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# $\Omega-$ ALGEBRAS ON CO-HEYTING VALUED SETS

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Abstract. As known, co-Heyting algebras are dual to Heyting algebras. co-Heyting algebra has many studying areas as topos theory, co-intuitionistic logic, linguistics, quantum theory, etc.

In this paper, we studied on co-Heyting valued sets. The co-Heyting valued $\Omega-$algebra, co-Heyting Valued Algebra Homomorphism are defined and some properties of these sets are examined.

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## 1. Introduction

co- Heyting algebra that is a lattice which dual is Heyting algebra. co-Heyting algebra has applications in many different areas.

**Definition 1.1.** [1]A Boolean algebra is an algebra $(H, \vee, \wedge, ^{-}, 0_H, 1_H)$ where $(H, \vee, \wedge, 0_H, 1_H)$ is a distributive lattice and for all $a \in H$,

$$a \wedge \overline{a} = 0 \text{ and } a \vee \overline{a} = 1$$

**Definition 1.2.** [1]A Heyting algebra is an algebra $(H, \vee, \wedge, \rightarrow, 0_H, 1_H)$ such that $(H, \vee, \wedge, 0_H, 1_H)$ is an lattice and for all $a, b, c \in H$,

$$a \leq b \rightarrow c \Leftrightarrow a \wedge b \leq c$$

$(H, \vee, \wedge, 0_H, 1_H)$ is a Heyting algebra with $\forall a, b \in H$,

$$a \rightarrow b = \bigvee \{c : a \wedge c \leq b, c \in H\}.$$

**Proposition 1.1.** [4]*An algebra* $(H, \vee, \wedge, \rightarrow, 0_H, 1_H)$ *is a Heyting algebra if and only if* $(H, \vee, \wedge, 0_H, 1_H)$ *is an lattice and the following identities hold for all* $a, b, c \in H$,

(1) $a \rightarrow a = 1$
(2) $a \wedge (a \rightarrow b) = a \wedge b$
(3) $b \wedge (a \rightarrow b) = b$
(4) $a \rightarrow (b \wedge c) = (a \rightarrow b) \wedge (a \rightarrow c)$

**Definition 1.3.** [1]A co-Heyting algebra is an algebra $(H^*, \vee, \wedge, \hookrightarrow, 0_{H^*}, 1_{H^*})$ such that $(H^*, \vee, \wedge, 0_{H^*}, 1_{H^*})$ is an lattice and for all $a, b, c \in H^*$,

$$a \hookrightarrow b \leq c \Leftrightarrow a \leq b \vee c$$

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$(H^*, \vee, \wedge, 0_{H^*}, 1_{H^*})$ is a co-Heyting algebra with $\forall a, b \in H^*$,

$$a \hookrightarrow b = \bigwedge \{c : a \vee c \geq b, c \in H^*\}.$$

A co-Heyting algebra with the ordering reversed will yield a Heyting algebra. The implication operation in this algebra will be $a \to b = b \hookrightarrow a$.

It is clear that $H$ and $H^*$ are same sets with different order relations. $1_{H^*}$ and $0_{H^*}$ are greatest and least elements of $H^*$,respectively.

**Proposition 1.2.** *An algebra* $(H^*, \vee, \wedge, \hookrightarrow, 0_{H^*}, 1_{H^*})$ *is a co-Heyting algebra if and only if* $(H^*, \vee, \wedge, 0_{H^*}, 1_{H^*})$ *is an lattice and the following identities hold for all* $a, b, c \in H^*$,

(1) $a \hookrightarrow a = 0$
(2) $a \vee (b \hookrightarrow a) = a \vee b$
(3) $b \vee (b \hookrightarrow a) = b$
(4) $(b \vee c) \hookrightarrow a = (b \hookrightarrow a) \vee (c \hookrightarrow a)$

