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THE DIRECT AND INVERSE SPECTRAL PROBLEM FOR STURM-LIOUVILLE OPERATOR WITH DISCONTINUOUS COEFFICIENT

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Abstract. In this study, for Sturm-Liouville operator with discontinuous coefficient encountered in the non-homogeneous materials, direct and inverse problems are investigated. The spectral properties of the Sturm-Liouville problem with discontinuous coefficient such as the orthogonality of its eigenfunctions and simplicity of its eigenvalues are examined. Asymptotic formula is found for eigenvalues, and resolvent operator is constructed. The expansion formula with respect to eigenfunctions is obtained. It is shown that its eigenvalues are in the form of a complete system. Also, the Weyl solution and Weyl function are defined. Uniqueness theorems for the solution of the inverse problem according to spectral date are proved.

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1. Introduction

In this study, the heat problem of a rod that consists of two parts with fixed cross section is examined. The side surfaces of the rod have been isolated and have different physical features [1]-[4]. When initial temperature is given arbitrary and the temperature at the ending points is not equal to zero, the heat problem of the rod takes the following form:

$$\rho(x)U_t = U_{xx} + q(x)U, \quad 0 \le x \le \pi,$$

$$U(x,0) = \phi(x), \quad U_t(x,0) = \psi(x), \quad 0 \le x\pi$$

$$U_x(0,t) = 0, \quad U_x(\pi,t) = 0, \quad t > 0$$

where the function $U(x,t)$ is the temperature in the bar at the time t, $\rho(x)$ is a piecewise constant function and refers to the density of the rod and $\phi(x)$, $\psi(x)$ are enough smooth functions. By the method of separation of variables, the preceding equation is reduced to a boundary value problem for Sturm-Liouville equation:

$$-y'' + q(x)y = \lambda^2 \rho(x)y, \quad 0 \le x \le \pi, \tag{1.1}$$

$$y'(0) = y'(\pi) = 0, \tag{1.2}$$

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where in particular, $\rho(x)$ is chosen as

$$\rho(x) = \begin{cases} 1 & 0 \leq x < a \\ \alpha^2 & a \leq x \leq \pi. \end{cases} \tag{1.3}$$

$q(x) \in L_2(0, \pi)$ is a real valued function and λ is a complex parameter. Then, in finding the solution of the above diffusion problem, spectral problem (1.1), (1.2) must be examined [1]-[5]. The spectral problems with discontinuous coefficient on the bounded interval are investigated in [6]-[15]. The similar problems on the half line by different authors have been studied (see [16]-[18]). Let $\varphi(x)$ and $\psi(x)$ be solutions of (1.1), (1.2) boundary value problem satisfying the initial conditions

$$\varphi(0, \lambda) = 1, \varphi'(0, \lambda) = 0 \tag{1.4}$$

and

$$\psi(\pi, \lambda) = 1, \psi'(\pi, \lambda) = 0. \tag{1.5}$$

Denote

$$\Delta(\lambda) = W[\varphi(x, \lambda), \psi(x, \lambda)] = \varphi'(x, \lambda)\psi(x, \lambda) - \varphi(x, \lambda)\psi'(x, \lambda). \tag{1.6}$$

The function $\Delta(\lambda)$ is called the characteristic function of the problem (1.1), (1.2), and substituting $x = 0$ and $x = \pi$ into (1.6), we get

$$\Delta(\lambda) = \varphi'(\pi, \lambda) = -\psi'(0, \lambda). \tag{1.7}$$

Lemma 1.1. *The eigenfunctions $y_1(x, \lambda_1)$ and $y_2(x, \lambda_2)$ corresponding to different eigenvalues $\lambda_1 \neq \lambda_2$ are orthogonal.*

Proof. Since $y_1(x, \lambda_1)$ and $y_2(x, \lambda_2)$ are eigenfunctions of problem (1.1), (1.2), we get

$$-y_1''(x, \lambda_1) + q(x)y_1(x, \lambda_1) = \lambda_1^2\rho(x)y_1(x, \lambda_1),$$
$$-y_2''(x, \lambda_2) + q(x)y_2(x, \lambda_2) = \lambda_2^2\rho(x)y_2(x, \lambda_2).$$

Multiplying these equalities by $y_1(x, \lambda_1$) and $-y_2(x, \lambda_2$, respectively, and adding together,

$$\frac{d}{dx}\{< y_2(x, \lambda_2), y_1(x, \lambda_1) >\} = (\lambda_1^2 - \lambda_2^2)\rho(x)y_1(x, \lambda_1)y_2(x, \lambda_2)$$

is found. Integrating from 0 to π and using the condition (1.2), we have

$$(\lambda_1^2 - \lambda_2^2)\int_0^\pi \rho(x)y_1(x, \lambda_1)y_2(x, \lambda_2)dx = 0.$$

Since $\lambda_1 \neq \lambda_2$,

$$\int_0^\pi \rho(x)y_1(x, \lambda_1)y_2(x, \lambda_2)dx = 0.$$

□

Corollary 1.1. *The eigenvalues of the boundary value problem (1.1), (1.2) are real.*

