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ON A NONLOCAL BOUNDARY VALUE PROBLEM

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Abstract. In this study, parabolic partial differential equation with two integral boundary conditions are considered for distribution of family savings for a family set. By seperation of variables method, eigenvalues and eigenfunctions of the problem are obtained and solution is written. Moreover, Method of limes method and Crank Nicolson method are applied to the problem and errors of numerical methods are presented.

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1. Introduction

Integral boundary conditions for parabolic equations are well known problem in applications (see, for example, Cannon[1], Ionkin[7], Kamynin[8], Day[3], Erofeeko and Kozlovski[6]). Such a boundary condition are called nonlocal boundary condition or nonclassical boundary condition. Similar problems are also used for hyperbolic equations.

In this study, we deal with a family saving model which can be represented by Kolmogorov equation with two integral boundary conditions.

Suppose that $x(t)$ denotes the saving of a family at time t and satisfy the differential equation

$$dx = F\left(x,t\right) dt + G\left(x,t\right) dX, \quad G \geq 0 \tag{1.1}$$

where X is the Markov process, $F(x,t)$ is the rate of the change for the family saving and $G\left(x,t\right) dX$ is the random change of the family income.

For a family set let us assume that equation (1.1) describes the saving of all families by ignoring the dynamic of individual family saving. The density distribution of the saving of families $u(x,t)$ satisfies

$$\frac{\partial u}{\partial t} = -\frac{\partial}{\partial x}\left(\left(c(x,t) + F(x,t)\right)\, u\right) + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left(b(x,t)u\right) + f(x,t) \tag{1.2}$$

with initial condition

$$u(x,0) = \varphi(x),\ 0 \leq x \leq l \tag{1.3}$$

and boundary conditions

$$\int_0^l u\left(x,t\right) dx = N\left(t\right), t \geq 0 \tag{1.4}$$

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$$\int_0^l x\,u\,(x,t)\,dx = K\,(t)\,, t \geq 0 \tag{1.5}$$

where $c(x,t)$, $b(x,t)$, $K(t)$, $N(t)$, $\varphi(x)$ anf $f(x,t)$ are continuously differentiable functions. $N(t)$, $K(t)$ denote total number of families and total amount of family saving in $[0,l]$ respectively [6].

2. Special Case of The Model

We will consider special case of problem (1.2)-(1.5) on region $D = (0 < t < \infty) \times (0 < x < l)$

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} + f(x,t), \tag{2.1}$$

$$u(x,0) = \varphi(x),\ 0 \leq t \leq T, \tag{2.2}$$

$$\int_0^1 u\,(x,t)\,dx = N\,(t)\,, t \geq 0, \tag{2.3}$$

$$\int_0^1 x\,u\,(x,t)\,dx = K\,(t)\,, t \geq 0, \tag{2.4}$$

where $f(x,t)$, $K(t)$, $N(t)$, $\varphi(x)$ are continuously differentiable functions on region D. Compatibility conditions of this problem are
$\int_0^1 x\varphi(x)dx = N(0)$ and $\int_0^1 \varphi(x)dx = K(0)$.

Using the transform

$$u(x,t) = v(x,t) + (12K(t) - 6N(t))x + 4N(t) - 6K(t)$$

boundary conditions of equation (2.1)-(2.4) become homogenous:

$$\frac{\partial v}{\partial t} = a^2 \frac{\partial^2 v}{\partial x^2} + F(x,t), \tag{2.5}$$

$$v(x,0) = \psi(x), \tag{2.6}$$

$$\int_0^1 v(x,t)dx = 0, \tag{2.7}$$

$$\int_0^1 xv(x,t)dx = 0, \tag{2.8}$$

where

$$F(x,t) = f(x,t) - (12K'(t) - 6N'(t))x + 4N'(t) - 6K'(t)$$

and

$$\psi(x) = \varphi(x) - (12K(0) - 6N(0))x + 4N(0) - 6K(0).$$

Equations (2.5)-(2.8) are linear with respect to v(x,t), then this problem can split into two auxiliary problems:

i)

$$\frac{\partial v}{\partial t} = a^2 \frac{\partial^2 v}{\partial x^2}, \tag{2.9}$$

$$v(x,0) = \psi(x), \tag{2.10}$$

$$\int_0^1 v(x,t)dx = 0 \tag{2.11}$$

$$\int_0^1 xv(x,t)dx = 0 \tag{2.12}$$

ii)

$$\frac{\partial v}{\partial t} = a^2 \frac{\partial^2 v}{\partial x^2} + F(x,t), \tag{2.13}$$

$$v(x,0) = 0, \tag{2.14}$$

$$\int_0^1 v(x,t)dx = 0 \tag{2.15}$$

$$\int_0^1 xv(x,t)dx = 0 \tag{2.16}$$

If solution of the problem (i) is $v_1(x,t)$ and solution of the problem (ii) is $v_2(x,t)$ then solution of the problem (2.5)-(2.8) is $v(x,t) = v_1(x,t) + v_2(x,t)$.