*Proof.* (1) $\forall\ a \in H^*$,

$$a \hookrightarrow a = \bigwedge \{c : a \vee c \geq a, c \in H^*\} = 0$$

(2) From definition it is obtained that,

$$a \vee (b \hookrightarrow a) \geq b \ \Rightarrow a \vee (b \hookrightarrow a) \geq a \vee b$$

and

$$\begin{aligned} (a \vee b) \vee a \quad &\geq \quad b \Rightarrow a \vee b \geq b \hookrightarrow a \text{ and } a \leq a \vee b \\ &\Rightarrow \quad a \vee b \geq a \vee (b \hookrightarrow a) \end{aligned}$$

(3)$\forall a, b \in H^*$,

$$b \leq a \vee b \Rightarrow b \vee (b \hookrightarrow a) = b$$

(4)$\forall a, b, c \in H^*$,

$$\begin{aligned} (b \quad &\hookrightarrow \quad a) \vee (c \hookrightarrow a) \vee a = (a \vee (b \hookrightarrow a)) \vee (a \vee (c \hookrightarrow a)) \geq b \vee c \\ &\Rightarrow \quad (b \hookrightarrow a) \vee (c \hookrightarrow a) \geq (b \vee c) \hookrightarrow a \end{aligned}$$

On the other hand, $(b \vee c) \hookrightarrow a \geq (b \hookrightarrow a) \vee (c \hookrightarrow a)$. □

## 2. co-Heyting Valued Sets

In this section, the concepts of co-Heyting valued set, co-Heyting valued function are defined and some properties of these structures are examined.

**Definition 2.1.** Let $H^*$ be a complete co-Heyting algebra and $X$ be a universal. $H^*-$ valued set is determined with $[=]$ function

$$[=] : X \times X \to H^*, [=] (a, b) = [a = b]$$

which satisfy the following conditions.

(1) $[a = b] \geq [b = a]$
(2) $[a = b] \vee [b = c] \geq [a = c]$

Let $X$ be a universal. $u \in X, E(u)$ means the degree of existence the elemet $u$. For $H^*-$valued sets we will use,

$$E(u) = [u \in X].$$

So, $[u \in X] = [u = u]$.

**Definition 2.2.** Let $A$ be a $H^*-$valued set. The subset of $A$ is a $s : A \to H^*$ function with following conditions.

(1) $[x \in s] \vee [x = y] \geq [y \in s]$
(2) $[x \in s] \geq [x \in A]$

**Definition 2.3.** Let $(X, =)$ and $(Y, =)$ are $H^* - valued\ sets$. If $f : X \times Y \to H^*$ function satisfy the following conditions then called $H^*-$valued function and it is shown $f : X \to Y$.

F1 $f(x, y) \geq [x = x] \vee [y = y]$
F2 $[x = x'] \vee f(x, y) \vee [y = y'] \geq f(x', y')$
F3 $f(x, y) \vee f(x, y') \geq [y = y']$
F4 $[x = x] \geq \bigwedge \{f(x, y) : y \in Y\}$

**Notation 1:** $f(x, y) := [f(x) = y]$

**Definition 2.4.** Let $(X, =)$ be an $H^*-$valued set. $I : X \times X \to H^*$ ,$I(x, x') = [x = x']$ function is called unit function.

**Definition 2.5.** Let $(X, =), (Y, =)$ and $(Z, =)$ are $H^*-$valued sets and $f : X \to Y$, $g : Y \to Z$ are $H^*-$valued functions. For $x \in X, z \in Z$,

$$(g \circ f)(x, z) = \bigwedge \{f(x, y) \vee g(y, z) : y \in Y\}.$$

**Proposition 2.1.** *Let $(X, =), (Y, =)$ and $(Z, =)$ are $H^*-$valued sets and $f : X \to Y$, $g : Y \to Z$ are $H^*-$valued functions. The function $(g \circ f) : X \to Z$ is a $H^*-$valued function.*

*Proof.* (i)Let $x \in X, z \in Z$,

$$\begin{aligned} (g \circ f)(x, z) &= \bigwedge \{f(x, y) \vee g(y, z) : y \in Y\} \\ &\geq \bigwedge \{[x = x] \vee [y = y] \vee [z = z] : y \in Y\} \\ &= [x = x] \vee [z = z] \vee \bigwedge \{[y = y] : y \in Y\} \\ &\geq [x = x] \vee [z = z] \end{aligned}$$

