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FUZZY RIGHT FRACTIONAL OSTROWSKI INEQUALITIES

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Abstract. In this paper, firstly fuzzy basic concept is studied.We investigated other Ostrowski type inequalities in literature. We obtained the very general fuzzy fractional Ostrowski type inequality with right fractional Caputo derivative using the Hölder inequality in this type.

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1. Introduction

Mathematical inequalities take an important place among mathematical concepts.These enable us to find the values of these quantities approximately. Mathematical inequalities have also important applications in functional analysis. For example when building norms on some linear spaces.

The following result is known in the literature as an Ostrowski's inequality. In 1938, the classical integral inequality was proved by A.M. Ostrowski [9] .

The inequality of Ostrowski gives us an estimation for the deviation of the values of a smooth function from its mean value. More precisely, if $f : [a, b] \to \mathbb{R}$ is a differentiable function with bounded derivative, then

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \leq \left[\frac{1}{4} + \frac{(x - \frac{a+b}{2})^2}{(b-a)^2} \right] (b-a) \left\| f' \right\|_\infty$$

for every $x \in [a, b]$. Moreover the constant $1/4$ in the right side of the inequality is the best possible value for the better result.

The theory of fractional calculus has known an intensive development over the last few decades. It is shown that derivatives and integrals of fractional type provide an adequate mathematical modelling of real objects and processes see $[7] - [8]$.

We notice that the first generalization of Ostrowski's inequality was given by Milanovic and Pecaric in [2].

In [10] Pachpatte has proved the Ostrowski inequality in three independent variables. In the past few years, many authors have obtained various generalizations of this type of inequality and many researchers worked on a fractional form of it as well as on time scale calculus [11] .

Univariate right fractional Ostrowski inequalities has been shown by Anatassiou [12].

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Fuzzy sets were defined in [1] A standard fuzzy set in X is characterized by a membership function $\mu : X \to [0,1]$ A standard fuzzy set is called normalized if $\sup_{x\in X} \mu(x) = 1$

Fuzzy fractional calculus and the Ostrowski inequalities have been studied by Anatassiou [5].

The main purpose of this manuscript is to establish Ostrowski-type inequality involving right Caputo differentiability. First of all; we give basic information about the fuzzy set .Then ,we introduce the very general univariate fuzzy fractional Ostrowski type inequality.We show this inequality in fuzzy space.

2. Background

We need the following basic concepts

Definition 2.1. [5] Let $\mu : \mathbb{R} \to [0,1]$ with the following properties

i) is normal ,i.e.,$\exists x_0 \in \mathbb{R}$; $\mu(x_0) = 1$
ii) $\mu(\lambda x + (1-\lambda) y) \geq \min\{\mu(x), \mu(y)\}$,$\forall x, y \in \mathbb{R}$, $\forall \lambda \in [0,1]$ (μ is called a convexs fuzzy subset).
iii) μ is upper semicontinouns on $\mathbb{R}$, i.e. $\forall$ $x_0 \in \mathbb{R}$ and $\forall \epsilon > 0, \exists$ neighborhood $V(x_0) : \mu(x) \leq \mu(x_0) + \epsilon, \forall x \in V(x_0)$
iv) The set $\overline{supp(\mu)}$ is compact in $\mathbb{R}$. where $(supp(\mu) = \{x \in \mathbb{R} \ : \mu(x) > 0\})$

We call μ a fuzzy real number.Denote the set of all μ with $\mathbb{R}_{\mathcal{F}}$.E.g., $\chi_{\{x_0\}} \in \mathbb{R}_{\mathcal{F}}$, for any $x_0 \in \mathbb{R}$, where $\chi_{\{x_0\}}$ is the characteristic function at x_0.

For $0 < r \leq 1$ and $\mu \in \mathbb{R}_{\mathcal{F}}$ define

$$[\mu]^r = \{x \in \mathbb{R} : \mu(x) \geq r\}$$

and

$$[\mu]^0 = \overline{\{x \in \mathbb{R} : \mu(x) \geq r\}}$$

Then it is well known that for each $r \in [0,1]$, $[\mu]^r$ is a closed and bounded interval of $\mathbb{R}$.For $u, v \in \mathbb{R}_{\mathcal{F}}$ and $\lambda \in \mathbb{R}$,we define uniquely the sum $u \oplus v$ and the product $\lambda \odot u$ by

$$[u \oplus v]^r = [u]^r + [v]^r , \ [\lambda \odot u]^r = \lambda [u]^r , \forall r \in [0,1]$$

