A NOTE ON POROSITY CLUSTER POINTS

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Abstract. Porosity cluster points of real valued sequences was defined and studied in [3]. In this paper we give the relation between porosity cluster points and distance function.

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1. Introduction

Porosity is appeared in the papers of Denjoy [5], [6], Khintchine [11] and, Dolzenko [7]. It has many applications in theory of free boundaries [10], generalized subharmonic functions [8], complex dynamics [12], quasisymmetric maps [14], infinitesimal geometry [4] and other areas of mathematics.

Let $A \subset \mathbb{R}^+ = [0, \infty)$, then the right upper porosity of $A$ at the point 0 is defined as

$$p^+(A) := \limsup_{h \to 0^+} \frac{\lambda(A, h)}{h}$$

where $\lambda(A, h)$ denotes the length of the largest open subinterval of $(0, h)$ that contains no point of $A$ (for more information look [13]). The notion of right lower porosity of $A$ at the point 0 is defined similarly.

In [1], the notation of porosity which was defined at zero for the subsets of real numbers, has been redefined at infinity for the subsets of natural numbers.

Let $\mu : \mathbb{N} \to \mathbb{R}^+$ be a strictly decreasing function such that $\lim_{n \to \infty} \mu(n) = 0$, (it is called scaling function) and let $E$ be a subset of $\mathbb{N}$.

Upper porosity and lower porosity of the set $E$ at infinity were defined respectively in [1] as follows:

$$(1.1) \quad p_\mu^+(E) := \limsup_{n \to \infty} \frac{\lambda_\mu(E, n)}{\mu(n)}, \quad p_\mu^-(E) := \liminf_{n \to \infty} \frac{\lambda_\mu(E, n)}{\mu(n)},$$

where

$$\lambda_\mu(E, n) := \sup\{|\mu(n^{(1)}) - \mu(n^{(2)})| : n^{(1)} \leq n < n^{(2)}, (n^{(1)}, n^{(2)}) \cap E = \emptyset\}.$$

Using the definition of upper porosity, all subsets of natural numbers can be classify as follows: $E \subseteq \mathbb{N}$ is

(i) porous at infinity if $p_\mu^+(E) > 0$;
(ii) strongly porous at infinity if $p_\mu^+(E) = 1$;
(iii) nonporous at infinity if $\mathcal{P}_\mu(E) = 0$. Throughout this paper, we will consider only the upper porosity of subsets of $\mathbb{N}$.

Let us recall the definition of $\mathcal{P}_\mu$-convergence of real valued sequences for any scaling function:

**Definition 1.1.** [2] A sequence $x = (x_n)_{n \in \mathbb{N}}$ is said to be $\mathcal{P}_\mu$-convergent to $l$ if for each $\varepsilon > 0$,

$$\mathcal{P}_\mu(A_\varepsilon) > 0,$$

where $A_\varepsilon := \{n : |x_n - l| \geq \varepsilon\}$. It is denoted by $x \to l(\mathcal{P}_\mu)$ or $\lim_{n \to \infty} x_n = l$.

Let $x' = (x_{n_k})$ be a subsequence of $x = (x_n)$ for monotone increasing sequence $(n_k)_{k \in \mathbb{N}}$ and $K := \{n_k : k \in \mathbb{N}\}$, then we abbreviate $x' = (x_{n_k})$ by $(x)_K$.

**Definition 1.2.** [3]. Let $x = (x_n)$ be a sequence and $(x)_K$ be a subsequence of $x = (x_n)$. If

(i) $\mathcal{P}_\mu(K) > 0$, then $(x)_K$ is called a $\mathcal{P}_\mu$-thin subsequence of $x = (x_n)$,

(ii) $\mathcal{P}_\mu(K) = 1$, then $(x)_K$ is called a strongly $\mathcal{P}_\mu$-thin subsequence of $x = (x_n)$,

(iii) $\mathcal{P}_\mu(K) = 0$, then $(x)_K$ is a $\mathcal{P}_\mu$-nonthin (or $\mathcal{P}_\mu$-dense) subsequence of $x = (x_n)$.

**Definition 1.3.** [3]. A number $\alpha$ is said to be a $\mathcal{P}_\mu$-limit point of the sequence $x = (x_n)$ if it has a $\mathcal{P}_\mu$-nonthin subsequence that converges to $\alpha$.

The set of all $\mathcal{P}_\mu$-limit points of $x = (x_n)$ is denoted by $L_{\mathcal{P}_\mu}(x)$.

**Definition 1.4.** [3]. A number $\beta$ is said to be a $\mathcal{P}_\mu$-cluster point of $x = (x_n)$ if for every $\varepsilon > 0$, the set

$$\{n : |x_n - \beta| < \varepsilon\}$$

is nonporous. i.e.,

$$\mathcal{P}_\mu(\{n : |x_n - \beta| < \varepsilon\}) = 0.$$

For a given sequence $x = (x_n)$; the symbol $\Gamma_{\mathcal{P}_\mu}(x)$ denotes the set of all $\mathcal{P}_\mu$-cluster points.

### 2. Main Results

Some results about the set of $L_{\mathcal{P}_\mu}(x)$-cluster, and $L_{\mathcal{P}_\mu}(x)$-limit points of given real valued sequences has been investigated in [3]. In this paper, as a continuation of [3] the same subject will be studied.