(ii)Let $x, x' \in X$ and $z, z' \in Z$,

$$\begin{aligned} [x = x'] \vee (g \circ f)(x, z) \vee [z = z'] &= [x = x'] \vee \bigwedge \{f(x, y) \vee g(y, z) : y \in Y\} \vee [z = z'] \\ &= \bigwedge \{[x = x'] \vee f(x, y) \vee g(y, z) \vee [z = z'] : y \in Y\} \\ &= \bigwedge \left\{ \begin{array}{c} [x = x'] \vee f(x, y) \vee [y = y] \\ \vee g(y, z) \vee [z = z'] : y \in Y \end{array} \right\} \\ &\geq \bigwedge \{f(x', y') \vee g(y', z') : y \in Y\} = f(x', y') \vee g(y', z') \\ &\geq \bigwedge \{f(x', y') \vee g(y', z') : y' \in Y\} = (g \circ f)(x', z') \end{aligned}$$

(iii)Let $x \in X$ and $z, z' \in Z$,

$$\begin{aligned} (g \circ f)(x, z) \vee (g \circ f)(x, z') &= \bigwedge \{f(x, y) \vee g(y, z) : y \in Y\} \vee \bigwedge \{f(x, t) \vee g(t, z') : t \in Y\} \\ &= \bigwedge \{f(x, y) \vee f(x, y) \vee g(y, z) \vee g(y, z') : y \in Y\} \\ &\geq \bigwedge \{[y = y] \vee [z = z'] : y \in Y\} \\ &\geq [z = z'] \end{aligned}$$

(iv)Let $x \in X$,

$$\begin{aligned}
\bigwedge \{(g \circ f)(x,z) : z \in Z\} &= \bigwedge \left\{ \bigwedge \{f(x,y) \vee g(y,z) : y \in Y\} : z \in Z \right\} \\
&= \bigwedge \left\{ \bigwedge \{f(x,y) : y \in Y\} : z \in Z \right\} \vee \\
&\quad \bigwedge \left\{ \bigwedge \{g(y,z) : y \in Y\} : z \in Z \right\} \\
&\leq [x = x] \vee \bigwedge \{[y = y] : y \in Y\} = [x = x]
\end{aligned}$$

□

**Definition 2.6.** Let $(X, =)$ and $(Y, =)$ are $H^*-$valued sets and $f : X \to Y$ is $H^*-$valued function.

(1) $f$ is a monomorphism. $\Leftrightarrow \forall x, x' \in X, y \in Y$,

$$f(x,y) \vee f(x',y) \geq [x = x']$$

(2) $f$ is a epimorphism. $\Leftrightarrow \forall y \in Y$,

$$[y = y'] \geq \bigwedge \{f(x,y) : x \in X\}$$

**Definition 2.7.** Let $(X, =)$ be a $H^*-$valued set. $R : X \times X \to H^*$ is called $H^*-$valued equivalence relation.$\Leftrightarrow$

R1 $R(x,y) \vee [x = x] = R(x,y), R(x,y) \vee [y = y] = R(x,y)$
R2 $R(x,y) \vee [x = x'] \geq R(x',y), R(x,y) \vee [y = y'] \geq R(x,y')$
R3 $[x = x] \geq R(x,x)$ :
R4 $R(x,y) \geq R(y,x)$
R5 $R(x,y) \vee R(y,z) \geq R(x,z)$

**Example 2.1.** Let $(X, =), (Y, =)$ are $H^*-$valued sets and $f : X \to Y$ is $H^*-$valued function.$\forall x_1, x_2 \in X$,

$$C_f(x_1, x_2) = [f(x_1) = f(x_2)]$$

function is a $H^*-$valued equivalence relation on $X$.

