5th IFSCOM2018 PROCEEDING BOOK
ISBN: 978-605-68670-0-2

IFSCOM2018
5th Ifs And Contemporary Mathematics Conference
September, 05-09, 2018, Kahramanmaras, Turkey
pp:115-124

EXISTENCE OF THE FIXED POINTS OF SELF MAPPINGS DEFINED IN PRODUCT OF UNIFORMLY CONVEX BANACH SPACE

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ABSTRACT

In this paper, the set of self mappings defined in the product of set which is a nonempty closed convex subset of a uniformly convex Banach space are presented. Also the definition of the asymptotically nonexpansiveness of this type of self mappings is given. And the existence of fixed point is proved by giving necessary conditions.

1. Introduction and Preliminaries

A mapping T on a subset D of a Banach space $(E, ||.||)$ is said to be nonexpansive if

$$||Tx - Ty|| \leq ||x - y||, \forall x, y \in D.$$

$x \in D$ is called fixed point of T, when $x = Tx$. If the literature is investigated one can see that the existance of fixed points for nonexpansive mappings in Banach spaces is studied in 1965 ([**?**],[**?**],[**?**]).

In [**?**], the authors introduced the theory of asymptotical nonexpansiveness of mappings defined in the algebraic product $E \times E$ and with values in the space E. And they proved the existence of coupled fixed points of such mappings when E is a uniformly convex Banach space.

The basic definitions which were given in [**?**] as follows:

Definition 1.1. *([**?**]) Let D be a nonempty subset of a real normed linear space E. A mapping $F : D \times D \to D$ is said to be nonexpansive if*

$$||F(x, y) - F(u, v)|| \leq \frac{1}{2}[||x - u|| + ||y - v||], \forall x, y, u, v \in D. \quad (1)$$

Definition 1.2. *([**?**])*

2000 *Mathematics Subject Classification.* 47H10, 54H25.

Let D be a nonempty subset of a real normed linear space E. A mapping $F : D \times D \to D$ is said to be asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$||F^n(x,y) - F^n(u,v)|| \leq \frac{k_n}{2}[||x-u|| + ||y-v||], \forall n \geq 1, \forall x,y,u,v \in D \tag{2}$$

where the sequence F^n is defined as follows:

$$F^0(x,y) = x$$

,

$$F^{n+1}(x,y) = F(F^n(x,y), F^n(y,x)), n \geq 0.$$

In this paper,the set of self mappings defined in the product of set which is a nonempty closed convex subset of a uniformly convex Banach space are presented. Also the definition of the asymptotically nonexpansiveness of this type of self mappings is given. And the existence of fixed point is proved by giving necessary conditions.

Now, assume that D be a nonempty subset of a real normed linear space E. We denote by Φ the set of functions $T_F : D \times D \to D \times D$ where $M = \{F | F : D \times D \to D$ is a mapping $\}$ and $T_F(x,y) = (F(x,y), F(y,x))$.
Also, since all the norms on the finite-dimensional normed spaces are equivalent, the norm on the $E \times E$ can be selected as $||(x,y)||_{E\times E} = ||x||_E + ||y||_E$ and then,

$$||T_F(x,y) - T_F(u,v)||_{E\times E} = ||F(x,y) - F(u,v)||_E + ||F(y,x) - F(v,u)||_E. \tag{3}$$

is obtained.

Definition 1.3. *Let D be a nonempty subset of a real normed linear space E. A mapping $T_F \in \Phi$ is said to be nonexpansive if*

$$||T_F(x,y) - T_F(u,v)|| \leq ||x-u|| + ||y-v||, \forall x,y,u,v \in X. \tag{4}$$

Definition 1.4. *T_F is said to be asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that*

$$||T_F^n(x,y) - T_F^n(u,v)|| \leq k_n[||x-u|| + ||y-v||], \forall n \geq 1, \forall x,y,u,v \in X. \tag{5}$$

where the sequence $\{T_F^n\}$ is defined as follows:

$$\begin{array}{l} T_F^0(x,y) = (x,y) \\ T_F^{n+1}(x,y) = T_F(T_F^n(x,y)), n \geq 0. \end{array} \tag{6}$$

Definition 1.5. *T_F is said to be uniformly L-Lipschitzian (where L is a positive constant) if*

$$||T_F^n(x,y) - T_F^n(u,v)|| \leq L[||x-u|| + ||y-v||], \forall n \geq 1, \forall x,y,u,v \in X. \tag{7}$$

When the equality is verified for $n = 1$,i.e.,when

$$||T_F(x,y) - T_F(u,v)|| \leq L[||x-u|| + ||y-v||], \forall x,y,u,v \in X, \tag{8}$$

T_F is said to be Lipschitz with the constant L (or L-Lipschitzian).

Remark 1.1. (1) *When $F : D \times D \to D$ is a mapping as described in [**?**], it is easy to see that if $T_F : D \times D \to D \times D$ is a nonexpansive mapping then T_F is an asymtotically nonexpansive mapping with a constant sequence $\{1\}$.*

(2) *If* $T_F : D \times D \to D \times D$ *is an asymtotically nonexpansive mapping with a sequence* $\{k_n\} \in [1,\infty)$ *such that* $k_n \to 1$*, then it must be uniformly L-Lipschitzian with* $L = \sup_{n\geq 1}\{k_n\}$.

