5th IFSCOM2018 PROCEEDING BOOK
ISBN: 978-605-68670-0-2

IFSCOM2018
5th Ifs And Contemporary Mathematics Conference
September, 05-09, 2018, Kahramanmaras, Turkey
pp: 132-139

FOCAL SURFACE OF A TUBULAR SURFACE WITH BISHOP FRAME IN $\mathbb{E}^3$

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Abstract. In this study, we focus on focal surface of a tubular surface in Euclidean 3−space $\mathbb{E}^3$. Firstly, we give the tubular surface with respect to Bishop frame. Then, we define focal surface of this tubular surface. We get some results for this type of surface to become flat and we show that there is no minimal focal surface of a tubular surface in $\mathbb{E}^3$. Further, we show that u-parameter curves cannot be asymptotic curves and we obtain some results about v-parameter curves of the focal surface M^*.

1. Introduction

Focal surfaces are known as line congruences. The concept of line congruences is defined for the first time in visualization in 1991 by Hagen and Pottman [7].

Let $M : X(u,v)$ be a surface defined as a real-valued function and $N(u,v)$ be a unit normal vector on the surface. The line congruence is defined as

$$C(u,v,z) = X(u,v) + zE(u,v), \tag{1.1}$$

where $E(u,v)$ is the set of unit vectors. For each (u,v), the equation (1.1) indicates a line congruence and called generatrix. Here, the parameter z is a marked distance. In addition, there exist two special points (real, imaginary or unit) on the generatrix of C. These points are called as focal points which are the osculator points with generatrix. Therefore, focal surfaces are defined as a geometric locus of focal points. In general, there exist two focal surfaces. If $E(u,v) = N(u,v)$, then $C = C_N$ is normal congruence. Thus, the focal surface C_N has the following parametric representation

$$X_i^*(u,v) = X(u,v) + \kappa_i^{-1}(u,v)\, N(u,v),$$

where κ_1 and κ_2 are the principle curvature functions of the surface $M : X(u,v)$ [6]. The center of curvature of the normal section curve corresponds to a certain level of the normal vector at a point $X(u_0,v_0)$ on M. The extreme values are the center of the curvature of two principle directions. These points correspond to the focal points. For this reason, the congruence of lines is considered as a set of

2010 *Mathematics Subject Classification.* Primary 53A05; Secondary 53A10.
Key words and phrases. Focal surface, Tubular surface, Bishop frame.

lines which are tangent to two surfaces. Also, these two surfaces are focal surfaces of congruence of the lines. Thus, focal points of the normal congruence are the centers of curvature of two principle directions. Some studies can be found about focal curves and focal surfaces in Euclidean spaces [6, 14, 15, 16].

Canal surface with a significant place in geometry provides benefits in showing human internal organs, long thin objects, surface modelling, CG/ CAD and graphics. A canal surface $X(u,v)$ obtained by spin curve $\gamma(u)$ is the combination of the spheres which are determined by the radius function $r(u)$ and center $\gamma(u)$. If $r(u)$ is constant, then this surface is called tubular(tube) surface. In [18], the authors studied tubular surface in Euclidean $3-$space. Recently, in [10, 11, 12], the authors have attended to tubular surfaces in Euclidean $4-$space $\mathbb{E}^4$. Thanks to geometric structure of these types of surfaces, they are also used in reshaping and planning the movement lines of the robots (see, [9, 13]).

In differential geometry, frame fields are important tools for analyzing curves and surfaces. Frenet frame is the most familier frame field but there are also the other frame fields. In [1], Bishop (1975) stated there exists an alternative frame namely Bishop frame for parallel transport of curve. Bishop frame is defined even if the second derivative vanishes. There have been so many studies about Bishop frame such as [3, 8, 19, 20].

In the current work, we study focal surface of a tubular surface which is constructed by the Bishop frame. We calculate Gaussian curvature and the mean curvature functions of the focal surfaces and get the necessary and sufficient conditions of these surfaces to become flat and minimal.

