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## ABOUT CERTAIN HOMOLOGICAL PROPERTIES OF SYMMETRIC DERIVATIONS OF KÄHLER MODULES

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ABSTRACT. In this study, we express more informations concerning with homological properties of symmetric derivations of Kahler modules acquainted by H. Osborn in [1].

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### 1. INTRODUCTION

The definition of  $n$ -th order symmetric derivations of Kähler modules were given by H. Osborn at 1965 in [1]. J. Johnson made known the structures of differential module on certain modules of Kähler differentials in [3]. Then, advanced principal theories about the calculus of high order derivations and a few functorial features of high order differential modules were presented by Y. Nakai in [4]. Higher derivations and universal differential operators of Kähler modules were studied by R. Hart in [2]. Olgun defined generalized symmetric derivations on high order Kähler modules in [6]. Komatsu presented right differential operators on a noncommutative ring extension in [10]. The more informations about these subjects were found in [5,7,8,9,11]. The aim of this study is to investigate these homological structures and is to give more knowldege about them.

### 2. PRELIMINARY

Throughout this paper we assume  $R$  be a commutative algebra over an algebraically closed field  $k$  with characteristic zero. When  $R$  is a  $k$ -algebra,  $J_n(R)$  denotes the universal module of  $n$ -th order differentials of  $R$  over  $k$  and  $\Omega_n(R)$  be the module of  $n$ -th order Kähler differentials of  $R$  over  $k$  and  $d_n$  be the canonical  $n$ -th order  $k$ -derivation  $R \rightarrow \Omega_n(R)$  of  $R$ . The pair  $\{\Omega_n(R), d_n\}$  has the universal mapping property with regard to the  $n$ -th order  $k$ -derivations of  $R$ .  $\Omega_n(R)$  is generated by the set  $\{d_n(r) : r \in R\}$ .

**Definition 2.1.** Let  $R$  be a commutative algebra over a field  $k$  of characteristic zero,  $A$  be an  $R$ -module,  $A \otimes_R A$  be the tensor product of  $A$  with itself and let  $K$  be the submodule of  $A \otimes_R A$  generated by the elements of the form  $a \otimes b - b \otimes a$  where  $a, b \in A$ . Consider the factor module  $\vee^2 A = A \otimes A / K$ . The module  $\vee^2 A$  is said to be the second symmetric power of  $A$ . The canonic balanced map is defined such that

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$$\begin{aligned}\otimes : A \times A &\longrightarrow A \otimes A \\ \otimes(a, b) &= a \otimes b\end{aligned}$$

and a natural surjective map defined such that  $\gamma : A \otimes A \longrightarrow \vee^2 A$ . Then the composite map is bilinear and called  $\gamma \otimes = \vee$ .

**Lemma 2.1.** *Let  $A$  and  $B$  be  $R$ -modules and let  $\zeta : A \times A \longrightarrow B$  be a bilinear alternating map. Then there exists an  $R$ -module homomorphism  $f : \vee^2 A \longrightarrow B$  such that the diagram*

$$\begin{array}{ccc} A \times A & \xrightarrow{\zeta} & B \\ \vee \searrow & & \nearrow f \\ & \vee^2 A & \end{array}$$

commutes.

**Definition 2.2.** Let  $R$  be any  $k$ -algebra (commutative with unit),  $R \rightarrow \Omega_1(R)$  be first order Kähler derivation of  $R$  and let  $\vee(\Omega_1(R))$  be the symmetric algebra  $\bigoplus_{p \geq 0} \vee^p(\Omega_1(R))$  generated over  $R$  by  $\Omega_1(R)$ .

A symmetric derivation is any linear map  $D$  of  $\vee(\Omega_1(R))$  into itself such that

- i)  $D(\vee^p(\Omega_1(R))) \subset \vee^{p+1}(\Omega_1(R))$
- ii)  $D$  is a first order derivation over  $k$  and
- iii) the restriction of  $D$  to  $R$  ( $R \simeq \vee^0(\Omega_1(R))$ ) is the Kähler derivation

$d_1 : R \rightarrow \Omega_1(R)$ .