(3) *The sequence* $\{T_F^n(x,y)\}$ *can be written as the sequence* $\{a_n\}$ *defined as follows:*

$$\begin{aligned} &a_0 = (x_0, y_0) = (x, y) \\ &a_{n+1} = (x_{n+1}, y_{n+1}) = (F(x_n, y_n), F(y_n, x_n)) = T_F(x_n, y_n), n \geq 0. \end{aligned} \tag{9}$$

Now we shall define demi-closed maps at the origin as follows.

Definition 1.6. *Let* E *be a real Banach space and* D *be a closed subset of* E*. A mapping* $T_F : D \times D \to D \times D$ *is said to be demi-closed at the origin if, for any sequence* $\{a_n\}$ *in* $D \times D$*, the conditions* $a_n \to (p,q)$ *weakly and* $T_F(a_n) \to (0,0)$ *strongly imply* $T_F(p,q) = (0,0)$.

Olaoluwa, Olaleru and Chang proved the following Lemma and Theorem in ([**?**]).

Lemma 1.1. *([**?**]) Let* E *be a uniformly convex Banach space,* C *be a nonempty bounded closed convex subset of* E*. Then there exists a strictly increasing continuous convex function* $f : [0,\infty) \to [0,\infty)$ *with* $f(0) = 0$ *such that for any Lipschitzian mapping* $T : C \times C \to E$ *with the Lipschitz constant* $L \geq 1$*, any finite many elements* $\{(x_i, y_i)\}_{i=1}^n$ *in* $C \times C$ *and any finite many nonnegative numbers* $\{t_i\}_{i=1}^n$ *with* $\sum_{i=1}^n t_i = 1$*, the following inequality holds:*

(10)

$$\begin{aligned} &||T(\textstyle\sum_{i=1}^n t_i(x_i,y_i)) - \sum_{i=1}^n t_i T(x_i,y_i)|| \\ &\quad \leq \tfrac{L}{2} f^{-1}\{\max_{1\leq i,j\leq n}(||x_i - x_j|| + ||y_i - y_j|| - (\tfrac{L}{2})^{-1}||T(x_i,y_i) - T(x_j,y_j)||)\}. \end{aligned}$$

Theorem 1.1. *([**?**]) Let* D *be a nonempty closed convex subset of a uniformly convex Banach space* E *and* $F : D \times D \to D$ *be an asymptotically nonexpansive map with the sequence* $\{k_n\}$ *(*$\{k_n\} \subseteq [1,\infty)$ *with* $\lim_{n\to\infty} k_n = 1$ *). Then* $p_1 - F$ *satisfies the demi-closedness at the origin property. In other terms, if any sequence* $\{(x_n, y_n)\}$ *in* $D \times D$ *is such that* $x_n \to q_1$ *weakly,* $y_n \to q_2$ *weakly,* $x_n - F(x_n, y_n) \to 0$ *strongly and* $y_n - F(y_n, x_n) \to 0$ *strongly, then* F *has a coupled fixed point* (q_1, q_2).

2. Main Results

Now, we shall give our main results:

Lemma 2.1. *Let* E *be a uniformly convex Banach space,* C *be a nonempty bounded closed convex subset of* E*. Then there exists a strictly increasing continuous convex function* $f : [0,\infty) \to [0,\infty)$ *with* $f(0) = 0$ *such that for any Lipschitzian mapping* $T_F : C \times C \to E \times E$ *with the Lipschitz constant* $L \geq 1$*, any finite many elements* $\{(x_i, y_i)\}_{i=1}^n \subseteq C \times C$ *and any finite many nonnegative numbers* $\{t_i\}_{i=1}^n t_i = 1$*, the following inequality holds:*

(11)

$$||T_F(\sum_{i=1}^n t_i(x_i,y_i)) - \sum_{i=1}^n t_i T_F(x_i,y_i)|| \leq L f^{-1}\{\max_{1\leq i,j\leq n}(||x_i - x_j|| + ||y_i - y_j|| - \frac{1}{L}||T_F(x_i,y_i) - T_F(x_j,y_j)||)\}.$$

Proof. We shall prove by induction. For $n = 1$, (**??**) is achieved.
For $n = 2$:

Let δ be the modulus of uniform convexity of E and define $d : \mathbb{R}^+ \to \mathbb{R}^+$ by

$$d(t) = \begin{cases} \frac{1}{2}\int_0^t \delta(s)ds & ,0 \leq t \leq 2, \\ d(2) + \frac{1}{2}\delta(2)(t-2) & ,t > 2. \end{cases} \tag{12}$$

It is simple to see that d is strictly increasing, continuous, convex, satisfying $d(0) = 0$ and

$$2t_1t_2d(||u - v||) \leq 1 - ||t_1u - t_2v|| \tag{13}$$

for all $t_1, t_2 \geq 0$ such that $t_1 + t_2 = 1$ and $u, v \in E$ such that $||u|| \leq 1$ and $||v|| \leq 1$ (see [?],[?]).

