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SOME CLASSIFICATIONS OF AN X FDK-SPACE

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Abstract. In this paper, we introduce definitions of double wedge and double conull FDK-spaces. Also, we give some important corollary for any double sequence space about these definitions.

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1. Introduction

In [4], the first study on double sequences was examined by Bromwich. And than it was investigated by many authors such as Hardy [6], Moricz [7], Tripathy [16], Başarır and Sonalcan [2]. The notion of regular convergence for double sequences was defined by Hardy [6]. After that both the theory of topological double sequence spaces and the theory of summability of double sequences were studied by Zeltser [17]. The statistical and Cauchy convergence for double sequences were examined by Mursaleen and Edely [8] and Tripathy [15] in recent years. Many recent improvements containing the summability by four dimensional matrices might be found in [10].

Ω denotes the space of all complex valued double sequences which is a vector space with coordinatewise addition and scalar multiplication. Any vector subspace of Ω is called as a double sequence space. The space $\mathcal{M}_u$ of all bounded double sequences is defined by

$$\mathcal{M}_u := \left\{ x \in \Omega \ : \ \|x\|_\infty := \sup_{k,l} |x_{kl}| < \infty \right\},$$

which is a Banach space with the norm $\|.\|_\infty$. In addition we consider the double sequence spaces

$$\Phi := span\{e^{kl} \ : \ k, l \in \mathbb{N}\} = \left\{x \in \Omega \ : \ \exists N_0 \in \mathbb{N}, \forall (k,l) \in \mathbb{N}^2/[1,N_0]^2 \ : \ x_{kl} = 0\right\}$$

$$\Phi_1 := \Phi \cup \{e\}$$

$$\mathcal{L}_u := \left\{ x \in \Omega \ : \ \sum_{k,l} |x_{kl}| < \infty \right\}$$

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$$\mathcal{BV} := \left\{ x \in \Omega \ : \ \|x\|_{\mathcal{BV}} := \sum_{k,l} |x_{kl} - x_{k+1,l} - x_{k,l+1} + x_{k+1,l+1}| < \infty \right\}$$

$$\mathcal{C}_0 := \left\{ x \in \mathcal{C} \ : \ \forall l \in \mathbb{N} \ : \ \lim_k x_{kl} = 0 \right\}$$

$$\mathcal{C}_c := \left\{ x \in \Omega \ : \ \forall l \in \mathbb{N} : \ \lim_{k,l} x_{kl} \text{ exist} \right\}.$$

Throughout this paper e denotes the double sequence of ones; (δ^{ij}), $i, j = 1, 2, \ldots$, with the one in the (i, j) position.

A subspace E of the vector space Ω is called DK-space, if all the seminorms $r_{kl} : E \to \mathbb{R}, x \mapsto |x_{kl}|$ $(k, l \in \mathbb{N})$ are continuous. An FDK space is a DK-space with a complete, metrizable, locally convex topology. A normable FDK-space is called BDK-space.

2. Main Results

In this section conull (strongly conull) and double wedge (weak double wedge) FDK-spaces are defined and several characterizations are given.

Definition 2.1. If (E, τ) is a FDK-space containing Φ, and $\delta^{ij} \to 0$ in τ, then (E, τ) is called a double wedge space.

Definition 2.2. If (E, τ) is a FDK-space containing Φ, and $\delta^{ij} \to 0$ (weakly) in τ, then (E, τ) is called a weak double wedge space.

Definition 2.3. Let (E, τ) is an FDK-space containing Φ_1. If $\forall f \in E'$,

$$f(e) = \lim_{m,n \to \infty} \sum_{k,l=1}^{m,n} f(\delta^{kl})$$

then (E, τ) is called a conull FDK-space.

Definition 2.4. Let (E, τ) is an FDK-space containing Φ_1. If $\forall f \in E'$,

$$e = \lim_{m,n \to \infty} \sum_{k,l=1}^{m,n} \delta^{kl}$$

then (E, τ) is called a strong conull FDK-space.