Integrating both sides of (2.9) with respect to x from 0 to 1 and using integration by parts, integral boundary conditions in (2.11) and (2.12) become, respectively,

$$v_x(1,t) - v_x(0,t) = 0,$$

$$v_x(1,t) - v(1,t) + v(0,t) = 0.$$

Substituting these equations in (2.9)-(2.12), we have

$$\frac{\partial v}{\partial t} = a^2 \frac{\partial^2 v}{\partial x^2}, \tag{2.17}$$

$$v(x,0) = \psi(x), \tag{2.18}$$

$$v_x(1,t) - v_x(0,t) = 0, \tag{2.19}$$

$$v_x(1,t) - v(1,t) + v(0,t) = 0. \tag{2.20}$$

By the separation of variables, a Sturm-Liouville problem and an ODE are, respectively, obtained as

$$X''(x) + \lambda X(x) = 0, \tag{2.21}$$

$$X'(1) - X'(0) = 0, \tag{2.22}$$

$$X'(1) - X(1) + X(0) = 0, \tag{2.23}$$

and

$$T'(t) + \lambda a^2 T(t) = 0. \tag{2.24}$$

Sturm-Liouville problem (2.21)-(2.23) is self adjoint and boundary conditions are regular, and also strongly regular. Therefore, the eigenfunctions of the Sturm-Liouville problem are the Riesz basis on $L^2[0,1]$ (Naimark[11], Kesselman[9], Mikhailov [10]).

Characteristic equation of the Sturm-Liouville problem is

$$2 - 2\cos k - k\sin k = 0, \tag{2.25}$$

where $\sqrt{\lambda} = k$.
It is easily seen that $k_0 = 0$ and $k_{2n} = 2n\pi, (n = 1, 2, \cdots)$ are roots of the equation (2.25). There is also another root of equation (2.25) in $\left[n\pi, \frac{(2n+1)}{2}\pi\right]$. By using Langrange-Burmann formula root is calculated asymptotically as

$$\begin{aligned} k_{2n+1} &= (2n+1)\pi - 4((2n+1)\pi)^{-1} - \frac{32}{3}((2n+1)\pi)^{-3} - \frac{832}{15}((2n+1)\pi)^{-5} \\ &+ O(\frac{1}{n^7}). \end{aligned}$$

Corresponding eigenfunctions are obtained by

$$X_0(x) = 1,$$

$$X_{2n} = \cos(2\pi n)x,\ n = 1, 2, \cdots.$$

$$X_{2n+1} = \frac{-k_n}{2}\cos(k_n x) + \sin(k_n x),\ n = 1, 2, \cdots.$$

Therefore, solution of the problem (2.17)-(2.20) is

$$v_1(x,t) = \sum_{n=0}^{\infty} A_{2n}\cos(2\pi n x)e^{-a^2 4\pi^2 n^2 t} + \sum_{n=1}^{\infty} B_n\left(\frac{-k_n}{2}\cos(k_n x) + \sin(k_n x)\right)e^{-a^2 {k_n}^2 t},$$

where

$$A_0 = \int_0^1 \psi(x)dx,$$

$$A_n = 2\int_0^1 \psi(x)\cos(2\pi n)x, ,\ n = 1, 2, \cdots.$$

$$B_n = \frac{1}{\|X_{2n+1}(x)\|^2}\int_0^1 \psi(x)\left(\frac{-k_n}{2}\cos(k_n x) + \sin(k_n x)\right), ,\ n = 1, 2, \cdots.$$

Solution of the problem (2.13)-(2.16) can be easily obtained by

$$v_2(x,t) = \sum_{n=0}^{\infty} \left[\int_0^t F_{2n}(\tau) e^{-k_n^2(t-\tau)} d\tau \right] X_{2n}(x) + \left[\int_0^t F_{2n+1}(\tau) e^{-k_n^2(t-\tau)} d\tau \right] X_{2n+1}(x),$$

where

$$F_0(\tau) = \int_0^1 F(x,\tau) x dx,$$

$$F_{2n}(\tau) = \int_0^1 F(x,\tau) X_{2n}(x) dx, \; n = 1, 2, \cdots .$$

$$F_{2n+1}(\tau) = \int_0^1 F(x,\tau) X_{2n+1}(x) dx. \; n = 1, 2, \cdots .$$

3. Numerical Solution

Method of Lines [12] and the Crank-Nicolson method [13] are used for numerical solution of problem (2.1)-(2-4). In both methods, the Simpson's rule is used to approximate the integral in (2.3) and (2.4) numerically. We display here a few of numerical results.