(14)

$$\begin{aligned} ||T_F(x_1,y_1) - T_F(\textstyle\sum_{i=1}^2 t_i(x_i,y_i))|| &= ||T_F(x_1,y_1) - T_F(t_1x_1 + t_2x_2, t_1y_1 + t_2y_2)|| \\ &\leq L\{||x_1 - t_1x_1 - t_2x_2|| + ||y_1 - t_1y_1 - t_2y_2||\} \\ &= L\{||(1-t_1)x_1 - t_2x_2|| + ||(1-t_1)y_1 - t_2y_2||\} \\ &= L\{||t_2x_1 - t_2x_2|| + ||t_2y_1 - t_2y_2||\} \\ &= L\{||t_2x_1 - t_2x_2|| + ||t_2y_1 - t_2y_2||\} \\ &= Lt_2[||x_1 - x_2|| + ||y_1 - y_2||]. \end{aligned}$$

Similarly

(15)

$$\begin{aligned} ||T_F(x_2,y_2) - T_F(\textstyle\sum_{i=1}^2 t_i(x_i,y_i))|| &= ||T_F(x_2,y_2) - T_F(t_1x_1 + t_2x_2, t_1y_1 + t_2y_2)|| \\ &\leq L\{||x_2 - t_1x_1 - t_2x_2|| + ||y_2 - t_1y_1 - t_2y_2||\} \\ &= L\{||(1-t_2)x_2 - t_1x_1|| + ||(1-t_2)y_2 - t_1y_1||\} \\ &= L\{||t_1x_2 - t_1x_1|| + ||t_1y_2 - t_1y_1||\} \\ &= L\{||t_1x_2 - t_1x_1|| + ||t_1y_2 - t_1y_1||\} \\ &= Lt_1[||x_1 - x_2|| + ||y_1 - y_2||]. \end{aligned}$$

Thus, if $u = \frac{T_F(x_2,y_2) - T_F(\sum_{i=1}^2 t_i(x_i,y_i))}{Lt_1[||x_1-x_2||+||y_1-y_2||]}$ and $v = \frac{T_F(\sum_{i=1}^2 t_i(x_i,y_i)) - T_F(x_1,y_1)}{Lt_2[||x_1-x_2||+||y_1-y_2||]}$, then from (??) and (??),

$$\begin{aligned} ||u|| \leq 1, &\quad ||v|| \leq 1, \\ t_1u + t_2v = &\quad \tfrac{T_F(x_2,y_2) - T_F(x_1,y_1)}{L[||x_1-x_2||+||y_1-y_2||]}, \\ u - v = &\quad \tfrac{t_1T_F(x_1,y_1) + t_2T_F(x_2,y_2) - T_F(\sum_{i=1}^2 t_i(x_i,y_i))}{Lt_1t_2[||x_1-x_2||+||y_1-y_2||]}. \end{aligned} \tag{16}$$

Taking (??) in (??), we have

(17)

$$2t_1t_2d\Big(\frac{||t_1T_F(x_1,y_1) + t_2T_F(x_2,y_2) - T_F(\sum_{i=1}^2 t_i(x_i,y_i))||}{Lt_1t_2[||x_1-x_2|| + ||y_1-y_2||]}\Big) \leq 1 - \frac{||T_F(x_2,y_2) - T_F(x_1,y_1)||}{L[||x_1-x_2|| + ||y_1-y_2||]},$$

and then

(18)

$$\begin{aligned} &2t_1t_2L[||x_1-x_2|| + ||y_1-y_2||]d\big(\tfrac{||t_1T_F(x_1,y_1)+t_2T_F(x_2,y_2)-T_F(\sum_{i=1}^2 t_i(x_i,y_i))||}{Lt_1t_2[||x_1-x_2||+||y_1-y_2||]}\big) \\ &\leq L[||x_1-x_2|| + ||y_1-y_2||] - ||T_F(x_2,y_2) - T_F(x_1,y_1)||. \end{aligned}$$

Since $\frac{d(t)}{t}$ is strictly increasing mapping , $t_1, t_2 \leq \frac{1}{4}$ and $||x_1 - x_2|| + ||y_1 - y_2|| \leq 2D$, where $D :=$ diam C, we hold

(19)

$$\begin{aligned} LDd &\big(\tfrac{||t_1T_F(x_1,y_1)+t_2T_F(x_2,y_2)-T_F(\sum_{i=1}^2 t_i(x_i,y_i))||}{\frac{L}{2}D}\big) \\ &\leq 2t_1t_2L[||x_1-x_2|| + ||y_1-y_2||]d\big(\tfrac{||t_1T_F(x_1,y_1)+t_2T_F(x_2,y_2)-T_F(\sum_{i=1}^2 t_i(x_i,y_i))||}{Lt_1t_2[||x_1-x_2||+||y_1-y_2||]}\big) \\ &\leq L[||x_1-x_2|| + ||y_1-y_2||] - ||T_F(x_2,y_2) - T_F(x_1,y_1)||. \end{aligned}$$

Let $f_2 : \mathbb{R}^+ \to \mathbb{R}^+$ be defined as $f_2(t) = Dd(\frac{2t}{D})$; thus

(20)
$$Lf_2(\frac{1}{L}||t_1T_F(x_1,y_1)+t_2T_F(x_2,y_2)-T_F(\sum_{i=1}^{2}t_i(x_i,y_i))||) \leq L[||x_1-x_2||+||y_1-y_2||]-||T_F(x_2,y_2)-T_F(x_1,y_1)||.$$

Hence,

(21)
$$||t_1T_F(x_1,y_1)+t_2T_F(x_2,y_2)-T_F(\sum_{i=1}^{2}t_i(x_i,y_i))|| \leq Lf_2{}^{-1}(||x_1-x_2||+||y_1-y_2||]-\frac{1}{L}||T_F(x_2,y_2)-T_F(x_1,y_1)||).$$

