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ON HERMITE-HADAMARD INEQUALITIES FOR GEOMETRIC-ARITHMETICALLY φ-s-CONVEX FUNCTIONS VIA FRACTIONAL NTEGRALS

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Abstract. In this paper, we established integral inequalities of Hermite-Hadamard type involving Riemann-Liouville fractional integrals motivated by the definition of geometric-arithmetically s-convex and v-convex functions. We give equality for fractional integral of φ-convex functions on twice differentiable mapping.

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1. Introduction

Fractional calculus was born in 1695. In the past three hundred years, fractional calculus developed in diverse fields from physical sciences and engineering to biological sciences and economics$[1-8]$. Fractional Hermite-Hadamard inequalities involving all kinds of fractional integrals have attracted by many researches. In $[12]$, Shuang et al. introduced a new concept of geometric-arithmetically s-convex functions and presented interesting Hermite-Hadamard type inequalities for integer integrals of such functions. In $[15]$,Youness introduced a new concept of φ -convex functions. In $[16-17]$, YuMei Liao and colleagues gave Riemann-Lioville Hermite-Hadamard integral inequalities for once and twice differentiable geometric-arithmetically $s-$convex functions. We establish on Hermite-Hadamard inequalities for twice differentiable geometric-arithmetically $\varphi-s-$convex functions via fractional integrals.

2. Preliminaries

In this section, we will give some definitions, lemmas and notations which we use later in this work.

Definition 2.1. $(see\,[3])$ Let $f\in L\,[a,b]$. The Riemann- Liouville fractional integral $\mathrm{J}^{\alpha}_{a^+}f$ and $\mathrm{J}^{\alpha}_{b^-}f$ of order $\alpha>0$ with $\alpha\geq 0$ are defined by

$$J^{\alpha}_{a^+}f\left(x\right)=\tfrac{1}{\Gamma(\alpha)}\int_a^x(x-t)^{\alpha-1}f\left(t\right)dt,0\leq a<x\leq b$$

and

$$J^{\alpha}_{b^-}f\left(x\right)=\tfrac{1}{\Gamma(\alpha)}\int_x^b(t-x)^{\alpha-1}f\left(t\right)dt,0\leq a<x\leq b \tag{2.1}$$

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Here Γ is the gamma function.

Definition 2.2. (*see* [12 − 16]) Let $f : I \subseteq \mathbb{R}^+ \to \mathbb{R}^+$and s$\in (0,1]$. A function $f(x)$ is said to be geometric-arithmetically s-convex on I if for every $x, y \in I$ and t$\in [0,1]$, we have:

$$f\left(x^t y^{1-t}\right) \leq t^s f(x) + (1-t)^s f(y) \tag{2.2}$$

Definition 2.3. (*see* [14]) The incomplete beta function is defined as follows:

$$B_x(a,b) = \int_0^x t^{a-1} (1-t)^{b-1} dt. \tag{2.3}$$

Where $x \in [0,1], a, b > 0$.

Definition 2.4. (*see* [15]) Let $\varphi : [a,b] \subset \mathbb{R} \to [a,b]$ A function $f : [a,b] \to \mathbb{R}$ is said to be $\varphi - convex$ on $[a,b]$ if, for every $x, y \in [a,b]$ and $t \in [0,1]$ the following inequality holds:

$$f(t\varphi(x) + (1-t)\varphi(y)) \leq tf(\varphi(x)) + (1-t) f(\varphi(y)) \tag{2.4}$$

Definition 2.5. Let $\varphi : [a,b] \subset \mathbb{R} \to [a,b]$ and $s \in (0,1]$. A function $f : [a,b] \to \mathbb{R}$ is said to be $\varphi - s - convex$ on $[a,b]$ if, for every $x, y \in [a,b]$ and $t \in [0,1]$ the following inequality holds:

$$f\left(\varphi(x)^t \varphi(y)^{(1-t)}\right) \leq t^s f(\varphi(x)) + (1-t)^s f(\varphi(y)) \tag{2.5}$$

Lemma 2.1. (*see* [11]) *For t$\in [0,1]$, we have*

$$\begin{aligned} (1-t)^n &\leq 2^{1-n} - t^n \quad for \quad n \in [0,1], \\ (1-t)^n &\geq 2^{1-n} - t^n \quad for \quad n \in [0,\infty). \end{aligned} \tag{2.6}$$

The following inequality was used in the proof directly in[12].