**Definition 2.8.** Let $R$ be a $H^*-$valued equivalence relation on $X$. $d : X \to H^*$ is called equivalence class of $R \Leftrightarrow$

d1 $d(x) \vee R(x,x') \geq d(x')$
d2 $d(x) \vee d(y) \geq R(x,y)$

$d(x)$ is the equivalence class of $x \in X$.

**Proposition 2.2.** *Let $R$ be a $H^*-$valued equivalence relation on $X$ and $d_1, d_2$ are equivalence class of $R$.*

$$\begin{aligned}
\bigwedge \{d_1(x) : x \in X\} &= \bigwedge \{d_2(x) : x \in X\} \text{ and} \\
d_1(x) &\geq d_2(x) \Rightarrow d_1 = d_2
\end{aligned}$$

*Proof.* $x_0 \in X$,

$$\begin{aligned}
d_2(x_0) &= \bigwedge \{d_2(x) \vee d_1(x) : x \in X\} \\
&\geq \bigwedge \{R(x_0, x) \vee d_1(x) : x \in X\} \\
&\geq d_1(x_0)
\end{aligned}$$

□

**Proposition 2.3.** *Let* $(X,=),(Y,=)$ *are* $H^*-$ *valued sets and* $f : X \to Y$ *is* $H^*-$*valued function.* $f$ *is surjective* $\Leftrightarrow \forall y \in Y$,

$$\bigwedge \{[f(x)=y] : x \in X\} = [y=y]$$

## 3. $H^*$-Valued $\Omega-$Algebras

Now, let $\Omega$ is defined as follows,

$$\Omega = \{\omega : X^n \times X \to H^* : \omega \text{ satisfy F1-F4 conditions}\}$$

It means that, if $\omega \in \Omega$, $\omega$ is $H^*-$valued function. The concept of $H^*-$valued $\Omega-$algebra can be defined as following;

**Definition 3.1.** $A = \langle X, \Omega \rangle$ is $H^*-$valued $\Omega-$algebra. $\Leftrightarrow$ For $\omega \in \Omega$ and $((x_1,x_2,...,x_n),c) \in X^n \times X$,

$$\bigwedge \left\{ \left\{ \begin{array}{c} \bigvee [x_i \in A] \vee \omega((x_1,x_2,...,x_n),d) \\ \vee [c=d] : i=1,2,...,n \end{array} \right\} : d \in X \right\} \geq \omega((x_1,x_2,...,x_n),c)$$

**Example 3.1.** Let $A = \langle X, \Omega \rangle$ be a $H^*-$valued $\Omega-$algebra.

$$\{\Theta\} : A \to H^*, [x \in \{\Theta\}] = 1_{H^*}$$

is a subset of $A$.

$$E = \langle \{\Theta\}, \Omega \rangle$$

is a $H^*-$valued $\Omega-$algebra. $E$ is called trivial $H^*-$valued $\Omega-$algebra.

**Definition 3.2.** Let $A = \langle X, \Omega \rangle$ be a $H^*-$valued $\Omega-$algebra. If $K \subseteq X, B : K \to H^*$ is $H^*-$valued set, $(B,=) \subseteq (A,=)$ and for all $\omega \in \Omega, \omega \downarrow_B$satisfy the (1) then $B$ is $H^*-$valued $\Omega-$subalgebra of $A$.

**Example 3.2.** Let $A = \langle X, \Omega \rangle$ be a $H^*-$valued $\Omega-$algebra.$E = \langle \{\Theta\}, \Omega \rangle$ is $H^*-$valued $\Omega-$ subalgebra.