**Definition 2.3.** Let  $R$  be any  $k$ -algebra (commutative with unit),  $R \rightarrow \Omega_n(R)$  be  $n$ -th order Kähler derivation of  $R$  and let  $\vee(\Omega_n(R))$  be the symmetric algebra  $\bigoplus_{p \geq 0} \vee^p(\Omega_n(R))$  generated over  $R$  by  $\Omega_n(R)$ .

A generalized symmetric derivation is any  $k$ -linear map  $D$  of  $\vee(\Omega_n(R))$  into itself such that

- i)  $D(\vee^p(\Omega_n(R))) \subset \vee^{p+1}(\Omega_n(R))$
- ii)  $D$  is a  $n$ -th order derivation over  $k$  and
- iii) the restriction of  $D$  to  $R$  ( $R \simeq \vee^0(\Omega_n(R))$ ) is the Kähler derivation

$d_n : R \rightarrow \Omega_n(R)$ .

**Proposition 2.1.** *Let  $R = k[x_1, \dots, x_s]$  be a polynomial algebra of dimension  $s$ .*

*Then  $\Omega_n(R)$  is a free  $R$ -module of rank  $\binom{n+s}{s} - 1$  with basis*

$$\{ d_n(x_1^{i_1} \dots x_s^{i_s}) : i_1 + \dots + i_s \leq n \}$$

$$\vee^2(\Omega_n(R)) \text{ is a free } R\text{-module of rank } \binom{t+1}{t-1}$$

where  $t = \binom{n+s}{s} - 1$  with basis  $\{ d_n(x_1^{i_1} \dots x_s^{i_s}) \otimes d_n(x_1^{i_1} \dots x_s^{i_s}) : i_1 + \dots + i_s \leq n \}$

### 3. SYMMETRIC POWERS OF KÄHLER MODULES

In this section, we consider the tensor, exterior and symmetric algebras of Kähler modules and define the symmetric powers of a given module  $A$  over a  $k$ -algebra and a few elementary properties.

**Definition 3.1.** Let  $A$  be a  $R$ -module.

i) By  $\otimes^n A$  we shall denote the  $R$ -module with a universal  $R$ -bilinear map of  $A^n \rightarrow \otimes^n A$  written  $(x_1, \dots, x_n) \rightarrow x_1 \otimes \dots \otimes x_n$ . This module is called the  $n$ -fold tensor power of  $A$ .

ii) By  $\Lambda^n A$  we shall denote the  $R$ -module with a universal alternating  $R$ -bilinear map of  $A^n \rightarrow \Lambda^n A$  written  $(x_1, \dots, x_n) \rightarrow x_1 \wedge \dots \wedge x_n$ . This module is called the  $n$ -fold exterior power of  $A$ .

iii) By  $\vee^n A$  we shall denote the  $R$ -module with a universal symmetric  $R$ -bilinear map of  $A^n \rightarrow \vee^n A$  written  $(x_1, \dots, x_n) \rightarrow x_1 \dots x_n$ . This module is called the  $n$ -fold symmetric power of  $A$ .

Let us the conventin that  $\otimes^1 A, \Lambda^1 A$  and  $\vee^1 A$  are all identified with  $A$ , while  $\otimes^0 A, \Lambda^0 A$  and  $\vee^0 A$  are all identified with  $R$ .