**Theorem 2.1.** Assume that $x = (x_n)$ is monotone increasing (or decreasing) sequence of real numbers. If $\sup x_n < \infty$ (or $\inf x_n < \infty$), then $\sup x_n \in \Gamma_{\mathcal{P}_\mu}(x)$ (or $\inf x_n \in \Gamma_{\mathcal{P}_\mu}(x)$).

**Proof.** The proof will be given only for monotone increasing sequences. The other case can be proved by following similar steps. From the definition of supremum for any $\varepsilon > 0$, there exists an $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that the inequality

$$\sup x_n - \varepsilon < x_{n_0} \leq \sup x_n$$

holds. Since the sequence is monotone increasing, then we have

$$\sup x_n - \varepsilon < x_{n_0} < x_n \leq \sup x_n < \sup x_n + \varepsilon$$

for all $n > n_0$. 
From (2.1), for any \( \varepsilon > 0 \) there exists an \( n_0 = n_0(\varepsilon) \in \mathbb{N} \) such that following inequality
\[
|x_n - \sup x_n| < \varepsilon
\]
holds for all \( n > n_0(\varepsilon) \).

From (2.2), following inclusion
\[
\mathbb{N}\{1, 2, 3, ..., n_0\} \subset \{n : |x_n - \sup x_n| < \varepsilon\}
\]
and the inequality
\[
\mathbb{P}_\mu(\{n : |x_n - \sup x_n| < \varepsilon\}) \leq \mathbb{P}_\mu(\mathbb{N}\{1, 2, ..., n_0\})
\]
hold.

Since \( \mathbb{P}_\mu(\mathbb{N}\{1, 2, 3, ..., n_0\}) = 0 \), then from Lemma 1.1 in [3] we have
\[
\mathbb{P}_\mu(\{n : |x_n - \sup x_n| < \varepsilon\}) = 0.
\]
This gives the desired proof. \(\square\)

**Corollary 2.1.** If \( x = (x_n) \) is a bounded sequence, then \( \sup x_n \) and \( \inf x_n \) are belong to \( \Gamma_{\mathbb{P}_\mu}(x) \).

Let \( A, B \subset \mathbb{R} \) and recall the distance between \( A \) and \( B \) is defined as
\[
d(A, B) := \inf\{|a - b| : a \in A, b \in B\}.
\]

**Theorem 2.2.** Let \( x = (x_n) \) be a real valued sequence. If \( \Gamma_{\mathbb{P}_\mu}(x) \neq \emptyset \), then \( d(\Gamma_{\mathbb{P}_\mu}(x), x) = 0 \).

**Proof.** Assume \( \Gamma_{\mathbb{P}_\mu}(x) \neq \emptyset \). Let us consider an arbitrary element \( y^* \in \Gamma_{\mathbb{P}_\mu}(x) \). Then, for an arbitrary \( \varepsilon > 0 \) we have
\[
\mathbb{P}_\mu(\{n : |x_n - y^*| < \varepsilon\}) = 0.
\]
So, the set \( \{x_n : |x_n - y^*| < \varepsilon\} \) has at least countable number elements of \( x = (x_n) \). Let us denote this set by \( D \) where \( D := \{n_k : |x_{n_k} - y^*| < \varepsilon\} \subset \mathbb{N} \). Therefore, we have
\[
0 \leq \text{dist}(\Gamma_{\mathbb{P}_\mu}(x), x) = \inf\{|y - x_n| : y \in \Gamma_{\mathbb{P}_\mu}(x), n \in \mathbb{N}\} \leq \inf\{|y^* - x_{n_k}| : n_k \in D\} < \varepsilon.
\]
So, for every \( \varepsilon > 0 \), we have \( 0 \leq d(\Gamma_{\mathbb{P}_\mu}(x), x) < \varepsilon \). \(\square\)

**Theorem 2.3.** Let \( x = (x_n) \) be a real valued sequence and \( \gamma \in \mathbb{R} \) be an arbitrary fixed point. If \( d(\gamma, x) \neq 0 \), then \( \gamma \not\in \Gamma_{\mathbb{P}_\mu}(x) \).

**Proof.** From the hypothesis we have
\[
d(\gamma, x) := \inf\{|x_k - \gamma| : k \in \mathbb{N}\} = m > 0.
\]
From the assumption the inequality
\[
|x_k - \gamma| \geq m
\]
holds for all \( k \in \mathbb{N} \). It means that the open interval \( (\gamma - m, \gamma + m) \) has no elements of the sequence \( x = (x_n) \). So, we have
\[
\mathbb{P}_\mu(\{k : |x_k - \gamma| < m\}) = 1.
\]
For \( 0 < \varepsilon < m \) we have
\[
\mathbb{P}_\mu(\{k : |x_k - \gamma| < \varepsilon\}) = 1.
\]
So, $\gamma \notin \Gamma_{\mu}(x)$.

Remark 2.1. If $d(\gamma, x) = 0$, it is not necessary for $\gamma \in \Gamma_{\mu}(x)$.

Let us consider $(x_n) = (\frac{1}{n})_{n \in \mathbb{N}}$. For $\gamma = \frac{1}{2}$, it is clear that $d(\frac{1}{2}, \frac{1}{n}) = 0$ holds, but $\frac{1}{2} \notin \Gamma_{\mu}(x) = \{0\}$.

References


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