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A NOTE ON POROSITY CLUSTER POINTS

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Abstract. Porosity cluster points of real valued sequences was defined and studied in [3]. In this paper we give the relation between porosity cluster points and distance function.

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1. Introduction

Porosity is appeared in the papers of Denjoy [5], [6], Khintchine [11] and, Dolzenko [7]. It has many applications in theory of free boundaries [10], generalized subharmonic functions [8], complex dynamics [12], quasisymmetric maps [14], infinitesimal geometry [4] and other areas of mathematics.

Let $A \subset \mathbb{R}^+ = [0, \infty)$, then the right upper porosity of A at the point 0 is defined as

$$p^+(A) := \limsup_{h\to 0^+} \frac{\lambda(A,h)}{h}$$

where $\lambda(A,h)$ denotes the length of the largest open subinterval of $(0,h)$ that contains no point of A (for more information look [13]). The notion of right lower porosity of A at the point 0 is defined similarly.

In [1], the notation of porosity which was defined at zero for the subsets of real numbers, has been redefined at infinity for the subsets of natural numbers.

Let $\mu : \mathbb{N} \to \mathbb{R}^+$ be a strictly decreasing function such that $\lim_{n\to\infty} \mu(n) = 0$, (it is called scaling function) and let E be a subset of $\mathbb{N}$.

Upper porosity and lower porosity of the set E at infinity were defined respectively in [1] as follows:

$$\overline{p}_\mu(E) := \limsup_{n\to\infty} \frac{\lambda_\mu(E,n)}{\mu(n)}, \quad \underline{p}_\mu(E) := \liminf_{n\to\infty} \frac{\lambda_\mu(E,n)}{\mu(n)}, \tag{1.1}$$

where

$$\lambda_\mu(E,n) := \sup\{|\mu(n^{(1)}) - \mu(n^{(2)})| : n \le n^{(1)} < n^{(2)},\ (n^{(1)}, n^{(2)}) \cap E = \emptyset\}.$$

Using the definition of upper porosity, all subsets of natural numbers can be classify as follows: $E \subseteq \mathbb{N}$ is
(i) porous at infinity if $\overline{p}_\mu(E) > 0$;
(ii) strongly porous at infinity if $\overline{p}_\mu(E) = 1$;

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(iii) nonporous at infinity if $\overline{p}_\mu(E) = 0$. Throughout this paper, we will consider only the upper porosity of subsets of $\mathbb{N}$.

Let us recall the definition of $\overline{p}_\mu$-convergence of real valued sequences for any scaling function:

Definition 1.1. [2] A sequence $x = (x_n)_{n\in\mathbb{N}}$ is said to $\overline{p}_\mu$-convergent to l if for each $\varepsilon > 0$,

$$\overline{p}_\mu(A_\varepsilon) > 0,$$

where $A_\varepsilon := \{n \ : \ |x_n - l| \geq \varepsilon\}$. It is denoted by $x \to l(\overline{p}_\mu)$ or $(\overline{p}_\mu - \lim_{n\to\infty} x_n = l)$.

Let $x' = (x_{n_k})$ be a subsequence of $x = (x_n)$ for monotone increasing sequence $(n_k)_{k\in\mathbb{N}}$ and $K := \{n_k : k \in \mathbb{N}\}$, then we abbreviate $x' = (x_{n_k})$ by $(x)_K$.

Definition 1.2. [3]. Let $x = (x_n)$ be a sequence and $(x)_K$ be a subsequence of $x = (x_n)$. If
(i) $\overline{p}_\mu(K) > 0$, then $(x)_K$ is called $\overline{p}_\mu$-thin subsequence of $x = (x_n)$,
(ii) $\overline{p}_\mu(K) = 1$, then $(x)_K$ is called a strongly $\overline{p}_\mu$-thin subsequence of $x = (x_n)$,
(iii) $\overline{p}_\mu(K) = 0$, then $(x)_K$ is a $\overline{p}_\mu$-nonthin (or $\overline{p}_\mu$-dense) subsequence of $x = (x_n)$.

Definition 1.3. [3]. A number α is said to be $\overline{p}_\mu$-limit point of the sequence $x = (x_n)$ if it has a $\overline{p}_\mu$-nonthin subsequence that converges to α.