Lemma 1.2. *The zeros λ_n of characteristic function $\Delta(\lambda)$ coincide with the eigenvalues of the boundary value problem (1.1), (1.2). The functions $\varphi(x, \lambda_n)$ and $\psi(x, \lambda_n)$ are eigenfunctions and there exists a sequence β_n such that*

$$\psi(x, \lambda_n) = \beta_n\varphi(x, \lambda_n), \quad \beta_n \neq 0. \tag{1.8}$$

Proof. 1) Let λ_0 be zero of $\Delta(\lambda)$. Then, because of (1.6), $\psi(x,\lambda_0)=\beta_0\varphi(x,\lambda_0)$ and the function $\psi(x,\lambda_0)$ and $\varphi(x,\lambda_0)$ satisfy the boundary condition (1.2). Thus, λ_0 is an eigenvalue and $\psi(x,\lambda_0)$, $\varphi(x,\lambda_0)$ are corresponding eigenfunctions.

2) Let λ_0 be an eigenvalue of the problem (1.1), (1.2) and let $y_0(x)$ be a corresponding eigenfunction. Then, $y_0(x)$ satisfies the boundary condition (1.2). Clearly, $y_0(x)\neq 0$. Without loss of generality, we put $y_0(0)=1$. Then $y_0'(0)=0$,and consequently, $y_0(x)\equiv\varphi(x,\lambda)$. Hence, from (1.7), $\Delta_0(\lambda)=0$. We have proved that for each eigenvalue there exists only one eigenfunction. □

Lemma 1.3. *The eigenvalues of the boundary value problem (1.1), (1.2) are simple and*

$$\dot{\Delta}(\lambda_n)=2\lambda_n\alpha_n\beta_n, \tag{1.9}$$

where

$$\alpha_n:=\int_0^\pi\rho(x)\varphi^2(x,\lambda_n)dx.$$

is the normalizing number of (1.1), (1.2).

Proof. Since $\varphi(x,\lambda_n)$ and $\psi(x,\lambda)$ are the solutions of this problem,

$$-\varphi''(x,\lambda_n)+q(x)\varphi(x,\lambda_n)=\lambda_n^2\rho(x)\varphi(x,\lambda_n),$$
$$-\psi''(x,\lambda)+q(x)\psi(x,\lambda)=\lambda^2\rho(x)\psi(x,\lambda),$$

are valid. Multiplying these equations by $\psi(x,\lambda\ )$ and $-\varphi(x,\lambda_n)$, respectively, and adding them together, we get

$$\frac{d}{dx}\{<\psi(x,\lambda),\varphi(x,\lambda_n)>\}=(\lambda_n^2-\lambda^2)\rho(x)\varphi(x,\lambda_n)\psi(x,\lambda).$$

Integrating from 0 to π and using the condition (1.2),

$$\int_0^\pi\rho(x)\varphi(x,\lambda_n)\psi(x,\lambda)=\frac{\Delta(\lambda_n)-\Delta(\lambda)}{\lambda_n^2-\lambda^2}$$

is found. From Lemma 2, since $\psi(x,\lambda_n)=\beta_n\varphi(x,\lambda_n)$, as $\lambda\to\lambda_n$, we obtain

$$\dot{\Delta}(\lambda_n)=2\lambda_n\alpha_n\beta_n$$

where $\beta_n=\psi(0,\lambda_n)$. Thus, it follows that $\dot{\Delta}(\lambda_n)\neq 0$. □

2. On the Eigenvalues of Problem (1.1), (1.2) at $q(x)\equiv 0$

Denote by $\varphi_0(x,\lambda)$ the solution equation $-y''=\lambda^2\rho(x)y$, satisfying the condition (1.4). It has the following form:

$$\varphi_0(x,\lambda)=\frac{1}{2}\left(1+\frac{1}{\sqrt{\rho(x)}}\right)\cos\lambda\mu^+(x)+\frac{1}{2}\left(1-\frac{1}{\sqrt{\rho(x)}}\right)\cos\lambda\mu^-(x), \tag{2.1}$$

where $\mu^\pm(x)=\pm x\sqrt{\rho(x)}+a(1\mp\sqrt{\rho(x)})$.

It is easy show that if $(\lambda_n^0)^2$ are eigenvalues of problem (1.1), (1.2) at $q(x)\equiv 0$, then λ_n^0 can be found from the equation $\varphi_0'(\pi,\lambda)=0$,that is, from the equation

$$\Delta_0(\lambda)=-\frac{1}{2}\lambda(\alpha+1)\sin\lambda\mu^+(\pi)+\frac{1}{2}(\alpha-1)\lambda\sin\lambda\mu^-(\pi)=0$$

$$\sin\lambda\mu^+(\pi)-\frac{\alpha-1}{\alpha+1}\sin\lambda\mu^-(\pi)=0. \tag{2.2}$$

At last, from (2.2), it follows that

$$\lambda_n^0 = \frac{1}{\mu^+(\pi)}\left(n\pi + \epsilon_n\right) \tag{2.3}$$

where

$$\epsilon_n = (-1)^n \frac{\alpha-1}{\alpha+1} sin\left(\frac{\mu^+(\pi)}{\mu^-(\pi)} n\pi\right) + O\left(\frac{1}{n}\right).$$