Notice $1 \odot u = u$ and its holds

$$u \oplus v = v \oplus u, \lambda \odot u = u \odot \lambda$$

İf $0 \leq r_1 \leq r_2 \leq 1$ then $[\mu]^{r_2} \subseteq [\mu]^{r_1}$ .Actually $[u]^r = \left[u_-^r, u_+^r\right]$, $u_-^r \leq u_+^r$, $u_-^r, u_+^r \in \mathbb{R}$, $\forall r \in [0,1]$

For $\lambda > 0$ one has $\lambda_\pm^r = (\lambda \odot u)_\pm^r$, respectively.

Definition 2.2. [5] $D : \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}} \to \mathbb{R}_+ \cup \{0\}$

$$D(u,v) = \sup\max\left\{\left| u_-^r - v_-^r\right|, \left| u_+^r - v_+^r\right|\right\}_{r\in[0,1]}$$

where $[u]^r = \left[u_-^r, u_+^r\right]$; $u, v \in \mathbb{R}_{\mathcal{F}}$. We have that D is a metric on $\mathbb{R}_{\mathcal{F}}$. Then $(\mathbb{R}_{\mathcal{F}},\ D)$ is a complete metric space with the following properties:

$i)$ $D(u \oplus w, v \oplus w) = D(u, v)$ $\forall u, v, w \in \mathbb{R}_{\mathcal{F}}$
$ii)$ $D(\lambda \odot u, \lambda \odot v) = |\lambda|\, D(u, v)$ $\forall \lambda \in \mathbb{R}, \forall u, v \in \mathbb{R}_{\mathcal{F}}$
$iii)$ $D(\ u \oplus v, w \oplus e) \leq D(u \oplus w) + D(v \oplus e), \forall u, v, w, e \in \mathbb{R}_{\mathcal{F}}$

Here $\sum^*$ is stands for fuzzy summation and $\widetilde{0} : \chi_{\{0\}} \in \mathbb{R}_{\mathcal{F}}$ is the neutral element with respect to $\oplus$,i.e.,

$$u \oplus \widetilde{0} = \widetilde{0} \oplus u = u\ , \forall u \in \mathbb{R}_{\mathcal{F}}$$

Denote

$$D^*(f, g) = \sup D(f, g)_{x \in [a,b]}$$

Where $f, g : [a, b] \to \mathbb{R}_{\mathcal{F}}$.

We define $C_F^U([a, b])$ the space of uniformly continuous functions from $[a, b] \to \mathbb{R}_{\mathcal{F}}$,also $C_F([a, b])$ the space of fuzzy continuous functions on $[a, b]$.İt is clear that

$$C_F^U([a, b]) = C_F([a, b])$$

and $L_F([a, b])$ is the space of Lebesque integrable functions.

Definition 2.3. [13] Let $u, v \in \mathbb{R}_{\mathcal{F}}$.If there exists $w \in \mathbb{R}_{\mathcal{F}}$ such that $u = v + w$, the w is called the Hukuhara difference of u and v, and it is denoted by $u \ominus v$.

Definition 2.4. [13] Let $u, v \in \mathbb{R}_{\mathcal{F}}$.If there exists $w \in \mathbb{R}_{\mathcal{F}}$ such that

$$u \ominus_{gH} v = w \Longleftrightarrow \begin{cases} (i) & u = v + w \\ or\ \ (ii) & u = v + (-1)w \end{cases}$$

Then w is called the generalized Hukuhara difference of u and v.

Please note that a function $f : [a, b] \to \mathbb{R}_{\mathcal{F}}$ so called fuzzy-valued function.The $r-$level representation of fuzzy-valued function f is expressed by

$$f_r(t) = \left[f_r^-(t), f_r^+(t)\right], t \in [a, b]\,, r \in [0, 1]$$

Here, $f^r(t) = f_r(t)$

Definition 2.5. [5] Let $f : [a, b] \to \mathbb{R}_{\mathcal{F}}$ We say that f is Fuzzy-Riemann integrable to $I \in \mathbb{R}_{\mathcal{F}}$ if for any $\epsilon > 0$,there exits $\delta > 0$ such that for any division $P = \{[u, v]\,; \xi\}$ of $[a, b]$ with the norms $\Delta(P) < \delta$,we have