2. Basic Concepts

Let $\gamma = \gamma(s) : I \to \mathbb{E}^3$ be a unit speed regular curve (parametrized by arc length function s) in the Euclidean space $\mathbb{E}^3$, where I is interval in $\mathbb{R}$. Then the derivatives of the Frenet frame $\{T, N_1, N_2\}$ of γ (Frenet-Serret formula);

$$\begin{bmatrix} T' \\ N_1' \\ N_2' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N_1 \\ N_2 \end{bmatrix},$$

where κ, τ are the curvature and torsion of curve γ, respectively [4].

The Bishop frame or parallel transport frame is an alternative approach to defining a moving frame that is well-defined even when the curve has vanishing second derivative. One can express parallel transport of an orthonormal frame along a curve simply by parallel transporting each component of the frame. The tangent vector and any convenient arbitrary basis for the remainder of the frame are used. Therefore, the Bishop (frame) formulas are expressed as

$$\begin{bmatrix} T' \\ M_1' \\ M_2' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & k_2 \\ -k_1 & 0 & 0 \\ -k_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} T \\ M_1 \\ M_2 \end{bmatrix},$$

where $\{T, M_1, M_2\}$ is the Bishop Frame and k_1, k_2 are called first and second Bishop curvatures, respectively [1].

The relation between Frenet frame and Bishop frame is given as follows:

$$\begin{bmatrix} T \\ N_1 \\ N_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} T \\ M_1 \\ M_2 \end{bmatrix},$$

where $\theta(s) = \arctan\left(\frac{k_2}{k_1}\right)$, $\tau(s) = \left(\frac{d\theta(s)}{ds}\right)$ and $\kappa(s) = \sqrt{k_1^2 + k_2^2}$. Here Bishop curvatures are defined by $k_1 = \kappa\cos\theta$, $k_2 = \kappa\sin\theta$.

Let M be a regular surface given with the parametrization $X(u,v) : (u,v) \in D \subset \mathbb{E}^2$ in $\mathbb{E}^3$. The tangent space of M at an arbitrary point $p = X(u,v)$ is spanned by the vectors X_u and X_v. The coefficients of the first fundamental form of M are defined as

$$E = \langle X_u, X_u\rangle, \quad F = \langle X_u, X_v\rangle, \quad G = \langle X_v, X_v\rangle, \tag{2.1}$$

where $\langle , \rangle$ is the Euclidean inner product. We assume that $W^2 = EG - F^2 \neq 0$, i.e. the surface patch $X(u,v)$ is regular.

Then the unit normal vector field of M is defined as

$$N = \frac{X_u \times X_v}{\|X_u \times X_v\|}. \tag{2.2}$$

The coeffiicients of the second fundamental form are given by

$$l = \langle X_{uu}, N\rangle, \quad m = \langle X_{uv}, N\rangle, \quad n = \langle X_{vv}, N\rangle. \tag{2.3}$$

[4].

The shape operator matrix of a surface is defined as

$$A_N = \begin{bmatrix} \frac{l}{E} & \frac{1}{W}\left(m - \frac{F}{E}l\right) \\ \frac{1}{W}\left(m - \frac{F}{E}l\right) & \frac{1}{W^2}\left(En - 2Fm + \frac{F^2}{E}l\right) \end{bmatrix},$$

[2].

The Gaussian curvature and the mean curvature functions of M are given by

$$K = \frac{ln - m^2}{EG - F^2}, \tag{2.4}$$

and

$$H = \frac{En + Gl - 2Fm}{2\left(EG - F^2\right)}, \tag{2.5}$$

respectively [4].

Let $\gamma : I \to M$ be a unit speed curve on the surface M. Then γ is an asymptotic curve for which the normal curvature vanishes in the direction γ'. Recall that γ is asymptotic if and only if γ' is perpendicular to the normal vector N of the surface M. Furthermore, γ is a geodesic curve on M if the tangential component $(\gamma'')^T$ of the acceleration of γ vanishes [4].

3. Focal Surface of a Tubular Surface with Bishop Frame

Tubular surface with the Bishop frame was studied by Doğan and Yaylı in [5]. In this section, we handle the tubular surface according to the Bishop frame and give the focal surface of this surface in $\mathbb{E}^3$.

