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THE RELATIONSHIP BETWEEN THE CATEGORIES OF ORDERED SETS, TOPOLOGICAL SPACES, METRIC SPACES AND APPROACH SPACES

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Abstract. A common extension of topological spaces and metric spaces is named as approach spaces in 1989. If a topological space generate an approach space it is named topological one, otherwise, it is named a metric one. In this study, I search the relationship between approach spaces and metric spaces is comparable to that between topological spaces and ordered sets in consideration of the study [5].

1. Introduction

Approach spaces, introduced by Lowen [6], are a common extension of topological spaces and metric spaces. Sober approach spaces, a counterpart of sober topological spaces in the metric setting. It is proved there that a topological space is sober as an approach space, if and only if it is sober as a topological space. So, it is natural to ask what kind of metric approach spaces are sober? This study presents a complete answer to this question. The answer is a bit surprising: a metric space is sober, as an approach space, if and only if it is Smyth complete. Sober approach spaces, a counterpart of sober topological spaces in the metric setting. I will search this answer.

A metric space is Smyth complete if every forward Cauchy net in it converges in its symmetrization. Smyth completeness originated in the works of Smyth that aimed to provide a common framework for the domain approach and the metric space approach to semantics in computer science. As advocated in [6], in this paper I emphasize that the relationship between approach spaces and metric spaces is analogous to that between topological spaces and ordered sets. This point of view has proved to be fruitful, and is well in accordance with the thesis of Smyth. The thesis of Smyth is: that domains are, or should be, a prime area for the application of quasi-uniform ideas, and can help us to get the definitions right. Apart form

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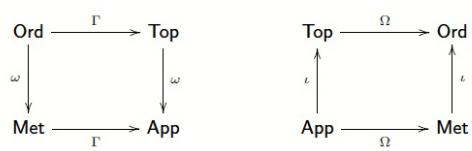
definitions and theorems are numbered, known concepts are mentioned in the text along with the reference [5].

2. Topological spaces, metric spaces, approach spaces and sobrification of metric approach spaces

A metric space is Smyth complete if every forward Cauchy net in it converges in its symmetrization [13]. Smyth completeness originated in the works of Smyth [14] that aimed to provide a common framework for the domain approach and the metric space approach to semantics in computer science. From now on topological spaces, metric spaces, approach spaces and sobrification of metric approach spaces will be studied. Detailed information can be found in [5].

An order on a set X is a map $X \times X \to \{0,1\}$ fulfilling certain requirements; a topology (identified with the corresponding closure operator) is a map $X \times 2^X \to \{0,1\}$ (the transpose of the closure operator) that satisfies certain conditions. Replacing the quantale $2 = (\{0,1\}, \wedge)$ by Lawveres quantale $(|0,\infty|^{op}, +)$ in the postulations of ordered sets and topological spaces, we obtain metric spaces and approach spaces.

The following commutative squares exhibit some basic relationship among the categories of ordered sets, topological spaces, metric spaces and approach spaces:



where,

- the involved categories are "self evident", and will be explained in the next section;
- the top row: Γ sends each ordered set $(X, \leq)$ to its Alexandroff topology, Ω sends a topological space to its specialization order;
- the bottom row: Γ sends a metric space to the corresponding metric approach space, Ω sends an approach space to its specialization metric;
- ω (in both cases) is a full and faithful functor with a right adjoint given by ι.

These facts can be found in [5]. The bottom row is an analogy of the top row in the metric setting. In particular, approach spaces extend metric spaces, via the functor Γ, in the same way as topological spaces extend ordered sets. The problem considered in this paper is to characterize those metric spaces (X, d) for which $\Gamma(X, d)$ are sober. To this end, some properties of the other functors will also be considered. The main results include:

(1) The specialization metric of a sober approach space is Yoneda complete. This is an analogy in the metric setting of the fact that the specialization order of a sober topological space is directed complete.
(2) For a metric space (X, d), the specialization metric space of the sobrification of $\Gamma(X, d)$ coincides with the Yoneda completion of (X, d).
(3) For a metric space (X, d), the approach space (X, d) is sober if and only if (X, d) is Smyth complete.

There are a lot of results about these theories. We continue our study with giving other main results are found in [5].

An ordered set is a set X together with a map $p : X \times X \to 2$ such that for all $x, y, z \in X$:

(P1) $p(x, x) = 1$,
(P2) $p(x, y) \wedge p(y, z) \leq p(x, z)$.

It is traditional to write $x \leq y$ for $p(x, y) = 1$ in order theory.