Lemma 2.1. *Roots of the function $\Delta_0(\lambda)$ are isolated, that is,*

$$\inf_{n\neq k}\left|\lambda_n^0 - \lambda_k^0\right| = \beta > 0.$$

3. Asymptotic Formulas of Eigenvalues

Using representation for solution $e(x,\lambda)$ of equation (1.1) satisfying the initial conditions $e(0,\lambda) = 1$, $e'(0,\lambda) = i\lambda$ (see[8]), it is easy to obtain the following integral representation for the solution $\varphi(x,\lambda)$:

$$\varphi(x,\lambda) = \varphi_0(x,\lambda) + \int_0^{\mu^+(x)} A(x,t)\cos\lambda t dt$$

where $K(x,.) \in L_1(-\mu^+(x), \mu^+(x))$ and $A(x,t) = K(x,t) - K(x,-t)$. The kernel $A(x,t)$ processes the following properties:

i) $A(\pi,\mu^+(\pi)) = \frac{1}{4}\int_0^\pi \frac{1}{\sqrt{\rho(t)}}\left(1 + \frac{1}{\sqrt{\rho(t)}}\right) q(t)dt,$

ii) $A(\pi,\mu^-(\pi)+0) - A(\pi,\mu^-(\pi)-0) = \frac{1}{4}\int_0^\pi \frac{1}{\sqrt{\rho(t)}}\left(1 - \frac{1}{\sqrt{\rho(t)}}\right) q(t)dt.$

Theorem 3.1. *Boundary value problem (1.1), (1.2) has a countable set of simple eigenvalues $\{\lambda_n^2\}_{n\geq 1}$, where*

$$\lambda_n = \lambda_n^0 + \frac{d_n}{\lambda_n^0} + \frac{k_n}{n}, \quad (\lambda_n > 0), \quad k_n \in l_2 \tag{3.1}$$

where λ_n^0 are zeros of the function

$$\Delta_0(\lambda) = -\frac{1}{2}\lambda(\alpha+1)\sin\lambda\mu^+(\pi) + \frac{1}{2}(\alpha-1)\lambda\sin\lambda\mu^-(\pi). \tag{3.2}$$

$\{\lambda_n^0\}^2$ are the eigenvalues of problem (1.1), (1.2), when $q(x) \equiv 0$, d_n is a bounded sequence

$$d_n = \frac{h^+\cos\lambda_n^0\mu^+(\pi) + h^-\cos\lambda_n^0\mu^-(\pi)}{\frac{1}{2}(\alpha+1)\mu^+(\pi)\cos\lambda_n^0\mu^+(\pi) - \frac{1}{2}(\alpha-1)\mu^-(\pi)\cos\lambda_n^0\mu^-(\pi)}. \tag{3.3}$$

Proof. Let $\varphi(x,\lambda)$ be the solution of equation (1.1) at initial conditions $\varphi(0,\lambda) = 1$, $\varphi'(0,\lambda) = 0$. Then the characteristic function $\Delta(\lambda) = \varphi'(\pi,\lambda)$ is entire with respect to λ and it has the most countable set of zeros λ_n and numbers λ_n^2 are eigenvalues of boundary value problem (1.1), (1.2). The standard method of variations of an arbitrary constants leads to the following integral equation for the solution $\varphi(x,\lambda)$

$$\varphi(x,\lambda) = \varphi_0(x,\lambda) + \int_0^x g(x,t;\lambda)q(t)\varphi(t,\lambda)dt \tag{3.4}$$

where

$$g(x,t;\lambda)=\frac{1}{2}\left(\frac{1}{\sqrt{\rho(x)}}+\frac{1}{\sqrt{\rho(t)}}\right)\frac{sin\lambda\left(\mu^{+}(x)-\mu^{+}(t)\right)}{\lambda}+$$

$$+\frac{1}{2}\left(\frac{1}{\sqrt{\rho(x)}}-\frac{1}{\sqrt{\rho(t)}}\right)\frac{sin\lambda\left(\mu^{-}(x)-\mu^{+}(t)\right)}{\lambda} \tag{3.5}$$

and $\varphi_0(x,\lambda)$ is the solution of equation (1.1) at q(x) = 0, satisfying the conditions $\varphi(0,\lambda)=1,\quad \varphi'(0,\lambda)=0$. From (3.4), after differentiating, we find

$$\varphi'(x,\lambda)=\varphi_0'(x,\lambda)+\int_0^x g_x(x,t;\lambda)q(t)\varphi(t,\lambda)dt \tag{3.6}$$

where

$$g_x(x,t;\lambda)=\sqrt{\rho(x)}\frac{1}{2}\left(\frac{1}{\sqrt{\rho(x)}}+\frac{1}{\sqrt{\rho(t)}}\right)\frac{cos\lambda\left(\mu^{+}(x)-\mu^{+}(t)\right)}{\lambda}+$$

$$+\sqrt{\rho(x)}\frac{1}{2}\left(\frac{1}{\sqrt{\rho(x)}}-\frac{1}{\sqrt{\rho(t)}}\right)\frac{cos\lambda\left(\mu^{-}(x)-\mu^{+}(t)\right)}{\lambda} \tag{3.7}$$

