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ABOUT CERTAIN HOMOLOGICAL PROPERTIES OF SYMMETRIC DERIVATIONS OF KÄHLER MODULES

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Abstract. In this study, we express more informations concerning with homological properties of symmetric derivations of Kahler modules acquainted by H. Osborn in [1].

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1. Introduction

The definition of n-th order symmetric derivations of Kähler modules were given by H.Osborn at 1965 in [1]. J.Johnson made known the structures of differential module on certain modules of Kähler differentials in [3]. Then, advanced principal theories about the calculus of high order derivations and a few functorial features of high order differential modules were presented by Y. Nakai in [4]. Higher derivations and universal differential operators of Kähler modules were studied by R. Hart in [2]. Olgun defined generalized symmetric derivations on high order Kähler modules in [6].Komatsu presented right differential operators on a noncommutative ring extension in [10]. The more informations about these subjects were found in [5,7,8,9,11].The aim of this study is to investigate these homological structures and is to give more knowldege about them.

2. Preliminary

Throughout this paper we assume R be a commutative algebra over an algebraically closed field k with characteristic zero. When R is a k-algebra , $J_n(R)$ denotes the universal module of n-th order differentials of R over k and $\Omega_n(R)$ be the module of n-th order Kähler differentials of R over k and d_n be the canonical $n-th$ order k-derivation $R \longrightarrow \Omega_n(R)$ of R.The pair $\{\Omega_n(R), d_n\}$ has the universal mapping property with regard to the $n-th$ order k-derivations of R. $\Omega_n(R)$ is generated by the set $\{d_n(r) : r \in R\}$.

Definition 2.1. Let R be a commutative algebra over a field k of characteristic zero,A be an R-module, $A \otimes_R A$ be the tensor product of A with itself and let K be the submodule of $A \otimes_R A$ generated by the elements of the form $a \otimes b - b \otimes a$ where $a, b \in A$. Consider the factor module $\vee^2 A = A \otimes A/K$. The module $\vee^2 A$ is said to be the second symmetric power of A. The canonic balanced map is defined such that

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$$\otimes : A \times A \longrightarrow A \otimes A$$
$$\otimes(a,b) = a \otimes b$$

and a natural surjective map defined such that $\gamma : A \otimes A \longrightarrow \vee^2 A$. Then the composite map is bilinear and called $\gamma\otimes = \vee$.

Lemma 2.1. *Let A and B be R-modules and let $\zeta : A \times A \longrightarrow B$ be a bilinear alternating map. Then there exists an R-module homomorphism $f : \vee^2 A \longrightarrow B$ such that the diagram*

$$\begin{array}{ccc} A \times A & \xrightarrow{\zeta} & B \\ \vee \searrow & & \nearrow f \\ & \vee^2 A & \end{array}$$

commutes.

Definition 2.2. Let R be any k-algebra (commutative with unit), $R \to \Omega_1(R)$ be first order Kähler derivation of R and let $\vee(\Omega_1(R))$ be the symmetric algebra $\bigoplus_{p\geq 0} \vee^p(\Omega_1(R))$ generated over R by $\Omega_1(R)$.

A symmetric derivation is any linear map D of $\vee(\Omega_1(R))$ into itself such that
i) $D(\vee^p(\Omega_1(R))) \subset \vee^{p+1}(\Omega_1(R))$
ii) D is a first order derivation over k and
iii) the restriction of D to R ($R \simeq \vee^0(\Omega_1(R))$) is the Kähler derivation
$d_1 : R \to \Omega_1(R)$.

Definition 2.3. Let R be any k-algebra (commutative with unit), $R \to \Omega_n(R)$ be n-th order Kähler derivation of R and let $\vee(\Omega_n(R))$ be the symmetric algebra $\bigoplus_{p\geq 0} \vee^p(\Omega_n(R))$ generated over R by $\Omega_n(R)$.