**Theorem 3.1.** *Let  $A$  be a free  $R$ -module on a basis  $X = \{x_1, \dots, x_d\}$ . If  $A$  is generated by  $x_1, \dots, x_d$  then  $A^{\otimes k}$  is generated by  $x_{i_1} \otimes \dots \otimes x_{i_k}$  as an  $R$ -module. Where  $1 \leq i_1, \dots, i_k \leq d$  and  $\dim_R(\otimes^k A) = d^k$ .*

*Since  $\Lambda^k(A)$  is factor module of  $A^{\otimes k}$ , so  $\Lambda^k(A)$  is generated by  $x_{i_1} \wedge \dots \wedge x_{i_k}$  as an  $R$ -module where  $1 \leq i_1, \dots, i_k \leq d$ . For any  $x_1, x_2 \in X$ , it satisfied  $x_1 \wedge x_1 = 0$  and  $x_1 \wedge x_2 + x_2 \wedge x_1 = 0$ . If  $0 \neq A$  is affine free with  $x_1, \dots, x_d$  then  $x_{i_1} \wedge \dots \wedge x_{i_k}$  is basis for  $\Lambda^k A$  and  $\dim_R(\Lambda^k A) = \binom{d}{k}$ .*

$\vee A$  may be presented by the generating set  $X$ , and relation  $xy = yx$  ( $x, y \in X$ ) and is the (commutative) polynomial algebra  $R[X]$ . Then an  $R$ -module basis for  $\vee A$  is given by those products  $x_1 \dots x_n$  with  $x_1 \leq \dots \leq x_n \in X$ . If  $X$  is a finite set  $\{x_1, \dots, x_r\}$ , then the elements of this basis can be written  $x_1^{i_1} \dots x_r^{i_r}$  with  $i_1, \dots, i_r \geq 0$ , and for each  $n$ ,  $\dim(\vee^n A) = \binom{r+n-1}{r-1}$ .

#### 4. HOMOLOGICAL PROPERTIES OF SYMMETRIC DERIVATIONS

**Theorem 4.1.** *Let  $R$  be an affine  $k$ -algebra. Then we have a long exact sequence of  $R$ -modules*

$$0 \rightarrow \ker \eta \rightarrow \Omega_{(2n)}(R) \xrightarrow{\eta} J_n(\Omega_n(R)) \rightarrow \operatorname{coker} \eta \rightarrow 0.$$

for all  $n \geq 0$ .

**Example 4.1.**  $R = k[a, b]$  be a polynomial algebra of dimension 2. Then  $\Omega_1(R)$  is a free  $R$ -module of rank 2 with basis  $\{d_1(a), d_1(b)\}$  and  $\Omega_2(R)$  is a free  $R$ -module of rank 5 with basis  $\{d_2(a), d_2(b), d_2(a^2), d_2(ab), d_2(b^2)\}$ .

$J_1(\Omega_1(R))$  is a free  $R$ -module generated by  $\{\Delta_1(d_1(a)), \Delta_1(d_1(b)), \Delta_1(ad_1(a)), \Delta_1(ad_1(b)), \Delta_1(bd_1(a)), \Delta_1(bd_1(b))\}$

**Theorem 4.2.** *Let  $R$  be an affine  $k$ -algebra. Then we have a long exact sequence of  $R$ -modules*

$$0 \rightarrow \ker \gamma \rightarrow J_n(\Omega_n(R)) \xrightarrow{\gamma} \vee^2(\Omega_n(R)) \rightarrow \operatorname{coker} \gamma \rightarrow 0.$$

for all  $n \geq 0$ .

**Lemma 4.1.** *Let  $R$  be an affine domain. Then  $\Omega_n(R)$  is a free  $R$ -module if and only if  $\vee^2(\Omega_n(R))$  is a free  $R$ -module.*

**Theorem 4.3.** *Let  $R$  be an affine  $k$ -algebra and  $\vee(\Omega_1(R))$  has at least one symmetric derivations.  $\Omega_1(R)$  is a projective  $R$ -module if and only if  $\Omega_2(R)$  is a projective  $R$ -module.*

**Corollary 4.1.** *Let  $R$  be an affine local  $k$ -algebra and  $\vee(\Omega_1(R))$  has at least one symmetric derivation.  $\Omega_1(R)$  is a free  $R$ -module if and only if  $\Omega_2(R)$  is a free  $R$ -module.*

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