)\, N + (-\beta^* \cos\theta - \gamma^* \sin\theta)\, C \end{aligned} \tag{3.7}$$

If we use the equations $\beta^* = \gamma\cos\theta$ and $\gamma^* = \gamma\sin\theta$ in Theorem 2, we obtain

$$\gamma^* = \frac{\beta^*\sin\theta}{\cos\theta}$$

and if we write this equation in (3.7)

$$W' = \left(-\beta^*\sin\theta + \frac{\beta^*\sin\theta}{\cos\theta}\cos\theta\right) N + \left(-\beta^*\cos\theta - \frac{\beta^*\sin\theta}{\cos\theta}\sin\theta\right) C$$

and

$$W' = -\frac{\beta^*}{\cos\theta} C$$

Also from the following equation

$$\frac{\gamma}{\beta} = -\frac{W'}{N'} = \frac{-\dfrac{\beta^*}{\cos\theta}C}{\beta C}$$

$$\frac{\gamma}{\beta} = -\frac{\beta^*}{\beta\cos\theta}$$

So this completes the proof. □

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