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ON A BOUNDARY VALUE PROBLEM WITH RETARDED ARGUMENT IN THE DIFFERENTIAL EQUATION

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Abstract. In this work a boundary value problem on the half axis is studied. The special solution of this boundary value problem is defined. The simplicity of the eigenvalues is shown and it is proven that all positive values of the parameter λ are the eigenvalues of this boundary value problem.

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1. Introduction

In this work we study the boundary value problem

$$y''(x) + \lambda y(x) + M(x)y(x - \Delta(x)) = 0 \quad (0 \le x < \infty), \tag{1}$$

$$y'(0) - hy(0) = 0, \tag{2}$$

$$y(x - \Delta(x)) \equiv \phi(x - \Delta(x)), \ \texttt{if} \ x - \Delta(x) < 0, \tag{3}$$

$$\sup_{[0,\infty)} |y(x)| < \infty, \tag{4}$$

where $M(x)$ and $\Delta(x) \ge 0$ are defined and continuous on the half axis $[0, \infty)$, λ is a real parameter $(-\infty < \lambda < \infty)$, h is an arbitrary real number and $\phi(x)$ is a continuous initial function on the initial set

$$E_0 = \{x - \Delta(x) : x - \Delta(x) < 0, \ x > 0\} \cup \{0\}$$

with $\phi(0) = 1$.

The literature for the boundary value problems for differential equations of the second order with retarded arguments begins with [1, 2, 3, 4, 5, 6, 7, 8, 9].

Differential equations with retarded argument, describe processes with aftereffect; they find many applications, particularly in the theory of automatic control, in the theory of self-oscillatory systems, in the study of problems connected with combustion in rocket engines, in a number of problems in economics, biophysics, and many other fields. Equations with retarded argument appear, for example each time when in some physical or technological problem, the force operating at the mass point depends on the velocity and the position of this point, not only at the given instant, but also at some given previous instant.

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The presence of retardations in the system studied often proves to be a result of a phenomenon which essentially influences the course of the process. For example, in automatic control systems the retardation is the time interval which the system requires to react to an input impulse. Various physical applications of such problems can be found in [8].

The rest of this paper is organized as follows. First, the equivalent integral representation for the solution of the boundary value problem (1)-(4) is constructed. Then, the simplicity of the eigenvalues is shown. It is proven that all positive values of the parameter λ are the eigenvalues of the boundary value problem (1)-(4).

2. The special solution

Let $w(x,\lambda)$ be a solution of (1) which satisfies the conditions

$$w(0,\lambda)=1,\ w'(0,\lambda)=h, \tag{5}$$

$$w\left(x-\Delta(x),\lambda\right)\equiv\phi(x-\Delta(x)),\ \texttt{if}\ x-\Delta(x)<0. \tag{6}$$

From Theorem I.2.1 (see [8]) it follows that under conditions (5), (6) there exists a unique solution of (1) on the half axis $[0,\infty)$.

Lemma 2.1. *Equation (1) together with the initial conditions (5), (6) are equivalent, for each value of $\lambda>0$ to the integral equation:*

$$w(x,\lambda)=\cos sx+\frac{h\sin sx}{s}-\frac{1}{s}\int_0^x M(t)\sin s(x-t)w(t-\Delta(t),\lambda)dt,\ (s^2=\lambda). \tag{7}$$

Proof. If we seek the solution of equation

$$w''(x,\lambda)+\lambda w(x,\lambda)=-q(x)w(x-\Delta(x),\lambda)$$

as

$$w(x,\lambda)=c_1 cossx+c_2 sinsx,$$

by applying the method of variation of parameters we have

$$w(x,\lambda)=\tilde{c_1}cossx+\tilde{c_2}sinsx-\frac{1}{s}\int_0^x q(t)sins(x-t)w(t-\Delta(t),\lambda)dt.$$

Taking condition (5) into consideration we find

$$\tilde{c_1}=1\quad\texttt{and}\quad\tilde{c_2}=\frac{h}{s}.$$

□

Before giving the following theorem, it will be useful to keep in mind that; here, the multiplicity of an eigenvalue of a boundary value problem is defined to be the number of linearly independent eigenfunctions corresponding to this eigenvalue.

Theorem 2.1. *The boundary value problem (1)-(4) can have only simple eigenvalues.*

Proof. Let $\tilde{\lambda}$ be an eigenvalue of the boundary value problem (1)-(4) and $\tilde{\varphi}(x,\tilde{\lambda})$ a corresponding eigenfunction. By (2) and (5),

$$W\left\{\tilde{\varphi}\left(0,\tilde{\lambda}\right),w\left(0,\tilde{\lambda}\right)\right\}=\begin{vmatrix}\tilde{\varphi}\left(0,\tilde{\lambda}\right) & 1\\ \tilde{\varphi}'\left(0,\tilde{\lambda}\right) & h\end{vmatrix}=0,$$

and according to Theorem II.2.2. in [8] the functions $\tilde{\varphi}(x, \tilde{\lambda})$ and $w(x, \tilde{\lambda})$ are linearly dependent on $[0, \infty)$. Hence it follows that $w(x, \tilde{\lambda})$ is an eigenfunction for the boundary value problem (1)-(4) and all eigenfunctions of this boundary value problem which correspond to the eigenvalue $\tilde{\lambda}$ are pairwise linearly dependent. □

3. Existence Theorem

Theorem 3.1. *Let*

$$\sup_{t \in E_0} |\phi(x)| = \phi_0 < \infty \tag{8}$$

and in equation (1) let

$$\int_0^\infty |M(t)| \, dt = M_\infty < \infty. \tag{9}$$

Then all positive values of the parameter λ are eigenvalues of the boundary value problem (1)-(4).

Proof. By (7), if $\lambda > 0$, then

$$w(x, \lambda) = R_\lambda \sin(st - \psi_\lambda) - \frac{1}{s} \int_0^x M(t) \sin s(x - t) w(t - \Delta(t), \lambda) dt, \tag{10}$$

where

$$R_\lambda = \sqrt{1 + \frac{h^2}{\lambda}}, \ \cos\psi_\lambda = \frac{1}{R_\lambda}, \ \sin\psi_\lambda = \frac{h}{sR_\lambda}, \ (0 \le \psi_\lambda < 2\pi).$$

Let $x_0 \in (0, \infty)$, and $N_\lambda(x_0) = \max\limits_{[0,x_0]} |w(x, \lambda)|$. Evidently, $N_\lambda(x_0) \ge N_\lambda(x)$ $(x_0 \ge x)$ and, from (10), (5), (6), (8) and (9) one of the following inequalities holds:

$$N_\lambda(x_0) \le R_\lambda + \frac{1}{s} \int_0^{x_0} |M(t)| \, N_\lambda(t) dt \tag{11}$$

or

$$N_\lambda(x_0) \le R_\lambda + \frac{\phi_0}{s} \int_0^{x_0} |M(t)| \, dt \le R_\lambda + \frac{\phi_0 M_\infty}{s}.$$

By Lemma II.3.5 in [8] it follows from (11) that

$$N_\lambda(x_0) \le R_\lambda exp \frac{1}{s} \int_0^{x_0} |M(t)| \, dt \le R_\lambda exp \frac{M_\infty}{s},$$

and for $s > 0$

$$N_\lambda(x_0) \le \max \left\{ R_\lambda exp \frac{M_\infty}{s}; R_\lambda + \frac{\phi_0 M_\infty}{s} \right\} < \infty.$$

The bound obtained is valid for any $\lambda > 0$ and is independent of x_0, proving the theorem. □

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