Let $\gamma(u) = (\gamma_1(u), \gamma_2(u), \gamma_3(u))$ be a unit speed curve in $\mathbb{E}^3$. The tubular surface according to Bishop frame has the following parametrization:

$$M : X(u,v) = \gamma(u) + r(\cos vM_1(u) + \sin vM_2(u)), \tag{3.1}$$

where $r = const.$ is the radius of the spheres. The tangent space of M at an arbitrary point $p = X(u,v)$ is spanned by the vectors

$$\begin{aligned} X_u &= (1-fr)T, \\ X_v &= -r\sin vM_1 + r\cos vM_2, \end{aligned} \tag{3.2}$$

where

$$f(u,v) = k_1(u)\cos v + k_2(u)\sin v.$$

Then coefficients of the first fundamental form become

$$E = (1-fr)^2, \quad F = 0, \quad G = r^2, \tag{3.3}$$

where $W^2 = EG - F^2 = (1-fr)^2 r^2$ [5].

Proposition 1. [5] $X(u,v)$ is a regular tubular surface patch if and only if $f \neq \frac{1}{r}$.

The unit normal vector field and the second partial derivatives of M are obtained as

$$N = -\cos vM_1 - \sin vM_2. \tag{3.4}$$

and

$$\begin{aligned} X_{uu} &= (-k_1' r\cos v - k_2' r\sin v)T + (1-fr)k_1M_1 + (1-fr)k_2M_2, \\ X_{uv} &= (k_1 r\sin v - k_2 r\cos v)T, \\ X_{vv} &= -r\cos vM_1 - r\sin vM_2, \end{aligned} \tag{3.5}$$

respectively.

Then the coefficients of the second fundamental form become

$$l = -(1-fr)f, \quad m = 0, \quad n = r. \tag{3.6}$$

Thus, from the equations (3.3) and (3.6), the Gaussian and mean curvature functions of M are calculated as

$$K = \frac{f}{r(fr-1)}, \tag{3.7}$$

and

$$H = rK - \frac{K}{2f}$$

respectively.

The shape operator matrix of the surface M is as follows:

$$A_N = \begin{bmatrix} \frac{-f}{1-fr} & 0 \\ 0 & \frac{1}{r} \end{bmatrix} \tag{3.8}$$

[5].

From now on, we can give the parametrization of the focal surface M^* of M by obtaining the principal curvature functions of M.

Using (3.8), we get the principal curvature functions as

$$\kappa_1 = \frac{1}{r}, \quad \kappa_2 = \frac{f}{fr-1}. \tag{3.9}$$

From the definition of the focal surface of a given surface and using the equation (3.9), we obtain the focal surface M^* of M as

$$X^*(u,v) = \gamma(u) + \frac{1}{f}\left(\cos v M_1 + \sin v M_2\right). \tag{3.10}$$

The tangent space of the focal surface M^* is spanned by the vectors

$$(X^*)_u = -\frac{f_u}{f^2}\cos v M_1 - \frac{f_u}{f^2}\sin v M_2, \tag{3.11}$$

and

$$(X^*)_v = -\frac{k_2}{f^2}M_1 + \frac{k_1}{f^2}M_2.$$

Thus from (3.11), the coefficients of the first fundamental form is obtained as follows:

$$E^* = \frac{f_u^2}{f^4}, \quad F^* = \frac{f_u k_2}{f^4}\cos v - \frac{f_u k_1}{f^4}\sin v, \quad G^* = \frac{k_1^2 + k_2^2}{f^4}, \tag{3.12}$$

where $(W^*)^2 = \frac{f_u^2}{f^6} \neq 0$. Using the first partial derivatives of $X^*(u,v)$, we see that

$$N^* = -T. \tag{3.13}$$

The second partial derivatives of $X^*(u,v)$ are as in the following:

$$\begin{aligned}
(X^*)_{uu} &= \frac{f_u}{f}T - \frac{f_{uu}f - 2f_u^2}{f^3}\cos v M_1 - \frac{f_{uu}f - 2f_u^2}{f^3}\sin v M_2, \\
(X^*)_{uv} &= \frac{-f_{uv}f + 2f_u f_v}{f^3}\sin v M_2 - \frac{f_u}{f^2}\cos v M_2, \\
(X^*)_{vv} &= \frac{2k_2 f_v}{f^3}M_1 + \frac{-2k_1 f_v}{f^3}M_2.
\end{aligned} \tag{3.14}$$