Consequently, if we put here $x=\pi$, we have

$$\Delta(\lambda)=\Delta_0(\lambda)+\int_0^\pi g_\pi'(x,t;\lambda)q(t)\varphi(t,\lambda)dt. \tag{3.8}$$

Now, from

$$\varphi(x,\lambda)=\varphi_0(x,\lambda)+O\left(\frac{e^{|Im\lambda|\mu^{+}(x)}}{|\lambda|}\right),\quad |\lambda|\to+\infty \tag{3.9}$$

we obtain

$$\Delta(\lambda)=\Delta_0(\lambda)+h^{+}\cos\lambda_n^0\mu^{+}(\pi)+h^{-}\cos\lambda_n^0\mu^{-}(\pi)+K_0(\lambda), \tag{3.10}$$

where

$$h^{\pm}=\frac{1}{4}(1\pm\alpha)\int_0^a q(t)dt+\frac{1}{4}\left(1\pm\frac{1}{\alpha}\right)\int_a^\pi q(t)dt, \tag{3.11}$$

and

$$K_0(\lambda)=\frac{1}{4}\int_0^a\left[(1+\alpha)cos\lambda\left(2\mu^{+}(t)-\mu^{+}(\pi)\right)+(1-\alpha)cos\lambda\left(2\mu^{+}(t)-\mu^{-}(\pi)\right)\right]q(t)dt$$

$$+\frac{1}{4}\int_a^\pi\left[\left(1+\frac{1}{\alpha}\right)cos\lambda\left(2\mu^{+}(t)-\mu^{+}(\pi)\right)\right]q(t)dt$$

$$+\frac{1}{4}\int_a^\pi\left[\left(1-\frac{1}{\alpha}\right)cos\lambda\left(\mu^{+}(\pi)+\mu^{-}(t)-\mu^{+}(t)\right)\right]q(t)dt$$

$$+0\left(\frac{e^{|Im\lambda|\mu^{+}(\pi)}}{|\lambda|}\right). \tag{3.12}$$

Let us denote $G_\delta=\{\lambda:\left|\lambda-\lambda_n^0\right|\geq\delta\}$, where δ is a sufficiently small positive number $\delta<\frac{\beta}{2}$ (see lemma 4). It is easy to show that (see [3])

$$|\Delta_0(\lambda)|\geq|\lambda|C_\delta e^{|Im\lambda|\mu^{+}(\pi)},\quad \lambda\in G_\delta,\quad C_\delta>0. \tag{3.13}$$

On the other hand, we obtain

$$\Delta(\lambda)-\Delta_0(\lambda)\le O\left(e^{|Im\lambda|\mu^+(\pi)}\right),\quad |\lambda|\to\infty. \tag{3.14}$$

Consider the contour $\Gamma_n=\{\lambda:|\lambda|=\left|\lambda_n^0\right|+\frac{\beta}{2}\}$, $(n=1,2,...)$. We have from (3.10)

$$|\Delta(\lambda)-\Delta_0(\lambda)|\le\tilde{C}e^{|Im\lambda|\mu^+(\pi)},\quad \lambda\in\Gamma_n, \tag{3.15}$$

for sufficiently large n, where $\tilde{C}>0$. Applying now Rouche's theorem, we have that the number of zeros of $\Delta_0(\lambda)$ inside Γ_n coincides with the number of zeros of $\Delta(\lambda)=\{\Delta(\lambda)-\Delta_0(\lambda)\}+\Delta_0(\lambda)$. Further applying the Rouche's theorem to the circle $\gamma_n(\delta)=\{\lambda:\left|\lambda-\lambda_n^0\right|\le\delta\}$, we conclude that for sufficiently large n, there exist only one zero λ_n of the function $\Delta(\lambda)$ in $\gamma_n(\delta)$. By virtue of the arbitrariness of $\delta>0$ we have

$$\lambda_n=\lambda_n^0+\varepsilon_n,\quad \varepsilon_n=o(1),\quad n\to\infty. \tag{3.16}$$

Substituting (3.16) into (3.10), we obtain and taking into our account the relations

$$\Delta_0(\lambda_n^0)=-\frac{1}{2}\lambda_n^0(\alpha+1)\sin\lambda_n^0\mu^+(\pi)+\frac{1}{2}\lambda_n^0(\alpha-1)\sin\lambda_n^0\mu^-(\pi)=0,$$

$$\sin\varepsilon_n\mu^+(\pi)\sim\varepsilon_n\mu^+(\pi),\quad \cos\varepsilon_n\mu^+(\pi)\sim1,\quad n\to\infty$$

we get

$$\varepsilon_n=\frac{d_n}{\lambda_n^0+\varepsilon_n}+\frac{\varepsilon_n}{\lambda_n^0+\varepsilon_n}\widetilde{d}_n+\frac{\widetilde{K}_n}{\lambda_n^0+\varepsilon_n} \tag{3.17}$$

where

$$d_n=\frac{h^+\cos\lambda_n^0\mu^+(\pi)+h^-\cos\lambda_n^0\mu^-(\pi)}{\frac{1}{2}(\alpha+1)\mu^+(\pi)\cos\lambda_n^0\mu^+(\pi)-\frac{1}{2}(\alpha-1)\mu^-(\pi)\cos\lambda_n^0\mu^-(\pi)},$$