The set of all $\overline{p}_\mu$-limit points of $x = (x_n)$ is denoted by $L_{\overline{p}_\mu}(x)$.

Definition 1.4. [3]. A number β is said to be a $\overline{p}_\mu$-cluster point of $x = (x_n)$ if for every $\varepsilon > 0$, the set

$$\{n : |x_n - \beta| < \varepsilon\}$$

is nonporous. i.e.,

$$\overline{p}_\mu(\{n : |x_n - \beta| < \varepsilon\}) = 0.$$

For a given sequence $x = (x_n)$; the symbol $\Gamma_{\overline{p}_\mu}(x)$ denotes the set of all $\overline{p}_\mu$-cluster points.

2. Main Results

Some results about the set of $L_{\overline{p}_\mu}(x)$-cluster, and $L_{\overline{p}_\mu}(x)$-limit points of given real valued sequences has been investigated in [3]. In this paper, as a continuation of [3] the same subject will be studied.

Theorem 2.1. *Assume that $x = (x_n)$ is monotone increasing (or decreasing) sequence of real numbers. If $\sup x_n < \infty$ (or $\inf x_n < \infty$), then $\sup x_n \in \Gamma_{\overline{p}_\mu}(x)$ (or $\inf x_n \in \Gamma_{\overline{p}_\mu}(x)$).*

Proof. The proof will be given only for monotone increasing sequences. The other case can be proved by following similar steps. From the definition of supremum for any $\varepsilon > 0$, there exists an $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that the inequality

$$\sup x_n - \varepsilon < x_{n_0} \leq \sup x_n$$

holds. Since the sequence is monotone increasing, then we have

$$\sup x_n - \varepsilon < x_{n_0} < x_n \leq \sup x_n < \sup x_n + \varepsilon \tag{2.1}$$

for all $n > n_0$.

From (2.1), for any $\varepsilon > 0$ there exists an $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that following inequality

$$|x_n - \sup x_n| < \varepsilon \tag{2.2}$$

holds for all $n > n_0(\varepsilon)$.

From (2.2), following inclusion

$$\mathbb{N} \backslash \{1, 2, 3, ..., n_0\} \subset \{n : |x_n - \sup x_n| < \varepsilon\}$$

and the inequality

$$\overline{p}_\mu(\{n : |x_n - \sup x_n| < \varepsilon\}) \leq \overline{p}_\mu(\mathbb{N} \backslash \{1, 2, ..., n_0\})$$

hold.

Since $\overline{p}_\mu(\mathbb{N} \backslash \{1, 2, 3, ..., n_0\}) = 0$, then from Lemma 1.1 in [3] we have

$$\overline{p}_\mu(\{n : |x_n - \sup x_n| < \varepsilon\}) = 0.$$

This gives the desired proof. □

Corollary 2.1. *If $x = (x_n)$ is a bounded sequence, then $\sup x_n$ and $\inf x_n$ are belong to $\Gamma_{\overline{p}_\mu}(x)$.*

Let $A, B \subset \mathbb{R}$ and recall the distance between A and B is defined as

$$d(A, B) := \inf\{|a - b| : a \in A, b \in B\}.$$

Theorem 2.2. *Let $x = (x_n)$ be a real valued sequence. If $\Gamma_{\overline{p}_\mu}(x) \neq \emptyset$, then $d(\Gamma_{\overline{p}_\mu}(x), x) = 0$.*

Proof. Assume $\Gamma_{\overline{p}_\mu}(x) \neq \emptyset$. Let us consider an arbitrary element $y* \in \Gamma_{\overline{p}_\mu}(x)$. Then, for an arbitrary $\varepsilon > 0$ we have

$$\overline{p}_\mu(\{n : |x_n - y*| < \varepsilon\}) = 0.$$

So, the set $\{x_n : |x_n - y*| < \varepsilon\}$ has at least countable number elements of $x = (x_n)$. Let us denote this set by D where $D := \{n_k : |x_{n_k} - y*| < \varepsilon\} \subset \mathbb{N}$. Therefore, we have

$$\begin{aligned} 0 \leq dist(\Gamma_{\overline{p}_\mu}(x), x) &= \inf\{|y - x_n| : y \in \Gamma_{\overline{p}_\mu}(x), n \in \mathbb{N}\} \\ &\leq \inf\{|y* - x_{n_k}| : n_k \in D\} < \varepsilon. \end{aligned}$$