Hence, from (3.13) and (3.14), we get the coefficients of the second fundamental form of the focal surface M^* as

$$l^* = \frac{-f_u}{f}, \quad m^* = 0, \quad n^* = 0. \tag{3.15}$$

Thus, we can give the following results:

Theorem 3.1. *Let M be a tubular surface according to Bishop frame given with the parametrization (3.1) and M^* be the focal surface of M with the parametrization (3.10) in $\mathbb{E}^3$. Then the Gaussian curvature of M^* vanishes, so the focal surface is flat.*

Proof. Let M^* be the focal surface of M with the parametrization (3.10) in $\mathbb{E}^3$. Using the equations (3.12), (3.15) and (2.4), we get $K^* = 0$, which completes the proof. □

Theorem 3.2. *Let M be a tubular surface according to Bishop frame given with the parametrization (3.1) and M^* be the focal surface of M with the parametrization (3.10) in $\mathbb{E}^3$. Then the mean curvature of M^* is*

$$H^* = \frac{-\left(k_1^2 + k_2^2\right)\left(k_1\cos v + k_2 \sin v\right)}{2\left(k_1'\cos v + k_2'\sin v\right)}. \tag{3.16}$$

Proof. Let M^* be the focal surface of M with the parametrization (3.10) in $\mathbb{E}^3$. Using the equations (2.5), (3.12) and (3.15), we get the result. □

Theorem 3.3. *There is no minimal focal surface M^* of the tubular surface M.*

Proof. Assume that the focal surface M^* of the surface M is minimal. From the equation (3.16), the curvature functions k_1, k_2 according to Bishop frame vanishes identically, which is a contradiction. Thus, there is no minimal focal surface of the tubular surface M. □

Example 3.4. Consider any given unit speed curve

$$\gamma(u) = \left(\cos\frac{u}{\sqrt{2}}, \sin\frac{u}{\sqrt{2}}, \frac{u}{\sqrt{2}}\right).$$

The Bishop vectors, curvatures k_1, and k_2 of this curve are determined by

$$T(u) = \gamma'(u) = \frac{1}{\sqrt{2}}\left(-\sin\frac{u}{\sqrt{2}}, \cos\frac{u}{\sqrt{2}}, 1\right),$$

$$M_1(u) = \frac{1}{\sqrt{2}}\left(-\cos\frac{u}{\sqrt{2}} - \frac{1}{\sqrt{2}}\sin\frac{u}{\sqrt{2}}, -\sin\frac{u}{\sqrt{2}} + \frac{1}{\sqrt{2}}\cos\frac{u}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right),$$

$$M_2(u) = \frac{1}{\sqrt{2}}\left(-\cos\frac{u}{\sqrt{2}} + \frac{1}{\sqrt{2}}\sin\frac{u}{\sqrt{2}}, -\sin\frac{u}{\sqrt{2}} - \frac{1}{\sqrt{2}}\cos\frac{u}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right),$$

$$k_1(u) = \frac{1}{2\sqrt{2}}, \quad k_2(u) = \frac{1}{2\sqrt{2}}.$$

Hence, tubular surface around the curve $\gamma(u)$ can be written by the use of these expressions:

$$\begin{aligned} X(u,v) = \Big(&\cos\tfrac{u}{\sqrt{2}} - \cos v\cos\tfrac{u}{\sqrt{2}} - \tfrac{1}{\sqrt{2}}\cos v\sin\tfrac{u}{\sqrt{2}} - \sin v\cos\tfrac{u}{\sqrt{2}} + \tfrac{1}{\sqrt{2}}\sin v\sin\tfrac{u}{\sqrt{2}}, \\ &\sin\tfrac{u}{\sqrt{2}} - \cos v\sin\tfrac{u}{\sqrt{2}} + \tfrac{1}{\sqrt{2}}\cos v\cos\tfrac{u}{\sqrt{2}} - \sin v\sin\tfrac{u}{\sqrt{2}} - \tfrac{1}{\sqrt{2}}\sin v\cos\tfrac{u}{\sqrt{2}}, \\ &\tfrac{u-\cos v+\sin v}{\sqrt{2}}\Big). \end{aligned}$$