$\widetilde{K}_n=K_0(\lambda_n^0+\varepsilon_n)$ and

$$\widetilde{d}_n=\frac{h^+\mu^+(\pi)\sin\lambda_n^0\mu^+(\pi)+h^-\mu^-(\pi)\sin\lambda_n^0\mu^-(\pi)}{\frac{1}{2}(\alpha+1)\mu^+(\pi)\cos\lambda_n^0\mu^+(\pi)-\frac{1}{2}(\alpha-1)\mu^-(\pi)\cos\lambda_n^0\mu^-(\pi)}.$$

Since $\frac{1}{\lambda_n^0+\varepsilon_n}=O(\frac{1}{n})$, $\frac{\varepsilon_n}{\lambda_n^0+\varepsilon_n}=o(\frac{1}{n})$, $n\to\infty$ we have that d_n, $\widetilde{d}_n$ are bounded and (3.17) implies

$$\varepsilon_n=O(\frac{1}{n}),\quad n\to\infty.$$

Using (3.17) once more, we can obtain more precisely as $n\to\infty$

$$\varepsilon_n=\frac{d_n}{\lambda_n^0}+\frac{k_n}{n},\quad k_n\in l_2 \tag{3.18}$$

where $k_n=\frac{\mu^+(\pi)}{\pi}\widetilde{K}_n+O(\frac{1}{n})$, $n\to\infty$ .The theorem is proved. □

4. Spectral Expansion Formula

Theorem 4.1. *1) The system of eigenfunctions* $\{\varphi(x,\lambda_n)\}_{n\geq 1}$ *of boundary value problem (1.1), (1.2) is complete in* $L_2(0,\pi;\rho)$;

2) If $f(x)$ *is an absolutely continuous function on the segment* $[0,\pi]$, *and* $f'(0) = f'(\pi) = 0$, *then*

$$f(x) = \sum_{n=1}^{\infty} a_n \varphi(x,\lambda_n), \tag{4.1}$$

where

$$a_n = \frac{1}{\alpha_n}\int_0^{\pi} f(t)\varphi(t,\lambda_n)\rho(t)dt, \tag{4.2}$$

and the series (4.1) converges uniformly on $[0,\pi]$;

3) For $f(x) \in L_2(0,\pi;\rho)$ *the series (4.1) converges in* $L_2(0,\pi;\rho)$, *moreover the Parseval equality*

$$\int_0^{\pi} |f(x)|^2 \rho(x)dx = \sum_{n=1}^{\infty} \alpha_n |a_n|^2 \tag{4.3}$$

holds.

Proof. Let $\psi(x,\lambda)$ be a solution of equation (1.1) under the initial conditions $\psi(\pi,\lambda) = 1, \quad \psi'(\pi,\lambda) = 0$. Denote

$$G(x,t;\lambda) = -\frac{1}{\Delta(\lambda)}\begin{cases} \psi(x,\lambda)\varphi(t,\lambda), & x \geq t \\ \varphi(x,\lambda)\psi(t,\lambda), & t \geq x \end{cases} \tag{4.4}$$

and let us consider the function

$$Y(x,\lambda) = \int_0^{\pi} \rho(t)f(t)G(x,t;\lambda)dt \tag{4.5}$$

which is a solution of the boundary value problem

$$-Y''(x,\lambda) + q(x)Y(x,\lambda) = \lambda^2\rho(x)Y(x,\lambda) - f(x)\rho(x), \tag{4.6}$$

$$Y'(0,\lambda) = 0, Y'(\pi,\lambda) = 0.$$

Using (1.9), we obtain

$$\underset{\lambda=\lambda_n}{Res}\, Y(x,\lambda) = \frac{1}{2\alpha_n\lambda_n}\varphi(x,\lambda_n)\int_0^{\pi} \rho(t)f(t)\varphi(t,\lambda_n)dt. \tag{4.7}$$

Let $f(x) \in L_2(0,\pi;\rho)$ be such that

$$\int_0^{\pi} \rho(t)f(t)\varphi(t,\lambda_n)dt = 0 \quad n = 1,2,3,\ldots.$$

Then, from (4.7), we have $\underset{\lambda=\lambda_n}{Res}\, Y(x,\lambda) = 0$; consequently, for fixed $x \in [0,\pi]$, the function $Y(x,\lambda)$ is entire with respect to λ. On the other hand, since

$$\Delta(\lambda) \geq |\lambda|\tilde{C}_\delta e^{|Im\lambda|\mu^+(\pi)}, \quad \lambda \in G_\delta, \quad \tilde{C}_\delta > 0. \tag{4.8}$$

(4.9)

$$\varphi(x,\lambda) = O\left(e^{|Im\lambda|\mu^+(x)}\right), \quad \psi(x,\lambda) = O\left(e^{|Im\lambda|(\mu^+(\pi)-\mu^+(x))}\right), \quad |\lambda| \to \infty,$$

from (4.5), it follows that for fixed $\delta > 0$ and sufficiently large $\lambda^* > 0$:

$$|Y(x,\lambda)| \le \frac{C_\delta}{|\lambda|}, \quad \lambda \in G_\delta, \quad |\lambda| \ge \lambda^*.$$

Using the maximum principle for module of analytic functions and Liouville theorem, we conclude that $Y(x,\lambda) \equiv 0$. This fact and (4.6) imply that $f(x) = 0$ a.e. on $[0,\pi]$. Thus, statement (1) of theorem is proved.