Therefore, (**??**) is true for $n = 2$.
Now suppose that

(22)
$$\begin{aligned}&||T_F(\textstyle\sum_{i=1}^n t_i(x_i,y_i)) - \sum_{i=1}^n t_iT_F(x_i,y_i)||\\ &\quad \leq Lf_n^{-1}\{\max_{1\leq i,j\leq n}(||x_i-x_j|| + ||y_i-y_j|| - \tfrac{1}{L}||T_F(x_i,y_i) - T_F(x_j,y_j)||)\}.\end{aligned}$$

Define a strictly increasing continuous convex function f_{n+1} satisfying $f_{n+1}(0) = 0$ and

$$f_{n+1}^{-1}(t) \geq f_2^{-1}(t) + f_n^{-1}(t + 2f_2^{-1}(t)). \tag{23}$$

An example is given by $*$ derives the inf-convolution, I the identity function on $\mathbb{R}$ and the functions in the functions f_n are extended to be $+\infty$ on $(-\infty, 0)$, then we could take (see [**?**])

$$f_{n+1} = \frac{1}{2}(I * f_2) \circ (\frac{1}{2}(f_2 * f_n)). \tag{24}$$

We derive also $\triangle^{n-1} = \{t = (t_1, t_2, ..., t_n) : t_i \geq 0, \sum_{i=1}^n t_i = 1\}$.
Let $t \in \triangle^n$ and $(x_1, y_1), ..., (x_{n+1}, y_{n+1}) \in C \times C$. We can ignore the state of the $t_{n+1} = 1$ which is trivial. For the remaining part of the proof, we get the subscript i range through $1 \leq i \leq n+1$, while j ranges through $1 \leq j \leq n$. Set

$$\begin{aligned}&(u_j, v_j) = (1-t_{n+1})(x_j,y_j) + t_{n+1}(x_{n+1},y_{n+1}),\\ &(x_i^{'}, y_i^{'}) = (T_F(x_i,y_i), T_F(y_i,x_i)),\\ &(u_j^{'}, v_j^{'}) = (1-t_{n+1})(x_j^{'},y_j^{'}) + t_{n+1}(x_{n+1}^{'},y_{n+1}^{'}),\\ &\mu_j = \tfrac{t_j}{1-t_{n+1}}.\end{aligned} \tag{25}$$

If $\mu \in \triangle^{n-1}$ then with the calculation

$$\begin{aligned}\textstyle\sum_{i=1}^{n+1} t_i(x_i,y_i) &= \textstyle\sum_{j=1}^n \mu_j(u_j,v_j),\\ \textstyle\sum_{i=1}^{n+1} t_i(x_i^{'},y_i^{'}) &= \textstyle\sum_{j=1}^n \mu_j(u_j^{'},v_j^{'}),\end{aligned} \tag{26}$$

(27)
$$\begin{aligned}||T_F(\Sigma_{i=1}^{n+1}t_i(x_i,y_i)) - \Sigma_{i=1}^{n+1}t_ix_i^{'}|| &= ||T_F(\textstyle\sum_{j=1}^n \mu_j(u_j,v_j)) - \Sigma_{j=1}^n\mu_ju_j^{'}||\\ &\leq ||T_F(\Sigma_{j=1}^n\mu_j(u_j,v_j)) - \Sigma_{j=1}^n\mu_jT_F(u_j,v_j)||\\ &\quad + \textstyle\sum_{j=1}^n \mu_j||T_F(u_j,v_j) - u_j^{'}||.\end{aligned}$$

Using the inequality (**??**), we can write

$$\begin{aligned}&f_n(\tfrac{1}{L}||T_F(\textstyle\sum_{j=1}^n \mu_j(u_j,v_j)) - \Sigma_{j=1}^n\mu_jT_F(u_j,v_j)||)\\ &\quad \leq \max_{1\leq j,k\leq n}(||u_j-u_k|| + ||v_j-v_k|| - \tfrac{1}{L}||T_F(u_j,v_j) - T_F(u_k,v_k)||)\\ &\quad = \max_{1\leq j,k\leq n}(||u_j-u_k|| + ||v_j-v_k|| - \tfrac{1}{L}||T_F(v_j,u_j) - T_F(v_k,u_k)||)\\ &\quad = \max_{1\leq j,k\leq n}(||u_j-u_k|| + ||v_j-v_k|| - \tfrac{1}{2L}||T_F(u_j,v_j) - T_F(u_k,v_k)||\\ &\qquad -\tfrac{1}{2L}||T_F(v_j,u_j) - T_F(v_k,u_k)||).\end{aligned} \tag{28}$$

By the triangle inequality,

(29)
$$\begin{aligned}&||u_j-u_k||-\tfrac{1}{2L}||T_F(u_j,v_j)-T_F(u_k,v_k)||\\ &\quad\le ||u_j-u_k||-\tfrac{1}{2L}||u_j^{'}-u_k^{'}||+\tfrac{1}{2L}||u_k^{'}-T_F(u_k,v_k)||+\tfrac{1}{2L}||u_j^{'}-T_F(u_j,v_j)||,\end{aligned}$$

(30)
$$\begin{aligned}&||v_j-v_k||-\tfrac{1}{2L}||T_F(v_j,u_j)-T_F(v_k,u_k)||\\ &\quad\le ||v_j-v_k||-\tfrac{1}{2L}||v_j^{'}-v_k^{'}||+\tfrac{1}{2L}||v_k^{'}-T_F(v_k,u_k)||+\tfrac{1}{2L}||v_j^{'}-T_F(v_j,u_j)||.\end{aligned}$$