$$D\left(\sum_P^* (v - u) \odot f(\xi)\,, I\right) < \epsilon$$

We write

$$I = (FR) \int_a^b f(x)\, dx$$

Theorem 2.1. [9] *Let $f:[a,b]\rightarrow\mathbb{R}_{\mathcal{F}}$ be fuzzy continuous.Then $(FR)\int\limits_a^b f(x)\,dx$ exists and belongs to $\mathbb{R}_{\mathcal{F}}$, furthermore it holds*

$$\left[(FR)\int\limits_a^b f(x)\,dx\right]^r=\left[\int\limits_a^b f_-^{(r)}(x)\,dx,\int\limits_a^b f_+^{(r)}(x)\,dx\right],\ r\in[0,1]$$

Theorem 2.2. [5] *Let $f\in C_F([a,b])$ and $c\in[a,b]$ .Then*

$$(FR)\int\limits_a^b f(x)\,dx=(FR)\int\limits_a^c f(x)\,dx+(FR)\int\limits_c^b f(x)\,dx$$

Theorem 2.3. [5] *Let $f,g\in C_F([a,b])$ and $c_1,c_2\in\mathbb{R}$.Then*

$$(FR)\int\limits_a^b (c_1f(x)+c_2g(x))\,dx=c_1(FR)\int\limits_a^b f(x)\,dx+c_2(FR)\int\limits_a^b g(x)\,dx$$

also we need

Lemma 2.1. [5] *If $f,g:[a,b]\subseteq\mathbb{R}\rightarrow\mathbb{R}_{\mathcal{F}}$ are fuzzy continuous functions,then the function $F:[a,b]\rightarrow\mathbb{R}_+$ defined by $F(x)=D(f(x),g(x))$ is continuous on $[a,b]$:*

$$D\left((FR)\int\limits_a^b f(x)\,dx,(FR)\int\limits_a^b g(x)\,dx\right)\leq(FR)\int\limits_a^b D(f(x),g(x))\,dx$$

Definition 2.6. [4] Let $f\in C_F([a,b])\cap L_F([a,b])$, $0<v\leq1$.

The fuzzy Riemann-Liouville integral of fuzzy-valued function f is defined as following:

$$\begin{aligned}\left(I_{\alpha+}^vf\right)(x)&=\frac{1}{\Gamma(v)}\odot\int\limits_a^x(x-t)^{v-1}\odot f(t)dt,x\in[a,b]\\ I_{\alpha+}^0f(x)&=f\end{aligned}$$

Let us consider the $r-level$ representation of fuzzy-valued function f as $f_r(t)=[f_r^-(t),f_r^+(t)]\,,t\in[a,b]\,,r\in[0,1]$

Also,we define the fuzzy fractional right Riemann-Liouville operator by

$$\begin{aligned}I_{b-}^vf(x)&=\frac{1}{\Gamma(v)}\odot\int\limits_x^b(t-x)^{v-1}\odot f(t)dt,x\in[a,b]\\ I_{b-}^0f(x)&=f\end{aligned}$$

Above, Γ denotes the gamma function:

$$\Gamma(v)=\int\limits_0^\infty e^{-t}t^{v-1}dt$$

Definition 2.7. [4] Let $f \in C_F([a,b]) \cap L_F([a,b])$, x_0 in (a,b) and $\Phi(x) = \frac{1}{\Gamma(1-v)} \int\limits_a^x \frac{f(t)}{(x-t)^v} dt$ .We say that f is Riemann-Liouville H-differentiable about order $0 < v < 1$ at x_0 , if there exists an element $\left({}^{RL}D^v_{\alpha^+}\right)(x_0) \in \mathbb{R}_F$,such that for $h > 0$ sufficiently small