Example 3.1.

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + (x^2 - 2)e^t,$$

$$u(x,0) = x^2,$$

$$\int_0^1 u(x,t) dx = (1/6) - 2t,$$

$$\int_0^1 x u(x,t) dx = (1/12) - t.$$

Exact solution of example 1 is $u(x,t) = x - x^2 - 2t$. The absolute relative errors at various spatial lengths for $u(0.5, 0.5)$ are shown in Table 1.

Relative Error at u(0.5,0.5) in Example 1

Spatial Length	MOL Method	Crank-Nicolson Method
h=0.1	1.3471E-14	2.9606E-16
h=0.05	8.4510E-13	2.9606E-16
h=0.025	1.7494E-12	6.8094E-15
h=0.0125	4.6876E-12	1.9244E-15

Example 3.2.

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2},$$
$$u(x,0) = \sin(\pi x),$$
$$\int_0^1 u(x,t)dx = \frac{2}{\pi}\exp(-\pi^2 t),$$
$$\int_0^1 xu(x,t)dx = (1/12) - t.$$

Exact solution of example 2 is $u(x,t) = \sin(\pi x)\exp(-\pi^2 t)$. The absolute relative errors at various spatial lengths for $u(0.5, 0.5)$ are shown in Table 2.

Relative Error at u(0.5,0.5) in Example 2

Spatial Length	MOL Method	Crank-Nicolson Method
h=0.1	0.0029	0.0075
h=0.05	4.5074E-4	0.0023
h=0.025	6.4370E-5	5.6888E-4
h=0.0125	4.4595E-6	9.1160E-5

4. Conclusion

Diffusion equation with two integral boundary conditions is studied. Integral boundary conditions are transformed to local one and by separation of variables, analytic solution of this problem is found. In addition, by applying the Method of Lines [12] and Crank Nicolson method [13], numerical solution of the problem is found.

References

[1] Cannon, J.R., *The solution of the heat equation subject to specification of energy*, Quarterly of Applied Mathematics, Vol. **21**, 155-160, 1963.

[2] Cannon, J. R. ve Van Der Hoek, J., *An implicit finite difference scheme for the diffusion of mass in a portion of the domain*, Numerical Solutions of Partial Differential Equations (J. Noye, Ed), 527-539. 1982

[3] Day, W.A., *Extension of a property of the equation to linear thermoelasticity and other theories*, Quarterly of Applied Mathematics, 1982, 40.3: 319-330.

[4] Dehghan, M. , *On the numerical solution of the diffusion equation with a nonlocal boundary condition*, Mathematical Problems in Engineering, 2003(2), 81-92, 2003

[5] Ekolin, G., *Finite difference methods for a nonlocal boundary value problem for the heat equation*, BIT Numerical Mathematics, 1991, 31.2: 245-261.

[6] Erofeenko, V.T. ve Kozlovski, I.Y., *Equation Partial Differential Equation and Mathematical Models in Economic*, URRS, 2011

[7] Ionkin N. I, *Solution of a boundary-value problem in heat conduction with a nonclassical boundary condition*, Diff. Eqs., Vol. 13, No. 2,1977, pp. 294-304.

[8] Kamynin, L. I., *A boundary value problem in the theory of heat conduction with a nonclassical boundary condition*, USSR Computational Mathematics and Mathematical Physics, 1964, 4.6: 33-59.

[9] Kesselman, G. M., *On the unconditional convergence of eigenfunction expansions of certain differential operators*, Izv. Vyssh. Uchebn. Zaved. Mat. 39(2),(1964), 8293, (Russian).

[10] Mikhailov, V. P., *On Riesz bases in* $L^2(0,1)$, Dokl. Akad. Nauk SSSR, 1962, 144, 981-984, (Russian)

[11] Naimark, M.A., *Linear Differential Operator*, 1968
[12] Rehmana, M. A. ve Taj, M. S., *A Numerical technique for heat equation subject to integral specification*, Sci.nt(Lahore),24(1),21-26,2011
[13] Shingmin W. ve Lin, Y. *A numerical method for the diffusion equation with nonlocal boundary specifications*, International Journal of Engineering Science, 1990, 28.6: 543-546.

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