By using (**??**) we get

(31)
$$\begin{aligned}f_2(\tfrac{1}{L}||T_F(u_j,v_j)-u_j^{'}||) &= f_2(\tfrac{1}{L}||T_F(u_j,v_j)-(1-t_{n+1})x_j^{'}-t_{n+1}x_{n+1}^{'}||)\\ &= f_2(\tfrac{1}{L}||T_F(u_j,v_j)-(1-t_{n+1})T_F(x_j,y_j)-t_{n+1}T_F(x_{n+1},y_{n+1})||)\\ &= f_2(\tfrac{1}{L}||T_F((1-t_{n+1})x_j+t_{n+1}x_{n+1},(1-t_{n+1})y_j+t_{n+1}y_{n+1})\\ &\quad -(1-t_{n+1})T_F(x_j,y_j)-t_{n+1}T_F(x_{n+1},y_{n+1})||)\\ &\le ||x_j-x_{n+1}||+||y_j-y_{n+1}||-\tfrac{1}{L}||T_F(x_j,y_j)-T_F(x_{n+1},y_{n+1})||\\ &= ||x_j-x_{n+1}||+||y_j-y_{n+1}||-\tfrac{1}{L}||x_j^{'}-x_{n+1}^{'}||.\end{aligned}$$

In an analogous way

$$f_2(\frac{1}{L}||T_F(v_j,u_j)-v_j^{'}||)\le ||x_j-x_{n+1}||+||y_j-y_{n+1}||-\frac{1}{L}||y_j^{'}-y_{n+1}^{'}||. \tag{32}$$

On the other hand,

(33)
$$\begin{aligned}||u_j-u_k||-\tfrac{1}{2L}||u_j^{'}-u_k^{'}|| &= ||(1-t_{n+1})(x_j-x_k)||-\tfrac{1}{2L}||(1-t_{n+1})(x_j^{'}-x_k^{'})||\\ &= (1-t_{n+1})[||x_j-x_k||-\tfrac{1}{2L}||x_j^{'}-x_k^{'}||]\\ &\le ||x_j-x_k||-\tfrac{1}{2L}||x_j^{'}-x_k^{'}||.\end{aligned}$$

Similarly,

$$||v_j-v_k||-\frac{1}{2L}||v_j^{'}-v_k^{'}||\le ||y_j-y_k||-\frac{1}{2L}||y_j^{'}-y_k^{'}||. \tag{34}$$

Take

(35)
$$\begin{aligned}t: &= \max\{||x_i-x_k||+||y_i-y_k||-\tfrac{1}{L}||x_i^{'}-x_k^{'}|| : 1\le i,k\le n+1\}\\ &= \max\{||x_i-x_k||+||y_i-y_k||-\tfrac{1}{L}||y_i^{'}-y_k^{'}|| : 1\le i,k\le n+1\}\\ &= \max\{||x_i-x_k||+||y_i-y_k||-\tfrac{1}{2L}||x_i^{'}-x_k^{'}||-\tfrac{1}{2L}||y_i^{'}-y_k^{'}|| : 1\le i,k\le n+1\}.\end{aligned}$$

Then, by (**??**) and (**??**),

$$\frac{1}{L}||T_F(u_j,v_j)-u_j^{'}||\le f_2^{-1}(t) \text{ and } \frac{1}{L}||T_F(v_j,u_j)-v_j^{'}||\le f_2^{-1}(t). \tag{36}$$

If we use (**??**) and (**??**) in (**??**) and also (**??**) and (**??**) in (**??**), we get

$$||u_j-u_k||-\frac{1}{2L}||T_F(u_j-v_j)-T_F(u_k-v_k)||\le ||x_j-x_k||-\frac{1}{2L}||x_j^{'}-x_k^{'}||+f_2^{-1}(t) \tag{37}$$

and

$$||v_j-v_k||-\frac{1}{2L}||T_F(v_j-u_j)-T_F(v_k-u_k)||\le ||y_j-y_k||-\frac{1}{2L}||y_j^{'}-y_k^{'}||+f_2^{-1}(t). \tag{38}$$

Then by summing inequalities (**??**) and (**??**),

(39)
$$\begin{aligned} &\|u_j - u_k\| + \|v_j - v_k\| - \tfrac{1}{2L}\|T_F(u_j, v_j) - T_F(u_k, v_k)\| - \tfrac{1}{2L}\|T_F(v_j, u_j) - T_F(v_k, u_k)\| \\ &\quad \le \|x_j - x_k\| + \|y_j - y_k\| - \tfrac{1}{2L}\|x_j^{'} - x_k^{'}\| - \tfrac{1}{2L}\|y_j^{'} - y_k^{'}\| + 2f_2^{-1}(t) \\ &\quad \le t + 2f_2^{-1}(t). \end{aligned}$$

By using inequality (**??**) in (**??**), we have

$$\|T_F(\sum_{j=1}^{n} \mu_j(u_j, v_j)) - \Sigma_{j=1}^{n}\mu_j T_F(u_j, v_j)\| \le L f_n^{-1}(t + 2f_2^{-1}(t)). \tag{40}$$

