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ENERGY SPECTRUM OF SPINLESS PARTICLES IN ELECTROMAGNETIC FIELDS

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Abstract. Dynamics of the non-relativistic and relativistic charged spinless particles subjected to space-dependent parallel and orthogonal electromagnetic fields is investigated by solving Schrödinger and Klein-Gordon equations. Exact solutions of the motion are used to obtain the quantized energy spectrum and momentum of the particles. Some numerical results for the first few quantum levels are determined with the help of MATHEMATICA software.

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1. Introduction

Finding the exact solutions of the wave equations for the external fields is one of the old problems. Among these equations Schrödinger and Klein-Gordon are the most studied ones. Besides by the increase in the applications of the electric and magnetic fields in fundamental areas of technology, especially in electromechanics, health physics and so forth ,a significant interest has been given to these solutions. Such efforts have been performed for different configurations of the external fields [1-3].

These studies provide remarkable information regarding the quantum mechanical systems. Some of these attempts are the interpretation of the physical processes. The most important ones are Compton scattering by a laser source, Brownian motion, coherent states, and energy levels of electrons.

There are very few studies in the literature on the solution of the wave equation of the spinless particles in the presence of both electric and magnetic fields. The aim of this study is to move this attempt one step further by obtaining the exact solutions of the spinless particles for two orientations of decaying electric and magnetic fields given by $Case(i)$ $A_0 = \frac{E_0}{z}$, $A_1 = \frac{B_0}{y}$ and $Case(ii)$ $A_0 = \frac{E_0}{y}$, $A_1 = \frac{B_0}{y}$, where E_0 and B_0 are constants. The first and second cases belong to the parallel and orthogonal fields, respectively. We note that y and z variables are defined in the region $(0, \infty)$ to keep the finite external fields. Such kind of varying electromagnetic field is encountered in semiconductor heterostructures.

In the following sections, the exact solutions for nonrelativistic and relativistic cases will be obtained, respectively. By comparing the solutions of the nonrelativistic and relativistic wave equations of the spinless particles, contributions coming

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from the relativistic effects will be considered and by using the mathematical properties of the wave functions we will obtain the energy spectrum and exact solutions for both cases.

2. Solution of the Schrodinger Equation

Motion of the nonrelativistic spinless particles is described by the Schrödinger equation and in the existence of the external electromagnetic fields, it is given by (we take $\hbar = 1$)

$$\left[\frac{\left(\vec{P} - e\vec{A}\right)^2}{2m}\right]\Phi = (i\frac{\partial}{\partial t} - eA_0)\Phi \tag{2.1}$$

where e is charge, m is mass ot the particle, $\vec{A}$ is the vector electromagnetic potential. In the following steps we solve the Schrödinger equation for the cases where electric and magnetic fields are parallel and perpendicular to each other.

2.1. **Case (i) Parallel EM Fields.** For the choice of $A_0 = \frac{E_0}{z}$, $A_1 = \frac{B_0}{y}$, $\vec{E} \parallel \vec{B}$. We define the solution of (2.1) by

$$\Phi_{\parallel} = e^{i(P_x - \epsilon t)} H(y) K(z) \tag{2.2}$$

Plugging this solution into (2.1) we obtain,

$$\left[\underbrace{\left(P_x - \frac{eB_0}{y}\right)^2 + P_y^2}_{\hat{Q}(y)} + \underbrace{P_z^2 - 2m\left(\epsilon - \frac{eE_0}{z}\right)}_{\hat{D}(z)}\right] H(y)K(z) = 0$$

In short we can write

$$\left[\hat{Q}(y) + \hat{D}(z)\right] H(y)K(z) = 0$$

Separating this equation with respect to y and z, we obtain

$$\left[\hat{Q}(y) + b\right] H(y) = 0 \tag{2.3}$$

$$\left[\hat{D}(z) - b\right] K(z) = 0 \tag{2.4}$$

where b is the constant of separation.