$i)$ $\left({}^{RL}D^v_{\alpha^+}\right)(x_0) = \lim\limits_{h\to 0^+} \frac{\Phi(x_0+h)\ominus\Phi(x_0)}{h} = \lim\limits_{h\to 0^+} \frac{\Phi(x_0)\ominus\Phi(x_0-h)}{h}$
or
$ii)$ $\left({}^{RL}D^v_{\alpha^+}\right)(x_0) = \lim\limits_{h\to 0^+} \frac{\Phi(x_0)\ominus\Phi(x_0+h)}{-h} = \lim\limits_{h\to 0^+} \frac{\Phi(x_0-h)\ominus\Phi(x_0)}{-h}$
or
$iii)$ $\left({}^{RL}D^v_{\alpha^+}\right)(x_0) = \lim\limits_{h\to 0^+} \frac{\Phi(x_0+h)\ominus\Phi(x_0)}{h} = \lim\limits_{h\to 0^+} \frac{\Phi(x_0-h)\ominus\Phi(x_0)}{-h}$
or
$iv)$ $\left({}^{RL}D^v_{\alpha^+}\right)(x_0) = \lim\limits_{h\to 0^+} \frac{\Phi(x_0)\ominus\Phi(x_0+h)}{-h} = \lim\limits_{h\to 0^+} \frac{\Phi(x_0)\ominus\Phi(x_0-h)}{h}$

3. Main results

Definition 3.1. [14] Let $f \in C_F([a,b]) \cap L_F([a,b])$ be a fuzzy set-valued function.Then f is said to be Caputo's H-differentiable at x when

$$\left(D^v_{\alpha+}\, f(x)\right) = \frac{1}{\Gamma(1-v)} \int\limits_a^x \frac{f'(t)}{(x-t)^v} dt$$

where $0 < \alpha < 1$ and $0 < v < 1$.

Also,we adopt the some procedure to present Caputo's H-differentiability,we say f is

$[(i)-v]$-differentiable if Eq. (8) holds while f is (i)-differentiable,and f is $[(ii)-v]$-differentiable if Eq. (21) holds while f is (ii)-differentiable.

Definition 3.2. [15] Let $f \in C_F([a,b]) \cap L_F([a,b])$, f^n is integrable.Then the right fuzzy Caputo derivate of f for $n-1 < v < n$, and $x \in [a,b]$, $D^v_{b-}f(x) \in \mathbb{R}_F$ and defined by

$$D^v_{b-}f(x) = \frac{(-1)^n}{\Gamma(n-v)} \odot \int\limits_x^b (t-x)^{-v+n-1} \odot f^n(t) dt$$

and for $n = 1$

$$D^v_{b-}f(x) = \frac{(-1)}{\Gamma(1-v)} \odot \int\limits_x^b (t-x)^{-v} \odot f'(t) dt$$

Theorem 3.1. [14] *Let* $f \in C_F([a,b]) \cap L_F([a,b])$, $0 < v < 1, 0 \le r \le 1,$

$i)$ Let f be (ii)-differentiable, then we have $[(i)-v]$ differentiable right fuzzy Caputo derivative and

$$\left(D^v_{b-}f\right)(x,r) = \left[\left(D^v_{b-}f_-\right)(x,r),\ \left(D^v_{b-}f_+\right)(x,r)\right]$$

ii) Let f be (i)-differentiable, then we have $[(ii)-v]$ differentiable right fuzzy Caputo derivative and

$$\left(\, D_{b-}^{v} f\right)(x,r) = \left[\, \left(\, D_{b-}^{v} f_{+}\right)(x,r)\,,\ \left(\, D_{b-}^{v} f_{-}\right)(x,r)\right]$$

Theorem 3.2. [14] *Let $0 < V < 1$, $D_{b-}^{v} f\,(x) = g(x, f(x))$ with the fuzzy initial condition $f_0 = f(b)$,the fuzzy fractional differential equation is equivalent to one of the following integral equations:*

i) if f is a $[(i)-v]$ differentiable fuzzy-valued function,then

$$f(x) = f(b) \oplus \frac{1}{\Gamma(v)} \odot \int_{x}^{b} (t-x)^{v-1} \odot (D_{b-}^{v} f)(t)dt$$

ii) if f is a $[(ii)-v]$ differentiable fuzzy-valued function,then

$$f(x) = f(b) \oplus \frac{-1}{\Gamma(v)} \odot \int_{x}^{b} (t-x)^{v-1} \odot (D_{b-}^{v} f)(t)dt$$

Theorem 3.3. *Let $f \in C_F\left([a,b]\right) \cap L_F\left([a,b]\right)$, $0 < v < 1$, $p, q > 0$such that $\frac{1}{p}+\frac{1}{q}= 1$,and $\left(D_{b-}^{v} f\right)(x) \in \mathbb{R}_{F}\,; (t \in [a,b])$*

$$D\left(\frac{1}{b-a} \odot (FR) \int_{a}^{b} f\,(x)\,dx, f(b)\right) \leq \frac{sup\ D((D_{b-}^{v} f)(t), \widetilde{0})}{\Gamma(v)(p(v-1)+1)^{\frac{1}{p}}(v+\frac{1}{p})}(b-a)^{v-1+\frac{1}{p}}$$