Let $f(x) \in AC[0,\pi]$ be an arbitrary absolutely continuous function. Let us transform the function $Y(x,\lambda)$ to the form

$$\begin{aligned} Y(x,\lambda) \quad = \quad & -\frac{1}{\lambda^2\Delta(\lambda)}\left\{\psi(x,\lambda)\int_0^x \left(-\varphi''(t,\lambda)+q(t)\varphi(t,\lambda)\right)f(t)dt - \right. \\ & \left. -\varphi(x,\lambda)\int_x^\pi \left(-\psi''(t,\lambda)+q(t)\psi(t,\lambda)\right)f(t)dt\right\}. \end{aligned}$$

Integrating by parts the addends with the second-order derivatives and taking into account conditions $f'(0) = 0$, $f'(\pi) = 0$, we have

$$Y(x,\lambda) = \frac{f(x)}{\lambda^2} - \frac{1}{\lambda^2}\left(Z_1(x,\lambda) + Z_2(x,\lambda)\right), \tag{4.10}$$

where

$$Z_1(x,\lambda) = \frac{1}{\Delta(\lambda)}\left[\psi(x,\lambda)\int_0^x g(t)\varphi'(t,\lambda)dt + \varphi(x,\lambda)\int_x^\pi g(t)\psi'(t,\lambda)dt\right],$$

$$\begin{aligned} Z_2(x,\lambda) = \frac{1}{\Delta(\lambda)} & \left[\psi(x,\lambda)\int_0^x \varphi(t,\lambda))f(t)\rho(t)dt + \right. \\ & \left. + \varphi(x,\lambda)\int_x^\pi \psi(t,\lambda))f(t)\rho(t)dt\right]. \end{aligned}$$

Here $g(t) = f'(t)$. Now consider the contour integral

$$I_N(x) = 2\sum_{n=1}^{N} \underset{\lambda=\lambda_n}{Res}\, Y(x,\lambda) = \sum_{n=1}^{N} a_n\varphi(x,\lambda_n) \tag{4.11}$$

where

$$a_n = \frac{1}{\alpha_n}\int_0^\pi \rho(t)f(t)\varphi(t,\lambda_n)dt.$$

On the other hand taking into account (4.10), we have

$$I_N(x) = f(x) - \frac{1}{2\pi i}\int_{\Gamma_n}\frac{1}{\lambda}\left(Z_1(x,\lambda) + Z_2(x,\lambda)\right)d\lambda. \tag{4.12}$$

Comparing (4.11) and (4.12), we obtain

$$f(x) = \sum_{n=1}^{\infty} a_n\varphi(x,\lambda_n) + \xi_N(x),$$

where

$$\xi_N(x) = \frac{1}{2\pi i}\int_{\Gamma_N}\frac{1}{\lambda}\left(Z_1(x,\lambda) + Z_2(x,\lambda)\right)d\lambda.$$

Therefore, in order to prove the item (2) of the theorem, it suffices to show that

$$\lim_{N\to\infty}\max_{0\le x\le\pi}|\xi_N(x)| = 0. \tag{4.13}$$

From the estimates of solution $\varphi(x,\lambda),\quad \psi(x,\lambda)$ and the function $\Delta(\lambda)$, it follows that for fixed $\delta>0$ and sufficiently large $\lambda^*>0$,

$$\max_{0\le x\le\pi}|Z_2(x,\lambda)|\le\frac{C_2}{|\lambda|},\quad \lambda\in G_\delta,\quad |\lambda|\ge\lambda^*, C_2>0. \tag{4.14}$$

Let us show that

$$\lim_{\substack{|\lambda|\to\infty\\ \lambda\in G_\delta}}\max_{0\le x\le\pi}|Z_1(x,\lambda)|=0. \tag{4.15}$$

At first, it was supposed that $g(t)$ is absolutely continuous on $[0,\pi]$. In this case, integration by parts gives

$$Z_1(x,\lambda)=-\frac{1}{\Delta(\lambda)}\left\{\psi(x,\lambda)\int_0^x\varphi(t,\lambda)g'(t)dt+\varphi(x,\lambda)\int_x^\pi\psi(t,\lambda)g'(t)dt\right\},$$

therefore, similarly to $Z_2(x,\lambda)$, we have

$$\max_{0\le x\le\pi}|Z_1(x,\lambda)|\le\frac{C_1}{|\lambda|},\quad \lambda\in G_\delta,\quad |\lambda|\ge\lambda^*, C_1>0.$$