When (**??**) and (**??**) are used in (**??**), we get

(41)
$$\begin{aligned} \|T_F(\textstyle\sum_{i=1}^{n+1} t_i(x_i, y_i)) - \sum_{i=1}^{n+1} t_i T_F(x_i, y_i)\| &\le L f_n^{-1}(t + 2f_2^{-1}(t)) + \textstyle\sum_{j=1}^{n} \mu_j L f_2^{-1}(t) \\ &\le L[f_n^{-1}(t + 2f_2^{-1}(t)) + L f_2^{-1}(t)] \\ &\le L f_{n+1}^{-1}(t) \text{ by the definition of } f_{n+1}. \end{aligned}$$

Thus,

(42)
$$\begin{aligned} &\|T_F(\textstyle\sum_{i=1}^{n+1} t_i(x_i, y_i)) - \sum_{i=1}^{n+1} t_i T_F(x_i, y_i)\| \\ &\quad \le L f_{n+1}^{-1}(\max_{1 \le i,k \le n+1}\{\|x_i - x_k\| + \|y_i - y_k\| - \tfrac{1}{L}\|T_F(x_i, y_i) - T_F(x_k, y_k)\|\}). \end{aligned}$$

In the sequel, if we see that the dependence of f_n on n can actually be omitted than the proof is completed.

Since E is uniformly convex, E is B-convex (see [**?**]) and since the product of B-convex spaces is also B-convex (see [**?**]), E^3 is B-convex, hence has the convex approximation property (C.A.P.)(see [**?**]), i.e., for each $\varepsilon > 0$, there exists a positive integer p such that

$$coM \subset co_p M + S(0, \frac{\varepsilon}{3L}) \times S(0, \frac{\varepsilon}{3L}) \times S(0, \frac{\varepsilon}{3L}), \tag{43}$$

where $S(0, r)$ is the open sphere centered at the origin and with r as radius, coM is the convex hall of M and

$$co_p M = \{\sum_{i=1}^{p} t_i X_i; t \in \triangle^{p-1}; X_i \in M \text{ for all } i \in \{1, ..., p\}, p \text{ fixed } \} \tag{44}$$

for every $M \subset C \times C \times C$.

Take $\gamma = f_p(\frac{2\varepsilon}{3L})$. Assume $x_1, ..., x_n, y_1, ..., y_n \in C$ satisfy

$$\|x_i - x_j\| + \|y_i - y_j\| - \frac{2}{L}\|T_F(x_i, y_i) - T_F(x_j, y_j)\| \le \gamma \text{ for all } i, j. \tag{45}$$

Think about $M = \{(x_i, y_i, T_F(x_i, y_i)) \in C^3 : i = 1, 2, ..., n\}$. Therefore, for each $t \in \triangle^{n-1}$, there exist $\mu \in \triangle^{p-1}$ and $i_1, ..., i_p \in \{1, ..., n\}$ such that

$$\begin{aligned} &\|\textstyle\sum_{i=1}^{n} t_i x_i - \sum_{j=1}^{p} \mu_j x_{i_j}\| < \frac{\varepsilon}{3L}, \\ &\|\textstyle\sum_{i=1}^{n} t_i y_i - \sum_{j=1}^{p} \mu_j y_{i_j}\| < \frac{\varepsilon}{3L}, \\ &\|\textstyle\sum_{i=1}^{n} t_i T_F(x_i, y_i) - \sum_{j=1}^{p} \mu_j T_F(x_{i_j}, y_{i_j})\| < \frac{\varepsilon}{3L} \le \frac{\varepsilon}{3}. \end{aligned} \tag{46}$$

Then,

(47)
$$\begin{aligned} \|T_F(\textstyle\sum t_i(x_i, y_i)) - \sum t_i T_F(x_i, y_i)\| &\le \|T_F(\textstyle\sum t_i(x_i, y_i)) - T_F(\sum \mu_j(x_{i_j}, y_{i_j}))\| \\ &\quad + \|T_F(\textstyle\sum \mu_j(x_{i_j}, y_{i_j})) - \Sigma\mu_j T_F(x_{i_j}, y_{i_j})\| \\ &\quad + \|\textstyle\sum \mu_j T_F(x_{i_j}, y_{i_j}) - \Sigma t_i T_F(x_i, y_i)\|. \end{aligned}$$

By using the inequalities

$$
\begin{aligned}
&||T_F(\sum t_i(x_i,y_i)) - T_F(\sum \mu_j(x_{i_j},y_{i_j}))|| \\
&\quad \leq \tfrac{L}{2}[||\sum t_i x_i - \sum \mu_j x_{i_j}|| + ||\sum t_i y_i - \sum \mu_j y_{i_j}||] \\
&\quad < \tfrac{L}{2}[\tfrac{\varepsilon}{3L} + \tfrac{\varepsilon}{3L}] = \tfrac{\varepsilon}{3}
\end{aligned}
\tag{48}
$$

and

$$
||T_F(\sum \mu_j(x_{i_j},y_{i_j})) - \Sigma\mu_j T_F(x_{i_j},y_{i_j})|| \leq \tfrac{L}{2} f_p^{-1}(\gamma) = \tfrac{L}{2} f_p^{-1}(f_p(\tfrac{2\varepsilon}{3L})) = \tfrac{\varepsilon}{3}, \tag{49}
$$

we get that

$$
||T_F(\sum t_i(x_i,y_i)) - \sum t_i T_F(x_i,y_i)|| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \tag{50}
$$

whenever $||x_i - x_j|| + ||y_i - y_j|| - \frac{2}{L}||T_F(x_i,y_i) - T_F(x_j,y_j)|| \leq \gamma$. Thus we can devise f such that $\varepsilon < \frac{L}{2} f^{-1}(\gamma)$, i.e., such that $f(t) \leq f_p(\frac{t}{3})$, which ensures (**??**) with f unrestricted from n. □