**Definition 3.3.** Let $A = \langle X, \Omega \rangle$ and $B = \langle Y, \Omega \rangle$ are similar $H^*-$valued $\Omega-$ algebras and $f : A \to B$ be $H^*-$valued function. $f$ is a $H^*-$valued $\Omega-$ algebra homomorphism$\Leftrightarrow$

- H1 $[x=x] = [f(x)=f(x)]$
- H2 $[x=x'] \leq [f(x)=f(x')]$
- H3 $f(\omega(x_1,x_2,...,x_n),y) = \bigvee \{f(x_i,y_i) : y = \omega(y_1,y_2,...,y_n), f(x_i,y_i) > 0\}$

**Example 3.3.** Let $A = \langle X, \Omega \rangle$ be a $H^*-$valued $\Omega-$algebra and $f : E \to A$, $g : E \to A$ are $H^*-$valued functions. $\forall x, I(\{\Theta\}, x) = 1_{H^*}$ $H^*-$valued $\Omega-$ algebra homomorphism exist. This homomorphism is unique.

**Proposition 3.1.** *Let* $A, B, C$ *are similar* $H^*-$*valued* $\Omega-$ *algebras. If* $f : A \to B$, $g : B \to C$ *are* $H^*-$*valued* $\Omega-$ *algebra homomorphisms then* $(g \circ f) : A \to C$ *is a* $H^*-$*valued* $\Omega-$ *algebra homomorphism.*

*Proof.* Let $x_1, x_2, ..., x_n \in A, z \in C,$

$$(g \circ f)(\omega(x_1, x_2, ..., x_n), z) = \bigwedge \{f(\omega(x_1, x_2, ..., x_n), y) \vee g(y, z) : y \in B\}$$

$$= \bigwedge \left\{ \begin{array}{c} \bigvee \{f(x_i, y_i) : y = \omega(y_1, y_2, ..., y_n), f(x_i, y_i) > 0\} \\ \vee g(y, z) : y \in B \end{array} \right\}$$

$$= \bigwedge \left\{ \begin{array}{c} \bigvee \{f(x_i, y_i) : i = 1, ..., n\} \vee \\ \{g(\omega(y_1, y_2, ..., y_n), z) : y = \omega(y_1, y_2, ..., y_n), f(x_i, y_i) > 0\} \end{array} \right\}$$

$$= \bigwedge \left\{ \begin{array}{c} \bigvee \{f(x_i, y_i) : i = 1, ..., n\} \vee \\ \left\{ \begin{array}{c} \bigvee g(y_i, z_i) : z = \omega(z_1, z_2, ..., z_n), g(y_i, z_i) > 0 : \\ y = \omega(y_1, y_2, ..., y_n), f(x_i, y_i) > 0 \end{array} \right\} \end{array} \right\}$$

$$= \bigwedge \left\{ \bigvee \left\{ \begin{array}{c} f(x_i, y_i) \vee g(y_i, z_i) : y = \omega(y_1, y_2, ..., y_n), f(x_i, y_i) > 0, \\ z = \omega(z_1, z_2, ..., z_n), g(y_i, z_i) > 0 : y_i \in B \end{array} \right\} \right\}$$

$$= \bigvee \left\{ \bigwedge \left\{ \begin{array}{c} f(x_i, y_i) \vee g(y_i, z_i) : y = \omega(y_1, y_2, ..., y_n), f(x_i, y_i) > 0, \\ z = \omega(z_1, z_2, ..., z_n), g(y_i, z_i) > 0 : y_i \in B \end{array} \right\} \right\}$$

$$= \bigvee \left\{ \left\{ \begin{array}{c} (g \circ f)(x_i, z_i) : y = \omega(y_1, y_2, ..., y_n), f(x_i, y_i) > 0, \\ z = \omega(z_1, z_2, ..., z_n), g(y_i, z_i) > 0 \end{array} \right\} : y_i \in B \right\}$$

$$= \bigvee \{(g \circ f)(x_i, z_i) : z = \omega(z_1, z_2, ..., z_n), (g \circ f)(x_i, z_i) > 0, i = 1, ..., n\}$$

□

## 4. Conclusion

In this paper, we introduced the $H^*-$valued $\Omega-$algebra and $H^*-$valued $\Omega-$algebra homomorphism. To study these concepts, firstly we defined $H^*-$ valued set and $H^*-$ valued function. In continuation of this study, injection and projection mappings on $H^*-$valued $\Omega-$algebra can be defined, isomorphism theorems can be proved.

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