Clearly, each strong conull (wedge) FDK-space is a conull (weak wedge) FDK-space and the following diagram is hold:

$$\begin{array}{ccccc} & & \text{conull FDK} & & \\ & \nearrow & & \searrow & \\ \text{strong conull FDK} & & & & \text{weak wedge FDK} \\ & \searrow & & \nearrow & \\ & & \text{wedge FDK} & & \end{array}$$

Indeed, let E be a strong conull FDK-space. Then we have $e^{(kl)} \to e$. Hence $P(e^{(kl)} - e) \to 0$ $(k, l \to \infty)$ for the seminorm P in τ. In this case, the following equation is hold

$$\begin{aligned}
\delta^{kl} &= e^{(k,l)} - e^{(k-1,l-1)} - \sum_{i=k,j=1}^{k,l} \delta^{ij} - \sum_{i=1,j=l}^{k,l} \delta^{ij} \\
&= e^{(k,l)} - e^{(k-1,l-1)} - \sum_{i=k,j=1}^{k,l} \delta^{ij} - \sum_{i=1,j=l}^{k,l} \delta^{ij} + e^{(k-1,l-1)} - e^{(k-1,l-1)} \\
&= e^{(k,l)} - e^{(k,l-1)} - e^{(k-1,l)} + e^{(k-1,l-1)} + e - e + e - e \; .
\end{aligned}$$

So, we have

$$\begin{aligned}
&P\left(\delta^{kl}\right) = P\left(e^{(k,l)} - e^{(k,l-1)} - e^{(k-1,l)} + e^{(k-1,l-1)} + e - e + e - e\right) \\
&\leq P\left(e^{(k,l)} - e\right) + P\left(e^{(k,l-1)} - e\right) + P\left(e^{(k-1,l)} - e\right) + P\left(e^{(k-1,l-1)} - e\right).
\end{aligned}$$

It is clearly that since $P\left(\delta^{kl}\right) \to 0$, $k,l \to \infty$, E is double wedge space.

We recall that the $\alpha-$dual of a subset E of Ω is defined to be ([14])

$$E^{\alpha} := \left\{ x = (x_{kl}) \; : \; \sum_{k,l=1}^{\infty,\infty} |x_{kl} y_{kl}| < \infty \; , \; \forall y = (y_{kl}) \in E \right\}.$$

Lemma 2.1. *If $z^{(mn)} \in \mathcal{C}_0$, $m,n = 1,2,\ldots$, then there exists $z \in \mathcal{C}_0$ such that*

$$\lim_{i,j\to\infty} \frac{z_{ij}^{(mn)}}{z_{ij}} = 0 \qquad (m,n = 1,2,\ldots).$$

Furthermore, for any such z, we have $z^{\alpha} \subseteq \bigcap_{m,n=1}^{\infty,\infty} \{z^{(mn)}\}^{\alpha}$.

Proof. Let $z^{(mn)} \in \mathcal{C}_0$. We can choose two sequences (i_k), (j_l) of positive integers such that

$$1 = i_0 < i_1 < i_2 < \ldots \qquad 1 = j_0 < j_1 < j_2 < \ldots$$

and

$$\max_{\substack{1\leq m\leq k \\ 1\leq n\leq l}} |z_{ij}^{(mn)}| < \frac{1}{4^{kl}} \qquad \left(\begin{array}{c} i \geq i_k, \; j \geq j_l \\ k,l = 1,2,\ldots \end{array}\right).$$

Define $z \in \Omega$ as follows:

$$z_{ij} = \frac{1}{2^{kl}} \qquad \left(\begin{array}{c} i_k \leq i < i_{k+1}, \; j_l \leq j < j_{l+1} \\ k,l = 0,1,2,\ldots \end{array}\right).$$

Clearly, $z \in \mathcal{C}_0$ and, fixing m,n

$$\left| \frac{z_{ij}^{(mn)}}{z_{ij}} \right| < \frac{1}{2^{kl}} \qquad \left(\begin{array}{c} i \geq i_k, \; j \geq j_l \\ k \geq m, \; l \geq n \end{array}\right).$$

Thus, $\lim_{i,j\to\infty} \frac{z_{ij}^{(mn)}}{z_{ij}} = 0 \;\; \forall m,n$.