A generalized symmetric derivation is any k-linear map D of $\vee(\Omega_n(R))$ into itself such that
i) $D(\vee^p(\Omega_n(R))) \subset \vee^{p+1}(\Omega_n(R))$
ii) D is a n-th order derivation over k and
iii) the restriction of D to R ($R \simeq \vee^0(\Omega_n(R))$) is the Kähler derivation
$d_n : R \to \Omega_n(R)$.

Proposition 2.1. *Let $R = k[x_1, ..., x_s]$ be a polynomial algebra of dimension s. Then $\Omega_n(R)$ is a free R-module of rank $\binom{n+s}{s} - 1$ with basis*
$\{ d_n(x_1^{i_1}.....x_s^{i_s}) : i_1 + ... + i_s \leq n\}$

$\vee^2(\Omega_n(R))$ is a free R-module of rank $\binom{t+1}{t-1}$

where $t = \binom{n+s}{s} - 1$ with basis $\{ d_n(x_1^{i_1}.....x_s^{i_s}) \otimes d_n(x_1^{i_1}.....x_s^{i_s}) : i_1 + ... + i_s \leq n\}$

3. Symmetric Powers of Kähler Modules

In this section, we consider the tensor, exterior and symmetric algebras of Kähler modules and define the symmetric powers of a given module A over a k-algebra and a few elementary properties.

Definition 3.1. Let A be a R-module.
i) By $\otimes^n A$ we shall denote the R-module with a universal R-bilinear map of $A^n \longrightarrow \otimes^n A$ written $(x_1, ..., x_n) \longrightarrow x_1 \otimes ... \otimes x_n$. This module is called the n-fold tensor power of A.
ii) By $\Lambda^n A$ we shall denote the R-module with a universal alternating R-bilinear map of $A^n \longrightarrow \Lambda^n$ written $(x_1, ...x_n) \longrightarrow x_1 \wedge ... \wedge x_n$. This module is called the n-fold exterior power of A.
iii) By $\vee^n A$ we shall denote the R-module with a universal symmetric R-bilinear map of $A^n \longrightarrow \vee^n A$ written $(x_1, ..., x_n) \longrightarrow x_1 ... x_n$. This module is called the n-fold symmetric power of A.

Let us the conventin that $\otimes^1 A$,$\Lambda^1 A$ and $\vee^1 A$ are all identified with A, while $\otimes^0 A$, $\Lambda^0 A$ and $\vee^0 A$ are all identified with R.

Theorem 3.1. *Let A be a free R-module on a basis* $X = \{x_1, ..., x_d\}$. *If A is generated by* $x_1, ..., x_d$ *then* $A^{\otimes k}$ *is generated by* $x_{i_1} \otimes ... \otimes x_{i_k}$ *as an R-module. Where* $1 \leq i_1, ..., i_k \leq d$ *and* $dim_R(\otimes^k A) = d^k$.

Since $\Lambda^k(A)$ *is factor module of* $A^{\otimes k}$, *so* $\Lambda^k(A)$ *is generated by* $x_{i_1} \wedge ... \wedge x_{i_k}$ *as an R-module where* $1 \leq i_1, ..., i_k \leq d$. *For any* $x_1, x_2 \in X$, *it satisfied* $x_1 \wedge x_1 = 0$ *and* $x_1 \wedge x_2 + x_2 \wedge x_1 = 0$. *If* $0 \neq A$ *is affine free with* $x_1, ..., x_d$ *then* $x_{i_1} \wedge ... \wedge x_{i_k}$ *is basis for* $\Lambda^k A$ *and* $dim_R(\Lambda^k A) = \binom{d}{k}$.