Let $\gamma^2 = (P_x^2 + b)$, and making $\rho = 2\gamma y$ change of variable (2.3) becomes Whittaker equation [4]

$$\left[\frac{d^2}{d\rho^2} - \frac{e^2 B_0^2}{\rho^2} + \frac{2eB_0 P_x}{\gamma\rho} - \frac{1}{4}\right] H(\rho) = 0 \tag{2.5}$$

So exact solution of (2.3) is

$$H(y) = W_{\lambda,\mu}(2\gamma y) \tag{2.6}$$

where $\mu^2 = \frac{1}{4} + e^2 B_0^2$, and $\lambda = \frac{eB_0 P_x}{\gamma}$.

In order Whittaker function to be bounded [4]

$$\mu - \lambda = -(n + \frac{1}{2}) = -N, \quad n = 0, 1, 2, \cdots$$

So from this equality we find

$$b = P_x^2 \left(\frac{1}{1 + \frac{(1/4+N^2)}{e^2 B_o^2} + 2NeB_o \sqrt{1 + \frac{1}{4e^2 B_0^2}}} - 1 \right)$$

Now for the solution of (2.4), this equation is written as

$$\left[P_z^2 - 2m \left(\epsilon - \frac{eE_0}{z} \right) - b \right] K(z) = 0$$

$$\left[\frac{d^2}{dz^2} - \frac{2meE_0}{z} + (2m\epsilon + b) \right] K(z) = 0 \tag{2.7}$$

(2.7) is similar to the below equation

Definition 2.1.

$$xy'' + (ax+b)y' + (cx+d)y = 0$$

For $a^2 > 4c$ solution is given by [5]

$$y = x^{-\frac{b}{2}} e^{-\frac{ax}{2}} {}_1F_1 \left(\frac{2d - ab}{2\sqrt{a^2 - 4c}}, \frac{b-1}{2}, x\sqrt{a^2 - 4c} \right)$$

Returning to the equation (2.7),

$$zK''(z) + \left[(2m\epsilon + b)z - 2meE_0 \right] K(z) = 0$$

for $0 > 2m\epsilon + b$

$$K(z) = {}_1F_1 \left(\frac{-meE_0}{\sqrt{-(2m\epsilon + b)}}, -\frac{1}{2}, z\sqrt{-4(2m\epsilon + b)} \right)$$

From the requirement of Hypergeometric functions to be finite

$$\frac{-2meE_0}{\sqrt{-4(2m\epsilon + b)}} = -n$$

where $n = 0, 1, 2, ...$ we obtain the energy spectrum of Schrödinger equation for the parallel case as

$$\epsilon_{\parallel} = \frac{P_x^2}{2m} \left[1 - \frac{1}{1 + \frac{(1/4+N^2)}{e^2 B_o^2} + 2NeB_o \sqrt{1 + \frac{1}{4e^2 B_0^2}}} \right] - \frac{me^2 E_0^2}{2n^2}$$

So the exact solution of (2.1) for parallel case is written as

$$\Phi_{\parallel} = e^{i(xP_x - \epsilon t)} W_{\lambda,\mu}(2\gamma y) {}_1F_1(z)$$

2.2. **Case (ii) Orthogonal EM Fields.** For the choice of $A_0 = \frac{E_0}{y}$, $A_1 = \frac{B_0}{y}$, $\vec{E} \perp \vec{B}$. In this case, we will look for the solution of (2.1) as

$$\Phi_{\perp} = e^{i(xP_x + zP_z - \epsilon t)} M(y)$$

Writing this in (2.1), we obtain

$$\left[\frac{d^2}{dy^2} - \frac{e^2 B_0^2}{y^2} + \frac{2e(P_x B_0 - mE_0)}{y} + (2m\epsilon - P_x^2 - P_z^2) \right] M(y) = 0$$

Again solution of this equation is given by Whittaker function as

$$M(y) = W_{\kappa,\sigma}(2uy)$$

where $\kappa = \frac{e(P_x - mE_0)}{iu}$, $\sigma^2 = \frac{1}{4} - e^2B_0^2$, $u^2 = (2m\epsilon - P_x^2 - P_z^2)$. For Whittaker functions,

$$\sigma - \kappa = -(n + 1/2)$$

should be satisfied. From this condition, we obtain the energy spectrum for the Schrödinger equation for the orthogonal case

$$\epsilon_\perp = \frac{1}{2m}\left[P_x^2 + P_z^2 - \frac{e^2P_x^2B_0^2 + e^2m^2E_0^2 - 2e^2mP_xB_0E_0}{\frac{1}{4} + e^2B_0^2 + N^2 + 2N\sqrt{\frac{1}{4} + e^2B_0^2}}\right]$$