So, for every $\varepsilon > 0$, we have $0 \leq d(\Gamma_{\overline{p}_\mu}(x), x) < \varepsilon$. □

Theorem 2.3. *Let $x = (x_n)$ be a real valued sequence and $\gamma \in \mathbb{R}$ be an arbitrary fixed point. If $d(\gamma, x) \neq 0$, then $\gamma \notin \Gamma_{\overline{p}_\mu}(x)$.*

Proof. From the hypothesis we have

$$d(\gamma, x) := \inf\{|x_k - \gamma| : k \in \mathbb{N}\} = m > 0.$$

From the assumption the inequality

$$|x_k - \gamma| \geq m$$

holds for all $k \in \mathbb{N}$. It means that the open interval $(\gamma - m, \gamma + m)$ has no elements of the sequence $x = (x_n)$. So, we have

$$\overline{p}_\mu(\{k : |x_k - \gamma| < m\}) = 1.$$

For $0 < \varepsilon < m$ we have

$$\overline{p}_\mu(\{k : |x_k - \gamma| < \varepsilon\}) = 1.$$

So, $\gamma \notin \Gamma_{\overline{p}_\mu}(x)$. □

Remark 2.1. If $d(\gamma, x) = 0$, it is not necessary for $\gamma \in \Gamma_{\overline{p}_\mu}(x)$.
Let us consider $(x_n) = (\frac{1}{n})_{n\in\mathbb{N}}$. For $\gamma = \frac{1}{2}$, it is clear that $d(\frac{1}{2}, \frac{1}{n}) = 0$ holds, but $\frac{1}{2} \notin \Gamma_{\overline{p}_\mu}(x) = \{0\}$.

References

[1] M. Altınok, O. Dovgoshey, M. Küçükaslan, *Local one-sided porosity and pretangent spaces*, Analysis, **36(3)** (2016), 147-171.

[2] M. Altınok, M. Küçükaslan, *On porosity-convergence of real valued sequences*, (Accepted in Alexandru Ioan Cuza Dn Ia Matematica in 2016).

[3] M. Altınok, *Porosity limit and cluster points of real valued sequences*, (Submitted 2016).

[4] V. Bilet, O. Dovgoshey, *Boundedness of pretangent spaces to general metric spaces*, Ann. Acad. Sci. Fenn. Math., **39** (2009), 73-82.

[5] A. Denjoy, *Sur une propriété des séries trigonométriques*, Verlag v.d. G. V. der Wie-en Natuur. Afd.,**29** (1920), 220-232.

[6] A. Denjoy, *Leçons sur le calcul des cofficients d'une série trigonométrique, Part II, Métrique et topologie d'ensembles parfaits et de fonctions*, Gauthier-Villars, Paris, 1941.

[7] E. P. Dolženko, *Boundary properties of arbitrary functions*, (Russian), Izv. Akad. Nauk SSSR Ser. Math. **31** (1967), 3-14.

[8] O. Dovgoshey and J. Riihentaus, *Mean value type inequalities for quasinearly subharmonic functions*, Glasgow Math. J., **55(2)** (2013), 349-368.

[9] J. A. Fridy *Statistical limit points*, Proceeding of the American Mathematical Society, **118(4)** (1993), 1187-1192.

[10] L. Karp, T. Kilpenläinen, A. Petrosyan and H. Shahgholian, *On the porosity of free boundaries in degenerate variational inequalities*, J. Differential Equations, **164** (2000), 110-117.

[11] A. Khintchine, *An investigation of the structure of measurable functions*, (Russian), Mat. Sbornik, **31** (1924), 265-285.

[12] F. Przytycki and S. Rohde, *Porosity of Collet-Eckmann Julia sets*, Fund. Math., **155** (1998), 189-199.

[13] B. S. Thomson, *Real Functions*, Lecture Notes in Mathematics, **1170**, Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, 1985.

[14] J. Väisälä, *Porous seta and quasisymmetric maps*, Trans. Amer. Math. Soc., **299** (1987), 525-533.

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