In the general case, we fix $\varepsilon>0$ and choose absolutely continuous function $g_\varepsilon(t)$ such that

$$\int_0^\pi|g_\varepsilon(t)-g(t)|\,dt<\varepsilon.$$

Then, using the estimates $\varphi(x,\lambda),\quad \psi(x,\lambda)\quad \Delta(\lambda)$, one can find $\lambda^{**}>0$ such that when $\lambda\in G_\delta$, $|\lambda|\ge\lambda^{**}$, from the relation

$$Z_1(x,\lambda)=\frac{1}{\Delta(\lambda)}\left[\psi(x,\lambda)\int_0^x\varphi'(t,\lambda)(g_\varepsilon(t)-g(t))dt+\right.$$

$$\left.+\,\varphi(x,\lambda)\int_x^\pi\psi'(t,\lambda)(g_\varepsilon(t)-g(t))dt\right]+$$

$$+\frac{1}{\Delta(\lambda)}\left[\psi(x,\lambda)\int_0^x\varphi(t,\lambda)g_\varepsilon'(t)dt-\varphi(x,\lambda)\int_x^\pi\psi(t,\lambda)g_\varepsilon'(t)dt\right],$$

we have

$$\max_{0\le x\le\pi}|Z_1(x,\lambda)|\le C\int_0^\pi|g_\varepsilon(t)-g(t)|\,dt+\frac{\widetilde{C}(\epsilon)}{|\lambda|}<C_\epsilon+\frac{\widetilde{C}(\epsilon)}{|\lambda|},\quad \lambda\in G_\delta,\quad |\lambda|\ge\lambda^{**}.$$

Consequently,

$$\overline{\lim_{\substack{|\lambda|\to\infty\\ \lambda\in G_\delta}}}\max_{0\le x\le\pi}|Z_1(x,\lambda)|\le C_\epsilon.$$

Since ε is an arbitrary positive number, we obtain the validity of equality (4.15). Relations (4.14), (4.15) immediately imply (4.13), thus, statement (2) of theorem is proved.

System of eigenfunction $\{\varphi(x,\lambda_n)\}_{n\ge1}$ is complete and orthogonal in $L_2(0,\pi;\rho)$. Therefore, it forms the orthogonal basis in $L_{2,\rho}(0,\pi)$ and Parseval equality from theorem is valid. □

5. Weyl solution, Weyl function

Let $\Phi(x,\lambda)$ be the solution of equation (1.1) that satisfied the conditions $\Phi'(0,\lambda) = 1$, $\Phi'(\pi,\lambda) = 0$. Denote by $C(x,\lambda)$ the solution of equation (1.1), which satisfied the initial conditions $C(0,\lambda) = 0$, $C'(0,\lambda) = 1$. Then, the solution $\psi(x,\lambda)$ can be represented as follows

$$\psi(x,\lambda) = \psi(0,\lambda)\varphi(x,\lambda) - \Delta(\lambda)C(x,\lambda) \tag{5.1}$$

or

$$-\frac{\psi(x,\lambda)}{\Delta(\lambda)} = C(x,\lambda) - \frac{\psi(0,\lambda)}{\Delta(\lambda)}\varphi(x,\lambda). \tag{5.2}$$

Denote

$$M(\lambda) := -\frac{\psi(0,\lambda)}{\Delta(\lambda)}. \tag{5.3}$$

It is clear that

$$\Phi(x,\lambda) = C(x,\lambda) + M(\lambda)\varphi(x,\lambda). \tag{5.4}$$

The function $\Phi(x,\lambda)$ and $M(\lambda) = \Phi(0,\lambda)$ are respectively called the Weyl solution and the Weyl function of the boundary value problem (1.1), (1.2). The Weyl function is a meromorphic function having simple poles at points λ_n eigenvalues of boundary value problem (1.1), (1.2). Relations (5.2), (5.4) yield

$$\Phi(x,\lambda) = -\frac{\psi(x,\lambda)}{\Delta(\lambda)}. \tag{5.5}$$

It can be shown that

$$< \varphi(x,\lambda), \Phi(x,\lambda) >= 1. \tag{5.6}$$

Theorem 5.1. *If $M(\lambda) = \widetilde{M}(\lambda)$, then $L = \widetilde{L}$; that is, the boundary value problem (1.1), (1.2), is unique by the Weyl function.*

Proof. We describe the matrix $P(x,\lambda) = [P_{ij}(x,\lambda)]_{i,j=1,2}$ with the formula

$$P(x,\lambda)\begin{pmatrix} \widetilde{\varphi}(x,\lambda) & \widetilde{\Phi}(x,\lambda) \\ \widetilde{\varphi}'(x,\lambda) & \widetilde{\Phi}'(x,\lambda) \end{pmatrix} = \begin{pmatrix} \varphi(x,\lambda) & \Phi(x,\lambda) \\ \varphi'(x,\lambda) & \Phi'(x,\lambda) \end{pmatrix}. \tag{5.7}$$

From (5.7), we have

$$\begin{aligned} \varphi(x,\lambda) &= P_{11}(x,\lambda)\widetilde{\varphi}(x,\lambda) + P_{12}(x,\lambda)\widetilde{\varphi}'(x,\lambda), \\ \Phi(x,\lambda) &= P_{11}(x,\lambda)\widetilde{\Phi}(x,\lambda) + P_{12}(x,\lambda)\widetilde{\Phi}'(x,\lambda), \end{aligned} \tag{5.8}$$

or

$$\begin{aligned} P_{11}(x,\lambda) &= \varphi(x,\lambda)\widetilde{\Phi}'(x,\lambda) - \Phi(x,\lambda)\widetilde{\varphi}'(x,\lambda), \\ P_{12}(x,\lambda) &= -\varphi(x,\lambda)\widetilde{\Phi}(x,\lambda) + \Phi(x,\lambda)\widetilde{\varphi}(x,\lambda). \end{aligned} \tag{5.9}$$