Now let $x \in z^{\alpha}$, then

$$\sum_{i,j=1}^{\infty,\infty} |x_{ij} z_{ij}| < \infty \; .$$

Moreover, for $m,n = 1,2,\ldots$ we have

$$\begin{aligned}\sum_{i,j=1}^{\infty,\infty} |x_{ij} z_{ij}^{(mn)}| &< \sum_{i,j=1}^{\infty,\infty} |x_{ij} z_{ij}| \frac{1}{2^{kl}} \\ &< \frac{1}{2^{kl}} \sum_{i,j=1}^{\infty,\infty} |x_{ij} z_{ij}| < \infty.\end{aligned}$$

Hence, $x \in \bigcap_{m,n=1}^{\infty,\infty} \{z^{(mn)}\}^{\alpha}$. The proof is completed. □

$s = (s_m)$, $t = (t_n)$ always denote strictly increasing of nonnegative integers with $s_1 = 0$, $t_1 = 0$. We will be interested in spaces of the form:

$$m|(s,t)| = \left\{ x \in \Omega \ : \ \sup_{m,n} \sum_{\substack{k=s_m+1 \\ l=t_n+1}}^{s_{m+1},t_{n+1}} |x_{kl}| < \infty \right\}$$

which becomes a BDK-space under the norm:

$$x \to \sup_{m,n} \sum_{\substack{k=s_m+1 \\ l=t_n+1}}^{s_{m+1},t_{n+1}} |x_{kl}|.$$

Theorem 2.1. *Let (E,τ) be an FDK-space. These are equivalent:*
i) E is a double wedge space,
ii) E contains z^{α} for some $z \in \mathcal{C}_0$,
iii) E contains $m|(s,t)|$ for some s,t and the inclusion mapping is compact,
iv) E contains $\mathcal{L}_u$ and the inclusion mapping is compact.

Proof. $(i \Rightarrow ii)$ Let $\{p_{mn}\}$ be a defining family of seminorms for the topology τ and let

$$z_{ij}^{(mn)} = p_{mn}(\delta^{ij}) \qquad (m,n,i,j = 1,2,\ldots).$$

Then $z^{(mn)} \in \mathcal{C}_0$, $m,n = 1,2,\ldots$, since E is a double wedge space. Suppose $y \in \bigcap_{m,n=1}^{\infty,\infty} \left\{z^{(mn)}\right\}^{\alpha}$, then for each m,n

$$\sum_{i,j} |y_{ij} z_{ij}^{(mn)}| < \infty \ .$$

Therefore,

$$\sum_{i,j} |y_{ij} p_{mn}(\delta^{ij})| = \sum_{i,j} p_{mn}(y_{ij}\delta^{ij}) < \infty$$

is obtained. Since the space E is complete $\sum_{i,j} y_{ij}\delta^{ij}$ converges in (E,τ) to, say x, or $y^{(mn)} \to x$. Thus $y_{ij}^{(mn)} \to x_{ij}$ for each i,j; also we always have $y_{ij}^{(mn)} \to y_{ij}$ for each i,j. Consequently $y = x$; that is

$$\bigcap_{m,n=1}^{\infty,\infty} \left\{z^{(mn)}\right\}^{\alpha} \subseteq E \ .$$

Choosing z as in Lemma 2.1, (ii) follows.

$(ii \Rightarrow iii)$ Let us choose strictly increasing sequences (s_m), (t_n) of positive integers such that $s_1 = 0$, $t_1 = 0$ and

$$|z_{ij}| \leq \frac{1}{2^{mn}}$$

whenever $i \geq s_m$, $j \geq t_n$, $m, n \geq 2$.

For $x \in m|(s,t)|$ and any positive integers k, l, u, v such that $l \geq k$ and $v \geq u$ we have

$$\sum_{\substack{i=s_k+1\\ j=t_p+1}}^{s_{l+1},t_{v+1}} |x_{ij}z_{ij}| = \sum_{m=k,n=u}^{l,v} \sum_{\substack{i=s_m+1\\ j=t_n+1}}^{s_{m+1},t_{n+1}} |x_{ij}z_{ij}| \leq \|x\| \sum_{m=k,n=u}^{l,v} \frac{1}{2^{mn}}.$$

Hence $x \in z^{\alpha}$. That is; $m|(s,t)| \subseteq z^{\alpha} \subseteq E$. Also, the inclusion theorem $i : (m|(s,t)|, \|.\|) \to (E,\tau)$ is compact.