Lemma 2.2. (*see* [13]) *for* $t \in [0,1]$ *and* $x, y > 0$,*we have*

$$tx + (1-t)y \geq y^{1-t} x^t. \tag{2.7}$$

Lemma 2.3. (*see* [10]) *Let* $f : [a,b] \to \mathbb{R}$ *be a twice differentiable mapping on* (a,b) *with* $a < b$. *If* $f'' \in L[a,b]$, *then the following equality for fractional integrals holds:*

$$\begin{aligned} &\tfrac{f(a)+f(b)}{2} - \tfrac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)\right] \\ &= \tfrac{(b-a)^2}{2} \int_0^1 \tfrac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{\alpha+1} f''(ta+(1-t)b)\, dt. \end{aligned} \tag{2.8}$$

Lemma 2.4. (*see* [9]) *Let* $f : [a,b] \to \mathbb{R}$ *be a twice differentiable mapping on* (a,b) *with* $a < b$. *If* $f'' \in L[a,b]$, *then*

$$\begin{aligned} &\tfrac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)\right] - f\left(\tfrac{a+b}{2}\right) \\ &= \tfrac{(b-a)^2}{2} \int_0^1 m(t) f''(ta+(1-t)b)\, dt. \end{aligned} \tag{2.9}$$

where

$$m(t) = \begin{cases} t - \frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{\alpha+1}, & t \in \left[0, \frac{1}{2}\right), \\ 1 - t - \frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{\alpha+1}, & t \in \left[\frac{1}{2}, 1\right). \end{cases}$$

Lemma 2.5. (*see* [9]) *Let* $f : [a,b] \to \mathbb{R}$ *be a differentiable mapping on* (a,b) *with* $a < b$. *If* $f'' \in L[a,b]$, $r > 0$, *then*

$$\begin{aligned} &\tfrac{f(a)+f(b)}{r(r+1)} + \tfrac{2}{r+1} f\left(\tfrac{a+b}{2}\right) - \tfrac{\Gamma(\alpha+1)}{r(b-a)^\alpha}\left[J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)\right] \\ &= (b-a)^2 \int_0^1 k(t) f''(ta+(1-t)b)\, dt. \end{aligned} \tag{2.10}$$

Where

$$k(t)=\begin{cases}\frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{\alpha+1}-\frac{t}{r+1}, & t\in\left[0,\frac{1}{2}\right),\\ \frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{\alpha+1}-\frac{1-t}{r+1}, & t\in\left[\frac{1}{2},1\right).\end{cases}$$

YuMei Liao and colleagues based on our study, they have provided the following theorems in [16],[17].

Theorem 2.1. *Let $f:[0,b]\to R$ be a differentiable mapping. If $|f'|$ is measurable and $|f'|$ is decreasing and geometric-arithmetically s-convex on $[0,b]$ for some fixed $\alpha\in(0,\infty)$, $s\in(0,1]$, $0\le a<b$, then the following inequality for fractional integrals holds:*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a^+}^{\alpha}f(b)+J_{b^-}^{\alpha}f(a)\right]\right|\\ &\le\frac{(b-a)\left(2^{\alpha+s}\left|f'(b)\right|-\left|f'(a)\right|-\left|f'(b)\right|\right)}{2^{\alpha+s+1}(\alpha+s+1)}\\ &+(b-a)\left|f'(a)\right|\left[0.5B(s+1,\alpha+1)-B_{0.5}(\alpha+1,s+1)\right]\\ &+(b-a)\left|f'(b)\right|\left[B_{0.5}(s+1,\alpha+1)-0.5B(a+1,\alpha+1)\right].\end{aligned}\tag{2.11}$$

Theorem 2.2. *Let $f:[0,b]\to R$ be a differentiable mapping. If $|f''|$ is measurable and $|f''|$ is decreasing and geometric-arithmetically s-convex on $[0,b]$ for some fixed $\alpha\in(0,\infty)$, $s\in(0,1]$, $0\le a<b$, then the following inequality for fractional integrals holds:*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a^+}^{\alpha}f(b)+J_{b^-}^{\alpha}f(a)\right]\right|\\ &\le\frac{(b-a)^2\left(\left|f''(a)\right|+\left|f''(b)\right|\right)}{2(\alpha+1)}\left(\frac{1}{s+1}-\frac{1}{\alpha+s+1}-B(s+1,\alpha+2)\right).\end{aligned}\tag{2.12}$$