So the exact solution of (2.1) in orthogonal case is written as

$$\Phi_\perp = e^{i(xP_x + zP_z - \epsilon t)}W_{\kappa,\sigma}(2uy)$$

3. Solution of the Klein-Gordon Equation

The Klein-Gordon equation for the relativistic spinless particles is given by (we take $\hbar = 1$)

$$\left[(\vec{P} - e\vec{A})^2 + m^2\right]\phi = (P_0 - eA_0)^2\phi \tag{3.1}$$

3.1. **Case (i) Parallel EM Fields.**

Again we will look for the solution as

$$\phi_\parallel = e^{i(xP_x - \epsilon t)}F(y)G(z)$$

writing this in (3.1) we obtain

$$\left[\underbrace{-\frac{d^2}{dy^2} + (P_x - \frac{eB_0}{y})^2}_{\hat{Q}(y)} - \underbrace{\frac{d^2}{dz^2} - (\epsilon - \frac{eE_0}{z})^2}_{\hat{R}(z)} + m^2\right]F(y)G(z) = 0$$

In short we can write

$$\left[\hat{Q}(y) + \hat{R}(z) + m^2\right]F(y)G(z) = 0$$

Separating this equation with respect to y and z, we obtain

$$\left[\hat{Q}(y) + s\right]F(y) = 0 \tag{3.2}$$

$$\left[\hat{R}(z) + m^2 - s\right]G(z) = 0 \tag{3.3}$$

where s is the separation constant.

Equation (3.2) is written as

$$\left[-\frac{d^2}{dy^2} + (P_x - \frac{eB_0}{y})^2 + s\right]F(y) = 0 \tag{3.4}$$

This equation is the same equation obtained in the Schrödinger case. So the solution is

$$F(y) = W_{\lambda,\mu}(2\gamma y)$$

where $\mu = \pm\sqrt{\frac{1}{4} + e^2B_0^2}$, $\lambda = \frac{eB_0}{\sqrt{1 + \frac{s}{P_x^2}}}$ and $\gamma = \sqrt{P_x^2 + s}$

As before

$$s = P_x^2 \left(\frac{1}{1 + \frac{(1/4+N^2)}{e^2 B_o^2} + 2NeB_o\sqrt{1 + \frac{1}{4e^2 B_0^2}}} - 1 \right)$$

Equation (3.3) is written as

$$\left[-\frac{d^2}{dz^2} - \left(\epsilon - \frac{eE_0}{z} \right)^2 + m^2 - s \right] G(z) = 0 \tag{3.5}$$

Again solution of this equation is given by Whittaker functions

$$G(z) = W_{\tilde{\lambda},\tilde{\mu}}(2\alpha z)$$

where $\tilde{\lambda} = \frac{ieE_0\epsilon}{\alpha}$, $\tilde{\mu} = \pm\sqrt{\frac{1}{4} - e^2 E_0^2}$, and $\alpha^2 = \epsilon^2 - m^2 + s$ and the energy spectrum for the parallel case is given by

$$\epsilon_{\|} = \pm \left[\frac{(m^2 - s)\left(\frac{1}{4} - e^2 E_0^2 + N^2 + 2\tilde{N}\sqrt{\frac{1}{4} - e^2 E_0^2}\right)}{\frac{1}{4} + \tilde{N}^2 + 2\tilde{N}\sqrt{\frac{1}{4} - e^2 E_0^2}} \right]^{\frac{1}{2}}$$

and the exact solution of Klein-Gordon equation for the parallel case is given by

$$\phi_{\|} = e^{i(xP_x - \epsilon t)} W_{\lambda,\mu}(2\gamma y) W_{\tilde{\lambda},\tilde{\mu}}(2\alpha z)$$

3.2. **Case (ii) Orthogonal EM Fields.** For the choice of $A_0 = \frac{E_0}{y}$, $A_1 = \frac{B_0}{y}$, $\vec{E} \perp \vec{B}$.