$(iii \Rightarrow iv)$ Since $\mathcal{L}_u \subset m|(s,t)|$ always true, the inclusion theorem $i : \mathcal{L}_u \to m|(s,t)|$ is continuous. By hypothesis $\mathcal{L}_u \subset m|(s,t)| \subset E$ and because of $i : (m|(s,t)|, \|.\|) \to (E,\tau)$ is compact $i* : \mathcal{L}_u \to E$ is compact. □

Corollary 2.1. *The intersection of all double wedge FDK-spaces is $\mathcal{L}_u$.*

Proof. Let E be a double wedge FDK-space. Then we have

$$\bigcap E = \bigcap \{z^{\alpha} \ : \ z \in \mathcal{C}_0\} = \mathcal{C}_0^{\alpha}.$$

Now we need show that $\mathcal{C}_0^{\alpha} = \mathcal{L}_u$. Since $\mathcal{C}_0 \subset \mathcal{M}_u$, $\mathcal{M}_u^{\alpha} \subset \mathcal{C}_0^{\alpha}$. That is, $\mathcal{L}_u = \mathcal{M}_u^{\alpha} \subset \mathcal{C}_0^{\alpha}$ is obtained. Suppose that $x = (x_{kl}) \in \mathcal{C}_0^{\alpha}$ but $x = (x_{kl}) \notin \mathcal{L}_u$. Then for $y = (y_{kl}) = \frac{1}{k!l!} \in \mathcal{C}_0$ we have

$$\sum |x_{kl}y_{kl}| = \sum |x_{kl}\frac{1}{k!l!}| = e^2 \sum |x_{kl}| = \infty.$$

This means that $x = (x_{kl}) \notin \mathcal{C}_0^{\alpha}$ which contradicts the hypothesis. So, $x = (x_{kl})$ must be in $\mathcal{L}_u$. That is, $\mathcal{L}_u \subset \mathcal{C}_0^{\alpha}$. This completes the proof. □

The one to one mapping $S^{(2)}$ of Ω to itself defined by

$$S^{(2)}x = \begin{pmatrix} x_{11} & x_{11}+x_{12} & \cdots & \cdots & \cdots \\ x_{11}+x_{21} & x_{11}+x_{12}+x_{21}+x_{22} & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \sum\limits_{k,l=1}^{m,n} x_{kl} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

$$(S^{(2)})^{-1}x = \begin{pmatrix} x_{11} & x_{12}-x_{11} & \cdots & \cdots & \cdots \\ x_{21}-x_{11} & x_{22}-x_{12}-x_{21}+x_{11} & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & x_{mn}-x_{m,n-1}-x_{m-1,n}+x_{m-1,n-1} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

Theorem 2.2. *i) (E,τ) is strongly conull FDK-space if and only if the space $(S^{(2)})^{-1}(E)$ is a double wedge FDK-space.*

ii) (E,τ) is conull FDK-space if and only if the space $(S^{(2)})^{-1}(E)$ is a weak double wedge FDK-space.

Proof. i) Let (E,τ) is strongly conull FDK-space and $\{P_{mn}\}$ is a set of seminorms on the topology τ. Then a topology with the set of seminorms $\{q_{mn}\}$ make $(S^{(2)})^{-1}(E)$ is a FDK-space such that

$$q_{mn}(x) := P_{mn}(S^{(2)}(x)).$$

Since (E,τ) is strongly conull FDK-space, for all $m,n \in \mathbb{N}$ $P_{mn}\left(e - e^{(mn)}\right) \to 0$. Otherwise,

$$S^{(2)}x = e - e^{(mn)} = \sum_{\substack{k=m+1\\l=n+1}}^{\infty,\infty} \delta^{kl} + \sum_{\substack{k=1\\l=n+1}}^{m,\infty} \delta^{kl} + \sum_{\substack{k=m+1\\l=1}}^{\infty,n} \delta^{kl}$$