3. Main Results

Lemma 3.1. *Let I be an interval $a,b\in I$ with $0\le a<b$ and $\varphi:I\to\mathbb{R}$ a continuous increasing function. Let $f:[\varphi(a),\varphi(b)]\to\mathbb{R}$ be a twice differentiable mapping on $(\varphi(a),\varphi(b))$. If $f''\in L[\varphi(a),\varphi(b)]$, then the following equality for fractional integral holds:*

$$\begin{aligned}&\frac{f(\varphi(a))+f(\varphi(b))}{2}-\frac{\Gamma(\alpha+1)}{2(\varphi(b)-\varphi(a))^{\alpha}}\left[J_{\varphi(a)^+}^{\alpha}f(\varphi(b))+J_{\varphi(b)^-}^{\alpha}f(\varphi(a))\right]\\ &=\frac{(\varphi(b)-\varphi(a))^2}{2}\int_0^1\left(\frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{\alpha+1}\right)f''(t\varphi(a)+(1-t)\varphi(b))\,dt.\end{aligned}\tag{3.1}$$

Proof. By using Lemma 3 and Definition 5, we have

$$\begin{aligned}I&=\int_0^1\left(\frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{\alpha+1}\right)\left(f''(t\varphi(a)+(1-t)\varphi(b))\right)dt\\ &=\frac{1}{\varphi(b)-\varphi(a)}\left[\int_0^1\left((1-t)^{\alpha}-t^{\alpha}\right)f'(t\varphi(a)+(1-t)\varphi(b))\,dt\right]\\ &=\frac{f(\varphi(a))+f(\varphi(b))}{(\varphi(b)-\varphi(a))^2}-\frac{\alpha}{(\varphi(b)-\varphi(a))^2}\\ &\quad\times\left(\int_0^1(1-t)^{\alpha-1}f(t\varphi(a)+(1-t)\varphi(b))\,dt\right.\\ &\quad\left.+\int_0^1 t^{\alpha-1}f(t\varphi(a)+(1-t)\varphi(b))\,dt\right)\\ &=\frac{f(\varphi(a))+f(\varphi(b))}{(\varphi(b)-\varphi(a))^2}-\frac{\alpha\Gamma(\alpha)}{(\varphi(b)-\varphi(a))^{\alpha+2}}\\ &\quad\times\left(\int_{\varphi(a)}^{\varphi(b)}(\varphi(x)-\varphi(a))^{\alpha-1}f(\varphi(x))\,d\varphi(x)\right.\\ &\quad\left.+\int_{\varphi(a)}^{\varphi(b)}(\varphi(b)-\varphi(x))^{\alpha-1}f(\varphi(x))\,d\varphi(x)\right)\\ &=\frac{f(\varphi(a))+f(\varphi(b))}{(\varphi(b)-\varphi(a))^2}-\frac{\Gamma(\alpha+1)}{(\varphi(b)-\varphi(a))^{\alpha+2}}\left[J_{\varphi(a)^+}^{\alpha}f(\varphi(b))+J_{\varphi(b)^-}^{\alpha}f(\varphi(a))\right].\end{aligned}$$

If I by multiplying $\frac{(\varphi(b)-\varphi(a))^2}{2}$, it obtain that

$$\begin{aligned}&\tfrac{(\varphi(b)-\varphi(a))^2}{2}\int_0^1\left(\tfrac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{\alpha+1}\right)f''t\varphi(a)+(1-t)\varphi(b)\,dt\\&=\tfrac{f(\varphi(a))+f(\varphi(b))}{2}-\tfrac{\Gamma(\alpha+1)}{2(\varphi(b)-\varphi(a))^{\alpha}}\left[J^{\alpha}_{\varphi(a)^{+}}f(\varphi(b))+J^{\alpha}_{\varphi(b)^{-}}f(\varphi(a))\right].\end{aligned}$$