Definition 2.1. *Let $I : E \times E \to E \times E$ is the identity function, that is, $I(x,y) = (x,y)$ for all $x,y \in X$ and $T_F \in \Phi$. We have that $I - T_F$ demi-closed at the origin if the existence of a sequence $\{(x_n,y_n)\} \subseteq D \times D$ such that*

$$
\begin{cases}
(x_n,y_n) \to (q_1,q_2) \text{ weakly }, \\
(x_n,y_n) - T_F(x_n,y_n) \to 0 \text{ strongly}
\end{cases} \tag{51}
$$

entails that

$$
(I - T_F)(q_1,q_2) = 0 \tag{52}
$$

i.e.,

$$
F(q_1,q_2) = q_1 \text{ and } F(q_2,q_1) = q_2. \tag{53}
$$

Theorem 2.1. *D is a nonempty closed convex subset of a uniformly convex Banach space E and $T_F \in \Phi$ be an asymptotically nonexpansive map with the sequence $\{k_n\}$ as defined in (**??**). And so $I - T_F$ satisfies the demi-closedness at the origin property. So, if any sequence $\{(x_n,y_n\} \subseteq D \times D$ is such that $(x_n,y_n) \to (q_1,q_2)$ weakly, $(x_n,y_n) - T_F(x_n,y_n) \to 0$ strongly, then T_F has a fixed point (q_1,q_2).*

Proof. Since $\{(x_n,y_n)\}$ converges weakly to $q = (q_1,q_2) \in D \times D$, $\{x_n\}$ and $\{y_n\}$ are bounded in D. Therefore, there exists $r > 0$ such that $\{x_n\},\{y_n\} \subset C =: D \cap B[0,r]$, where $B[0,r]$ is the closed ball of E of E of radius r centered in 0. Hence C is a nonempty bounded closed convex subset in D.
Now, we shall show that as $n \to \infty$, ${T_F}^n(q_1,q_2) \to (q_1,q_2)$.
Since $\{(x_n,y_n)\}$ converges weakly to (q_1,q_2), by Mazur's theorem (see, i.e., [**?**]), for all $n \geq 1$, there exists sequences $\{A_n\}$ and $\{B_n\}$ such that $A_n = \sum_{i=1}^{m(n)} t_i^{(n)} x_{i+n}$ and $B_n = \sum_{i=1}^{m(n)} t_i^{(n)} y_{i+n}$ where $t_i^{(n)} \geq 0, \sum_{i=1}^{m(n)} t_i^n = 1$ and $||A_n - q_1|| < \frac{1}{n}, ||B_n - q_2|| < \frac{1}{n}$.
Since the sequences $\{(x_n,y_n) - T_F(x_n,y_n)\}$ converges strongly to 0, for any given $\varepsilon > 0$ and positive integer $j \geq 1$, there is an integer $N^0 = N^0(\varepsilon,j)$ such that $\frac{1}{N^0} < \varepsilon$ and

$$
||(x_n,y_n) - T_F(x_n,y_n)|| \leq \frac{1}{1 + \sum_{l=1}^{j-1} k_l} < \varepsilon, n \geq \mathbb{N}. \tag{54}
$$

Since T_F is asymptotically nonexpansive,
(55)
$$\begin{aligned}||T_F^j(x_n,y_n)-T_F^{j+1}(x_n,y_n)|| &= ||T_F^j(x_n,y_n)-T_F^j(T_F(x_n,y_n))||\\ &= ||T_F^j(x_n,y_n)-T_F^j(F(x_n,y_n),F(y_n,x_n))||\\ &\le k_j[||x_n-F(x_n,y_n)||+||y_n-F(y_n,x_n)||.\end{aligned}$$

Thus, for any $n \geq \mathbb{N}$,
(56)
$$\begin{aligned}||(x_n,y_n)-T_F^j(x_n,y_n)|| &\le ||(x_n,y_n)-T_F(x_n,y_n)||+||(T_F-T_F^2)(x_n,y_n)||+...\\ &\quad +||(T_F^{j-1}-T_F^j)(x_n,y_n)||\\ &\le (1+\textstyle\sum_{l=1}^{j-1}k_l)[||x_n-F(x_n,y_n)||+||y_n-F(y_n,x_n)||]\\ &< \varepsilon.\end{aligned}$$

Since $T_F : D \times D \to D \times D$ is asymptotically nonexpansive, $T_F : C \times C \to D \times D$ is also asymptotically nonexpansive, hence $T_F^j : C \times C \to D \times D$ is a Lipschitzian mapping with the Lipschitz constant $k_j \geq 1$. Then we obtain the following inequality:
(57)
$$\begin{aligned}||T_F^j(A_n,B_n)-(A_n,B_n)|| &= ||T_F^j(A_n,B_n)-\textstyle\sum_{i=1}^{m(n)}t_i^nT_F^j(x_{i+n},y_{i+n})\\ &\quad +\textstyle\sum_{i=1}^{m(n)}t_i^nT_F^j(x_{i+n},y_{i+n})-(\sum_{i=1}^{m(n)}t_i^{(n)}x_{i+n},\sum_{i=1}^{m(n)}t_i^{(n)}y_{i+n})||\\ &\le ||T_F^j(A_n,B_n)-\textstyle\sum_{i=1}^{m(n)}t_i^nT_F^j(x_{i+n},y_{i+n})||\\ &\quad +\textstyle\sum_{i=1}^{m(n)}t_i^n||T_F^j(x_{i+n},y_{i+n})-(x_{i+n},y_{i+n})||.\end{aligned}$$