is hold. Thus we have

$$\begin{aligned} x &= (S^{(2)})^{-1}\left(\sum_{\substack{k=m+1\\l=n+1}}^{\infty,\infty} \delta^{kl} + \sum_{\substack{k=1\\l=n+1}}^{m,\infty} \delta^{kl} + \sum_{\substack{k=m+1\\l=1}}^{\infty,n} \delta^{kl}\right) \\ &= (S^{(2)})^{-1}\begin{pmatrix} 0 & \dots & 0 & 1 & \dots & 1 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 1 & \dots & 1 \\ 1 & \dots & 1 & 1 & \dots & 1 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & \dots & \dots & \dots & \dots & 1 \end{pmatrix} \\ &= \delta^{m+1,1} + \delta^{1,n+1} - \delta^{m+1,n+1}. \end{aligned}$$

From the definition of (q_{mn}) the following equation is obtained:

$$q_{mn}(\delta^{m+1,1} + \delta^{1,n+1} - \delta^{m+1,n+1}) = P_{mn}\left(e - \sum_{k,l=1}^{m,n} \delta^{kl}\right).$$

Since $P_{mn}\left(e - \sum_{k,l=1}^{m,n} \delta^{kl}\right) \to 0$ $(m,n \to \infty)$, we can say $q_{mn}(\delta^{m+1,1} + \delta^{1,n+1} - \delta^{m+1,n+1}) \to 0$ $(m,n \to \infty)$.

In this case, we have $(\delta^{m+1,1} + \delta^{1,n+1} - \delta^{m+1,n+1}) \to 0$ $(m,n \to \infty)$ according to the topology of the space $(S^{(2)})^{-1}(E)$. Since $\delta^{m+1,1} \to 0$ and $\delta^{1,n+1} \to 0$, $\delta^{m+1,n+1} \to 0$ is hold. This gives that $(S^{(2)})^{-1}(E)$ is a double wedge FDK-space.

Now we suppose that $(S^{(2)})^{-1}(E)$ is a double wedge FDK-space. Then we have for all $m,n \in \mathbb{N}$ $q_{mn}(\delta^{m+1,1} + \delta^{1,n+1} - \delta^{m+1,n+1}) \to 0$ $(m,n \to \infty)$. Since

$$q_{mn}(\delta^{m+1,1} + \delta^{1,n+1} - \delta^{m+1,n+1}) = P_{mn}\left(e - \sum_{k,l=1}^{m,n} \delta^{kl}\right),$$

$P_{mn}\left(e - \sum_{k,l=1}^{m,n} \delta^{kl}\right) \to 0$ $(m,n \to \infty)$ is obtained. So E strongly conull FDK-space.

ii) Let (E,τ) is conull FDK-space and let us define the topology of $(S^{(2)})^{-1}(E)$ as the proof of (i). Then $q_{mn}(x) := P_{mn}(S^{(2)}(x))$ and since $(S^{(2)})^{-1}\left(e - \sum_{k,l=1}^{m,n} \delta^{kl}\right) = \delta^{m+1,1} + \delta^{1,n+1} - \delta^{m+1,n+1}$, we have

$$q_{mn}(\delta^{m+1,1} + \delta^{1,n+1} - \delta^{m+1,n+1}) = P_{mn}\left(e - \sum_{k,l=1}^{m,n} \delta^{kl}\right).$$

Because of E is conull FDK space, $P_{mn}\left(e-\sum_{k,l=1}^{m,n}\delta^{kl}\right)\to 0$ (weak) $(m,n\to\infty)$. Hence $\delta^{m+1,1}+\delta^{1,n+1}-\delta^{m+1,n+1}\to 0$ (weak) $(m,n\to\infty)$. Consequently $\delta^{m+1,n+1}\to 0$ (weak) $(m,n\to\infty)$ is obtained. That is, $(S^{(2)})^{-1}(E)$ is a weak double wedge FDK-space. The other hand of the proof is as the proof of (i). $\square$

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