Taking equation (5.5) into consideration in (5.9), we get (5.4) into (5.9), then we get

(5.10)

$$P_{11}(x,\lambda) = 1 + \tfrac{1}{\Delta(\lambda)}\left[\varphi(x,\lambda)(\widetilde{\psi}'(x,\lambda) - \psi'(x,\lambda)) - \psi(x,\lambda)(\widetilde{\varphi}'(x,\lambda) - \varphi'(x,\lambda))\right]$$

$$P_{12}(x,\lambda) = \tfrac{1}{\Delta(\lambda)}\left[\psi(x,\lambda)\widetilde{\varphi}(x,\lambda) - \varphi(x,\lambda)\widetilde{\psi}(x,\lambda)\right].$$

Now, from (4.8)and (4.9), we have from equation (5.10)

$$\lim_{|\lambda|\to\infty} \max_{0\le x\le\pi} |P_{11}(x,\lambda)-1| = \lim_{|\lambda|\to\infty} \max_{0\le x\le\pi} |P_{12}(x,\lambda)| = 0. \tag{5.11}$$

Now, if we take into consideration equation (5.4) into (5.9), we get

$$P_{11}(x,\lambda) = \varphi(x,\lambda)\widetilde{C}'(x,\lambda) - C(x,\lambda)\widetilde{\varphi}'(x,\lambda) + (\widetilde{M}(\lambda) - M(\lambda))\varphi(x,\lambda)\widetilde{\varphi}'(x,\lambda)$$
$$P_{12}(x,\lambda) = C(x,\lambda)\widetilde{\varphi}(x,\lambda) - \widetilde{C}(x,\lambda)\varphi(x,\lambda) - (\widetilde{M}(\lambda) - M(\lambda))\varphi(x,\lambda)\widetilde{\varphi}(x,\lambda).$$

Therefore if $M(\lambda) = \widetilde{M}(\lambda)$, then $P_{11}(x,\lambda)$ and $P_{12}(x,\lambda)$ are entire functions for every fixed x. It can be easily seen from (5.11) that $P_{11}(x,\lambda) = 1$ and $P_{12}(x,\lambda) = 0$. Substituting into (5.8), we get $\varphi(x,\lambda) \equiv \widetilde{\varphi}(x,\lambda)$ and $\Phi(x,\lambda) \equiv \widetilde{\Phi}(x,\lambda)$ for every x and λ. Hence, we arrive at $q(x) \equiv \widetilde{q}(x)$. □

Theorem 5.2. *The expression*

$$M(\lambda) = -\sum_{n=0}^{\infty} \frac{1}{2\alpha_n\lambda_n(\lambda-\lambda_n)} \tag{5.12}$$

holds.

Proof. Using (5.3), we get for sufficiently large $\lambda^* > 0$,

$$M(\lambda) \le \frac{C_\delta}{|\lambda|}, \quad \lambda\in G_\delta, \quad |\lambda| > \lambda^*. \tag{5.13}$$

Further using (1.9) and (5.4) we calculate:

$$\operatorname*{Res}_{\lambda=\lambda_n} M(\lambda) = -\frac{\psi(0,\lambda_n)}{\dot{\Delta}(\lambda_n)} = -\frac{\beta_n}{\dot{\Delta}(\lambda_n)} = -\frac{1}{2\lambda_n\alpha_n}. \tag{5.14}$$

Now, let's consider the contour integral

$$J_N(\lambda) = \frac{1}{2\pi i}\int_{\Gamma_N} \frac{M(\mu)}{\lambda-\mu}d\mu, \quad \lambda\in Int\Gamma_N,$$

where $\Gamma_N = \left\{\mu : |\mu| = \left|\lambda_N^0\right| + \frac{\gamma}{2}\right\}$ is a contour of counter-clockwise by pass.

By virtue of (5.13) we have $\lim_{N\to\infty} J_N(\lambda) = 0$. On the other hand, by residue theorem and (5.14) yield

$$J_N(\lambda) = -M(\lambda) - \sum_{n=-N}^{N} \frac{1}{2\lambda_n\alpha_n(\lambda-\lambda_n)}$$

and when $N\to\infty$ we arrive at (5.12). □

Theorem 5.3. *If $\lambda_n = \widetilde{\lambda}_n, \alpha_n = \widetilde{\alpha}_n$ for all $n\in Z$ then $L = \widetilde{L}$. That is, the problem (1.1), (1.2) is uniquely determined by spectral date.*

Proof. Since $\lambda_n = \widetilde{\lambda}_n, \alpha_n = \widetilde{\alpha}_n$ for all $n\in Z$ and considering the formula (5.12), we have $M(\lambda) = \widetilde{M}(\lambda)$. Using Theorem 4, $L = \widetilde{L}$ is obtained. □

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