Again we will look for the solution of (3.1) as

$$\phi_{\perp} = e^{i(xP_x + zP_z - \epsilon t)} N(y)$$

Writing this in (3.1), we obtain

$$\left[\frac{d^2}{dy^2} + \frac{e^2(E_0^2 - B_0^2)}{y^2} + \frac{2e(P_x B_0 - \epsilon E_0)}{y} + (\epsilon^2 - m^2 - P_x^2 - P_z^2) \right] N(y) = 0$$

Again solution of this equation is given by Whittaker function as

$$N(y) = W_{\tilde{\kappa},\tilde{\sigma}}(2vy)$$

where $\tilde{\kappa} = \frac{e(P_x - \epsilon E_0)}{iv}$, $\tilde{\sigma}^2 = \frac{1}{4} - e^2(E_0^2 - B_0^2)$, $v^2 = (\epsilon^2 - m^2 - P_x^2 - P_z^2)$. For Whittaker functions,

$$\tilde{\sigma} - \tilde{\kappa} = -(\tilde{n} + 1/2)$$

should be satisfied. From this condition, we obtain the energy spectrum for the Klein-Gordon equation for the orthogonal case from below quadratic equation

$$\epsilon^2 \underbrace{(w^2 + e^2 E_0^2)}_{a} + \epsilon \underbrace{(-2e^2 P_x B_0 E_0)}_{b} + \underbrace{e^2 P_x^2 B_0^2 - w^2(m^2 + P_x^2 + P_z^2)}_{c} = 0$$

where $w = \sqrt{\frac{1}{4} - e^2(E_0^2 - B_0^2)} + \tilde{N}$.

$$\epsilon_{\perp} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Exact solution of the (3.1) for the orthogonal case is

$$\phi_{\perp} = e^{i(xP_x + zP_z - \epsilon t)} W_{\tilde{\kappa},\tilde{\sigma}}(2vy)$$

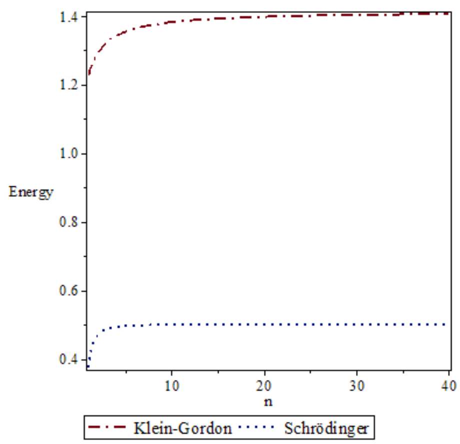

FIGURE 1. Parallel Case

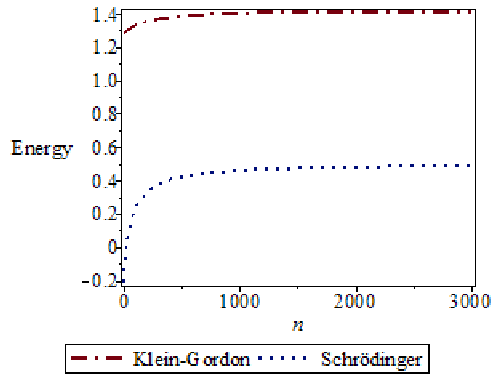

FIGURE 2. Orthogonal Case

4. Conclusion

We investigated the motion of the spin-0 particles in electromagnetic fields for parallel and orthogonal orientations. Analysis is performed for Schrödinger and Klein-Gordon cases and that present us the contribution of the relativistic effects. In the case of $\overrightarrow{E} \parallel \overrightarrow{B}$, the relativistic effects arise only for the motion in the z-direction. In that case the Whittaker functions that occurred in the relativistic solutions are replaced by the confluent hypergeometric function for nonrelativistic solutions. In case of the orthogonal fields $\overrightarrow{E} \perp \overrightarrow{B}$, exact solutions of the Schrödinger and Klein-Gordon equations are found in terms of the Whittaker functions.

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