Also, the parameterization of the focal surface of this surface is obtained as

$$\begin{aligned} X^*(u,v) = \Big(&\cos\tfrac{u}{\sqrt{2}} + \tfrac{2\cos v}{\cos v+\sin v}\left(-\cos\tfrac{u}{\sqrt{2}} - \tfrac{1}{\sqrt{2}}\sin\tfrac{u}{\sqrt{2}}\right) + \tfrac{2\sin v}{\cos v+\sin v}\left(-\cos\tfrac{u}{\sqrt{2}} + \tfrac{1}{\sqrt{2}}\sin\tfrac{u}{\sqrt{2}}\right), \\ &\sin\tfrac{u}{\sqrt{2}} + \tfrac{2\cos v}{\cos v+\sin v}\left(-\sin\tfrac{u}{\sqrt{2}} + \tfrac{1}{\sqrt{2}}\cos\tfrac{u}{\sqrt{2}}\right) + \tfrac{2\sin v}{\cos v+\sin v}\left(-\sin\tfrac{u}{\sqrt{2}} - \tfrac{1}{\sqrt{2}}\cos\tfrac{u}{\sqrt{2}}\right), \\ &\tfrac{u}{\sqrt{2}} - \tfrac{\sqrt{2}\cos v-\sqrt{2}\sin v}{\cos v+\sin v}\Big). \end{aligned}$$

Further, by taking radius $r = \sqrt{2}$, we plot the tubular surface and its focal surface in $\mathbb{E}^3$:

FIGURE 1. Tubular surface M and the focal surface M^*

Theorem 3.5. *Let M be a tubular surface according to Bishop frame given with the parametrization (3.1) and M^* be the focal surface of M with the parametrization (3.10) in $\mathbb{E}^3$. Then,*

i) u-parameter curves of the focal surface M^* cannot be asymptotic curves.

ii) v-parameter curves of the focal surface M^* are asymptotic curves.

Proof. Let M^* be the focal surface of M with the parametrization (3.10) in $\mathbb{E}^3$.

i) By the use of (3.14), (3.13) and with the help of the definition of an asymptotic curve, we obtain $\langle (X^*)_{uu}, N^* \rangle = \frac{f_u}{f} = 0$ which contradicts the regularity of M^*. Thus, u-parameter curves of the focal surface M^* cannot be asymptotic curves.

ii) Similarly, from the same equations, we get $\langle (X^*)_{vv}, N^* \rangle = 0$ which means v-parameter curves of the focal surface M^* are asymptotic curves. □

Theorem 3.6. *Let M be a tubular surface according to Bishop frame given with the parametrization (3.1) and M^* be the focal surface of M with the parametrization (3.10) in $\mathbb{E}^3$. Then,*

i) u-parameter curves of the focal surface M^* are geodesic curves if and only if

$$-f_{uu}f + 2\left(f_u\right)^2 = 0 \tag{3.17}$$

holds.

ii) v-parameter curves of the focal surface M^* cannot be geodesic curves.

Proof. i) By the use of (3.14), (3.13), we have $(X^*)_{uu} \wedge N^* = 0$ if and only if the equation (3.17) is satisfied.

ii) Using the same equations, we have $(X^*)_{vv} \wedge N^* = 0$ if and only if $k_1 f_v = 0$ and $k_2 f_v = 0$. Then, we get $k_1 = 0$ and $k_2 = 0$, and so f vanishes identically, which contradicts the representation of the focal surface. Thus, v-parameter curves of the focal surface M^* cannot be geodesic curves. □

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