The proof is done. □

Lemma 3.2. *Let I be an interval $a,b\in I$ with $0\le a<b$ and $\varphi:I\to\mathbb{R}$ a continuous increasing function. Let $f:[\varphi(a),\varphi(b)]\to\mathbb{R}$ be a twice differentiable mapping on $(\varphi(a),\varphi(b))$. If $f''\in L[\varphi(a),\varphi(b)]$, then the following equality for fractional integral holds:*

$$\begin{aligned}&\tfrac{\Gamma(\alpha+1)}{2(\varphi(b)-\varphi(a))^{\alpha}}\left[J^{\alpha}_{\varphi(a)^{+}}f(\varphi(b))+J^{\alpha}_{\varphi(b)^{-}}f(\varphi(a))\right]-f\left(\tfrac{\varphi(a)+\varphi(b)}{2}\right)\\&=\tfrac{(\varphi(b)-\varphi(a))^2}{2}\int_0^1 m(t)\,f''(t\varphi(a)+(1-t)\varphi(b))\,dt,\end{aligned}\tag{3.2}$$

where

$$m(t)=\begin{cases}t-\frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{\alpha+1}, & t\in\left[0,\frac12\right),\\ 1-t-\frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{\alpha+1}, & t\in\left[\frac12,1\right).\end{cases}$$

Proof. By using Lemma 4 and Definition 5, we have:

$$\begin{aligned}I&=\int_0^1 m(t)\,f''(t\varphi(a)+(1-t)\varphi(b))\,dt\\&=\int_0^{\frac12}\left(t-\tfrac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{\alpha+1}\right)(f''(t\varphi(a)+(1-t)\varphi(b)))\,dt\\&\quad+\int_{\frac12}^1\left(1-t-\tfrac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{\alpha+1}\right)(f''(t\varphi(a)+(1-t)\varphi(b)))\,dt\\&=I_1+I_2.\end{aligned}$$

If use twice the partial integration method for I_1, we have

$$\begin{aligned}I_1&=\int_0^{\frac12}\left(t-\tfrac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{\alpha+1}\right)f''(t\varphi(a)+(1-t)\varphi(b))\,dt\\&=\tfrac{\left(\alpha-1+2^{1-\alpha}\right)f'\left(\frac{\varphi(a)+\varphi(b)}{2}\right)}{2(\alpha+1)(\varphi(a)-\varphi(b))}-\tfrac{f\left(\frac{\varphi(a)+\varphi(b)}{2}\right)}{(\varphi(a)-\varphi(b))^2}+\tfrac{\Gamma(\alpha+1)}{(\varphi(a)-\varphi(b))^{\alpha+2}}\\&\quad\times\left[\int_{\frac{\varphi(a)+\varphi(b)}{2}}^{\varphi(b)}(\varphi(x)-\varphi(a))^{\alpha-1}f(\varphi(x))\,d\varphi(x)\right.\\&\quad\left.+\int_{\frac{\varphi(a)+\varphi(b)}{2}}^{\varphi(b)}(\varphi(b)-\varphi(x))^{\alpha-1}f(\varphi(x))\,d\varphi(x)\right].\end{aligned}$$

If use twice the partial integration method for I_2, we have

$$\begin{aligned}I_2&=\int_{\frac12}^1\left(1-t-\tfrac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{\alpha+1}\right)f''(t\varphi(a)+(1-t)\varphi(b))\,dt\\&=-\left(\tfrac{\left(\alpha-1+2^{1-\alpha}\right)f'\left(\frac{\varphi(a)+\varphi(b)}{2}\right)}{2(\alpha+1)(\varphi(a)-\varphi(b))}\right)-\tfrac{f\left(\frac{\varphi(a)+\varphi(b)}{2}\right)}{(\varphi(a)-\varphi(b))^2}\\&\quad+\tfrac{\Gamma(\alpha+1)}{(\varphi(a)-\varphi(b))^{\alpha+2}}\left[\int_{\varphi(a)}^{\frac{\varphi(a)+\varphi(b)}{2}}(\varphi(x)-\varphi(a))^{\alpha-1}f(\varphi(x))\,d\varphi(x)\right.\\&\quad\left.+\int_{\varphi(a)}^{\frac{\varphi(a)+\varphi(b)}{2}}(\varphi(b)-\varphi(x))^{\alpha-1}f(\varphi(x))\,d\varphi(x)\right],\end{aligned}$$

by add uping I_1 and I_2 , and by multiplying $\frac{(\varphi(b)-\varphi(a))^2}{2}$ with I, it obtain that:

$$\begin{aligned}&\tfrac{\Gamma(\alpha+1)}{2(\varphi(b)-\varphi(a))^{\alpha}}\left[J^{\alpha}_{\varphi(a)^{+}}f(\varphi(b))+J^{\alpha}_{\varphi(b)^{-}}f(\varphi(a))\right]-f\left(\tfrac{\varphi(a)+\varphi(b)}{2}\right)\\&=\tfrac{(\varphi(b)-\varphi(a))^2}{2}\int_0^1 m(t)\,f''(t\varphi(a)+(1-t)\varphi(b))\,dt.\end{aligned}$$

The proof is done.