$\vee A$ *may be presented by the generating set X, and relation* $xy = yx$ $(x, y \in X)$ *and is the (commutative) polynomial algebra* $R[X]$. *Then an R-module basis for* $\vee A$ *is given by those products* $x_1 ... x_n$ *with* $x_1 \leq ... \leq x_n \in X$. *If X is a finite set* $\{x_1, ..., x_r\}$, *then the elements of this basis can be written* $x_1^{i_1} ... x_r^{i_r}$ *with* $i_1, ..., i_r \geq 0$, *and for each n,* $dim(\vee^n A) = \binom{r+n-1}{r-1}$.

4. Homological Properties of Symmetric Derivations

Theorem 4.1. *Let R be an affine k-algebra. Then we have a long exact sequence of R-modules*

$$0 \longrightarrow \; ker\eta \longrightarrow \Omega_{(2n)}(R) \xrightarrow{\eta} J_n(\Omega_n(R)) \longrightarrow \; coker\eta \longrightarrow 0.$$

for all $n \geq 0$.

Example 4.1. $R = k[a, b]$ be a polynomial algebra of dimension 2.Then $\Omega_1(R)$ is a free R-module of rank 2 with basis $\{d_1(a), d_1(b)\}$ and $\Omega_2(R))$ is a free R-module of rank 5 with basis $\{d_2(a), d_2(b)), d_2(a^2), d_2(ab), d_2(b^2)$.
$J_1(\Omega_1(R))$ is a free R-module generated by
$\{\Delta_1(d_1(a)), \Delta_1(d_1(b)), \Delta_1(ad_1(a)), \Delta_1(ad_1(b)), \Delta_1(bd_1(a)), \Delta_1(bd_1(b))\}$

Theorem 4.2. *Let R be an affine k-algebra. Then we have a long exact sequence of R-modules*

$$0 \longrightarrow \; ker\gamma \longrightarrow \; J_n(\Omega_n(R)) \xrightarrow{\gamma} \vee^2(\Omega_n(R)) \longrightarrow \; coker\gamma \longrightarrow 0.$$

for all $n \geq 0$.

Lemma 4.1. *Let R be an affine domain.Then* $\Omega_n(R)$ *is a free R-module if and only if* $\vee^2(\Omega_n(R))$ *is a free R-module.*

Theorem 4.3. *Let R be an affine k-algebra and $\vee(\Omega_1(R))$ has at least one symmetric derivations. $\Omega_1(R)$is a projective R-module if and only if $\Omega_2(R)$ is a projective R-module.*

Corollary 4.1. *Let R be an affine local k-algebra and $\vee(\Omega_1(R))$ has at least one symmetric derivation. $\Omega_1(R)$ is a free R-module if and only if $\Omega_2(R)$ is a free R-module.*

References

[1] Osborn, H., Modules of Differentials II, Math. Ann.175, 146-158, (1968).
[2] Hart, R., Higher Derivations and Universal Differentials, Journal of Alg. 184, 175-181, 1996.
[3] Johnson J., Kähler Differentials and differential algebra, The Annals of Mathematics,Second series,Vol. 89, No 1, 92-98, (1969).
[4] Nakai, Y., High Order Derivations, Osaka J. Math. 7, 1-27, (1970).
[5] Erdogan, A., Differential Operators and Their Universal Modules, Phd. Thesis, Universty of Leeds, (1993). .
[6] Olgun N., Symmetric Derivations On *Kähler* Modules, `arXiv:1602.01343 [math.AC]`
[7] Bourbaki, N., Algebra, Hermann, Paris, 1974.
[8] Sweedler, M.E. and Heyneman, R.G., Affine Hopf Algebras, J.Algebra 13, 192-241, (1969).
[9] Vasconcelos, W.V., A note On Normality And The Module of Differentials, Math. Z., 105, 291-293, 1968.
[10] Komatsu, H., High order Kähler Modules of Noncommutative ring extensions, Comm.Algebra 29, 5499-5524, (2001)
[11] Bergman,G., Tensor algebras, exterior algebras and symmetric algebras, http://math.berkeley.edu/gbergman, (1997).

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