Proof. We have

$$\begin{aligned}
D\left(\tfrac{1}{b-a} \odot (FR) \int_{a}^{b} f\,(x)\,dx, f(b)\right) &= D\left(\tfrac{1}{b-a} \odot (FR) \int_{a}^{b} f\,(x)\,dx, \tfrac{f(b)}{b-a} \int_{a}^{b} dx\right) \\
&= D\left(\tfrac{1}{b-a} \odot (FR) \int_{a}^{b} f\,(x)\,dx, \tfrac{1}{b-a} \odot (FR) \int_{a}^{b} f\,(b)\,dx\right) \\
&= \tfrac{1}{b-a} D\left((FR) \int_{a}^{b} f\,(x)\,dx, (FR) \int_{a}^{b} f\,(b)\,dx\right) \\
&\leq \tfrac{1}{b-a} \int_{a}^{b} D\left(f\,(x)\,, f(b)\right) dx \qquad\qquad (*)
\end{aligned}$$

Here $[(i)-v]$ differentiable .

We notice that $f \in C_F\left([a,b]\right) \cap L_F\left([a,b]\right)$, $0 < v < 1$,

$$f(x) = f(b) \oplus \frac{1}{\Gamma(v)} \odot \int_{x}^{b} (t-x)^{v-1} \odot (D_{b-}^{v} f)(t)dt$$

For $a \leq x \leq b$,we have

$$\begin{aligned}
D\left(f\left(x\right), f(b)\right) &= D\left(f(b) \oplus \frac{1}{\Gamma(v)} \odot \int_x^b (t-x)^{v-1} \odot (D_{b-}^v f)(t)dt, f(b)\right) \\
&= D\left(\frac{1}{\Gamma(v)} \odot \int_x^b (t-x)^{v-1} \odot (D_{b-}^v f)(t)dt, \widetilde{0}\right) \\
&\leq \frac{1}{\Gamma(v)} D\left(\int_x^b (t-x)^{v-1} \odot (D_{b-}^v f)(t)dt, \widetilde{0}\right) \\
&\leq \frac{1}{\Gamma(v)} D\left(\int_x^b (t-x)^{v-1} \odot (D_{b-}^v f)(t)dt, \int_x^b \widetilde{0} dt\right) \\
&\leq \frac{1}{\Gamma(v)} \int_x^b (t-x)^{v-1} \left(D\left((D_{b-}^v f)(t), \widetilde{0}\right)\right) dt \\
&\leq \frac{1}{\Gamma(v)} \left(\int_x^b (t-x)^{p(v-1)}\right)^{\frac{1}{p}} \left(\int_x^b \left(D\left((D_{b-}^v f)(t), \widetilde{0}\right)\right)^q dt\right)^{\frac{1}{q}} \\
&\leq \frac{1}{\Gamma(v)} \frac{(b-x)^{v-1+\frac{1}{p}}}{(p(v-1)+1)^{\frac{1}{p}}} \left(\int_x^b \left(D\left((D_{b-}^v f)(t), \widetilde{0}\right)\right)^q dt\right)^{\frac{1}{q}}
\end{aligned}$$

Now , $\forall x \in [a, b]$ and for $(*)$

$$\begin{aligned}
D\left(\frac{1}{b-a} \odot (FR) \int_a^b f\left(x\right) dx, f(b)\right) &\leq \frac{1}{b-a} \int_a^b D\left(f\left(x\right), f(b)\right) dx \\
&\leq \frac{sup\left(D\left((D_{b-}^v f)(t), \widetilde{0}\right)\right)}{(b-a)\Gamma(v)(p(v-1)+1)^{\frac{1}{p}}} \left(\int_a^b \left((b-x)^{v-1+\frac{1}{p}}\right) dx\right) \\
&= \frac{sup\left(D\left((D_{b-}^v f)(t), \widetilde{0}\right)\right)}{\Gamma(v)(p(v-1)+1)^{\frac{1}{p}}} \frac{(b-a)^{v-1+\frac{1}{p}}}{v+\frac{1}{p}} \quad (t \in [a, b])
\end{aligned}$$

□

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