Given a topological space X, the closure operator on X induces a map $c : X \times 2^X \to 2$, given by

$$c(x, A) = \begin{cases} 1, & x \in \bar{A}, \\ 0, & x \notin \bar{A}. \end{cases}$$

This map satisfies the following conditions:

(C1) $c(x, \{x\}) = 1$,
(C2) $c(x, \emptyset) = 0$,
(C3) $c(x, A \cup B) = c(x, A) \vee c(x, B)$,
(C4) $c(x, A) \geq c(x, B) \wedge \bigwedge_{y \in B} c(y, A)$.

The condition (C4) expresses the idempotency of the closure operator. Topologies on a set X correspond bijectively to maps $c : X \times 2^X \to 2$ that satisfy the conditions (C1)(C4).

The specialization order [4] of a topological space X is the composite

$$X \times X \xrightarrow{(x,y) \to (x, \{y\})} X \times 2^X \xrightarrow{c} 2;$$

or equivalently, $x \leq y$ if $x \in \overline{\{y\}}$. Taking specialization order defines a functor

$$\Omega : \text{Top} \to \text{Ord}$$

from the category of topological spaces and continuous maps to the category **Ord** of ordered sets and order-preserving maps. The functor Ω has a left adjoint

$$\Gamma : \text{Ord} \to \text{Top}$$

that maps an ordered set $(X, \leq)$ to the space obtained by endowing X with the Alexandroff topology of $(X, \leq)$ (i.e., the topology whose closed sets are the lower subsets in $(X, \leq)$).

Definition 2.1. ([5]) A metric space is a category enriched over the Lawvere quantale $([0, \infty]^{op}, +)$. Explicitly, a metric space (X, d) consists of a set X and a map $d : X \times X \to [0, \infty]$ such that $d(x, x) = 0$ and $d(x, y) + d(y, z) \geq d(x, z)$ for all $x, y, z \in X$. The map d is called a metric, and the value $d(x, y)$ the distance from x to y.

A metric space (X, d) is symmetric if $d(x, y) = d(y, x)$ for all $x, y \in X$; separated if $x = y$ whenever $d(x, y) = d(y, x) = 0$; finitary if $d(x, y) < \infty$ for all $x, y \in X$. A metric in the usual sense is exactly a symmetric, separated and finitary one. Given a metric d on a set X, the opposite d^{op} of d refers to the metric given by $d^{op}(x, y) = d(y, x)$; the symmetrization d^{sym} of d is given by

$d^{sym}(x,y) = \max\{d(x,y), d(y,x)\}$.

A non-expansive map $f : (X,d) \to (Y,p)$ between metric spaces is a map $f : X \to Y$ such that $d(x,y) \geq p(f(x), f(y))$ for all x, y in X. Metric spaces and non-expansive maps form a category, denoted by **Met**. A map $f : (X,d) \to (Y,p)$ between metric spaces is isometric if $d(x,y) = p(f(x), f(y))$ for all $x, y \in X$.

Example (The Lawvere metric, [5]).
For any a, b in $[0,\infty]$, the Lawvere distance, $d_L(a,b)$, from a to b is defined to be the truncated minus $b \ominus a$, i.e.,

$$d_L(a,b) = b \ominus a = \max\{0, b-a\},$$

where we take by convention that $\infty - \infty = 0$ and $\infty - a = \infty$ for all $a < \infty$. It is clear that $([0,\infty], d_L)$ is a separated, non-symmetric, and non-finitary metric space.

The opposite of the Lawvere metric is denoted by d_R, i.e., $d_R(x,y) = x \ominus y$.

Given a metric space (X,d), let PX be the set of all weights of (X,d). It is obvious that PX has the following properties:

(W1) For each $x \in X, d(-,x) \in PX$. Such weights are said to be representable.
(W2) For each subset $\{\phi_i\}_{i \in I}$ of PX, both $\inf_{i \in I} \phi_i$ and $\sup_{i \in I} \phi_i$ are in PX.
(W3) For all $\phi \in PX$ and $\alpha \in [0,\infty]$, both $\phi + \alpha$ and $\phi \ominus \alpha$ are in PX.

For all $\phi, \psi \in PX$, let

$$\bar{d}(\phi,\psi) = \sup_{x \in X} d_L(\phi(x), \psi(x))$$

Then $\bar{d}$ is a separated metric on PX. For all $x \in X$ and $\phi \in PX$, it holds that

$$\bar{d}(d(-,x), \phi) = \phi(x).$$

In particular, the correspondence $x \to d(,x)$ defines an isometric map $(X,d) \to (PX, \bar{d})$. That is, $d(x,y) = \bar{d}(d(,x), d(,y))$ for all $x, y \in X$. These facts are instances of the Yoneda lemma and the Yoneda embedding in enriched category theory, see e.g. [5].

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