By using the equation (**??**), we get that

$$\sum_{i=1}^{m(n)} t_i^n||T_F^j(x_{i+n},y_{i+n})-(x_{i+n},y_{i+n})|| < \varepsilon, \forall n \geq \mathbb{N}. \tag{58}$$

If we use Lemma **??** with inequality (**??**), we obtain
(59)
$$\begin{aligned}&||T_F^j(A_n,B_n)-\textstyle\sum_{i=1}^{m(n)}t_i^nT_F^j(x_{i+n},y_{i+n})||\\ &\quad\le k_jf^{-1}\{\max_{1\le i,k\le m(n)}(||x_{i+n}-x_{k+n}||+||y_{i+n}-y_{k+n}||-\tfrac{1}{k_j}||T_F^j(x_{i+n},y_{i+n})-T_F^j(x_{k+n},y_{k+n})||)\}\\ &\quad\le k_jf^{-1}\{\max_{1\le i,k\le m(n)}[||x_{i+n}-F^j(x_{i+n},y_{i+n})||+||y_{i+n}-F^j(y_{i+n},x_{i+n})||\\ &\qquad +||F^j(x_{i+n},y_{i+n})-F^j(x_{k+n},y_{k+n})||+||F^j(y_{i+n},x_{i+n})-F^j(y_{k+n},x_{k+n})||\\ &\qquad +||F^j(x_{k+n},y_{k+n})-x_{n+k}||+||F^j(y_{k+n},x_{k+n})-y_{k+n}||\\ &\qquad -\tfrac{1}{k_j}||T_F^j(x_{i+n},y_{i+n})-T_F^j(x_{k+n},y_{k+n})||]\}\\ &\quad\le k_jf^{-1}\{\max_{1\le i,k\le m(n)}[2\varepsilon+(1-k_j^{-1})k_j(||x_{i+n}-x_{k+n}||+||y_{i+n}-y_{k+n}||)]\}\\ &\quad\le k_jf^{-1}(2\varepsilon+2r(1-k_j^{-1})k_j), n\geq\mathbb{N}.\end{aligned}$$

By using inequalities (**??**) and (**??**) in (**??**), we have

$$||T_F^j(A_n,B_n)-(A_n,B_n)|| \le k_jf^{-1}[2\varepsilon+2rk_j(1-k_j^{-1})]+\varepsilon. \tag{60}$$

If we take limit superior as $n \to \infty$ in (**??**) and $\varepsilon > 0$ is arbitrary, we get

$$\limsup_{n\to\infty}||T_F^j(A_n,B_n)-(A_n,B_n)|| \le k_jf^{-1}[2rk_j(1-k_j^{-1})]. \tag{61}$$

And then, if we use the definition of A_n and B_n, for all $j \geq 1$ we obtain

(62)

$$\begin{aligned} ||T_F^j(q_1,q_2)-(q_1,q_2)|| &= ||(F^j(q_1,q_2),F^j(q_2,q_1))-(q_1,q_2)|| \\ &= ||F^j(q_1,q_2)-q_1||+||F^j(q_2,q_1))-q_2|| \\ &\leq ||F^j(q_1,q_2)-F^j(A_n,B_n)||+||F^j(A_n,B_n)-A_n||+||A_n-q_1|| \\ &\quad +||F^j(q_2,q_1)-F^j(B_n,A_n)||+||F^j(B_n,A_n)-B_n||+||B_n-q_2|| \\ &\leq (k_j+1)[||A_n-q_1||+||B_n-q_2||]+||F^j(A_n,B_n)-A_n||+||F^j(B_n,A_n)-B_n|| \\ &\leq \frac{2}{n}(k_j+1)+||F^j(A_n,B_n)-A_n||+||F^j(B_n,A_n)-B_n|| \\ &= \frac{2}{n}(k_j+1)+||T_F^j(A_n,B_n)-(A_n,B_n)||. \end{aligned}$$

If we take limit superior in the inequality (**??**), from (**??**) we get

$$||T_F^j(q_1,q_2)-(q_1,q_2)|| \leq k_j f^{-1}[2rk_j(1-k_j^{-1})]. \tag{63}$$

Therefore, the limit superior in the above inequality derives

$$\limsup_{j\to\infty} ||T_F^j(q_1,q_2)-(q_1,q_2)|| \leq f^{-1}(0)=0, \tag{64}$$

which implies that $||T_F^j(q_1,q_2)-(q_1,q_2)|| \to 0$ as $j\to\infty$.
Since T_F is continuous, we have

$$(q_1,q_2)=\lim_{j\to\infty} T_F(q_{1_{j+1}},q_{2_{j+1}})=\lim_{j\to\infty} T_F(T_F(q_{1_j},q_{2_j}))=T_F(q_1,q_2). \tag{65}$$

So, T_F has a fixed point (q_1,q_2). □

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