Lemma 3.3. *Let I be an interval $a,b\in I$ with $0\leq a<b$ and $\varphi:I\to\mathbb{R}$ a continuous increasing function. Let $f:[\varphi(a),\varphi(b)]\to\mathbb{R}$ be a twice differentiable mapping on $(\varphi(a),\varphi(b))$. If $f''\in L[\varphi(a),\varphi(b)]$, then the following equality for fractional integral holds:*

$$\begin{aligned}
&\tfrac{f(\varphi(a))+f(\varphi(b))}{r(r+1)}+\tfrac{2}{r+1}f\left(\tfrac{\varphi(a)+\varphi(b)}{2}\right)-\tfrac{\Gamma(\alpha+1)}{r(\varphi(b)-\varphi(a))^{\alpha}}\\
&\qquad\times\left[J^{\alpha}_{\varphi(a)^{+}}f(\varphi(b))+J^{\alpha}_{\varphi(b)^{-}}f(\varphi(a))\right]\\
&=(\varphi(b)-\varphi(a))^{2}\int_0^1 k(t)\,f''(t\varphi(a)+(1-t)\varphi(b))\,dt.
\end{aligned}\tag{3.3}$$

Where

$$k(t)=\begin{cases}\frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{\alpha+1}-\frac{t}{r+1}, & t\in\left[0,\frac{1}{2}\right),\\ \frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{\alpha+1}-\frac{1-t}{r:+1}, & t\in\left[\frac{1}{2},1\right).\end{cases}$$

□

Proof. By using Definition 5 and Lemma 5, we have

$$\begin{aligned}
I&=\int_0^1 k(t)\,f''(t\varphi(a)+(1-t)\varphi(b))\,dt\\
&=\tfrac{1}{r(r+1)(\alpha+1)}\\
&\quad\times\int_0^{\frac{1}{2}}\left[(r+1)\left[1-(1-t)^{\alpha+1}-t^{\alpha+1}\right]-rt(\alpha+1)\right]\\
&\quad\times f''(t\varphi(a)+(1-t)\varphi(b))\,dt\\
&\quad+\tfrac{1}{r(r+1)(\alpha+1)}\\
&\quad\times\int_{\frac{1}{2}}^{1}\left[(r+1)\left[1-(1-t)^{\alpha+1}-t^{\alpha+1}\right]-r(1-t)(\alpha+1)\right]\\
&\quad\times f''(t\varphi(a)+(1-t)\varphi(b))\,dt\\
&=I_1+I_2.
\end{aligned}$$

If use twice the partial integration method for I_1, we have

$$\begin{aligned}
I_1&=\int_0^{\frac{1}{2}}\left[(r+1)\left[1-(1-t)^{\alpha+1}-t^{\alpha+1}\right]-rt(\alpha+1)\right]\\
&\quad\times f''(t\varphi(a)+(1-t)\varphi(b))\,dt\\
&=\left[(r+1)\left(1-2^{-\alpha}\right)-\tfrac{r(\alpha+1)}{2}\right]\tfrac{f'\left(\frac{\varphi(a)+\varphi(b)}{2}\right)}{\varphi(a)-\varphi(b)}\\
&\quad+\tfrac{r(\alpha+1)f\left(\frac{\varphi(a)+\varphi(b)}{2}\right)}{(\varphi(b)-\varphi(a))^{2}}+\tfrac{(\alpha+1)f(\varphi(b))}{(\varphi(b)-\varphi(a))^{2}}-\tfrac{(r+1)(\alpha+1)\alpha\Gamma(\alpha)}{(\varphi(b)-\varphi(a))^{\alpha+2}}\\
&\quad\times\left[\int_{\frac{\varphi(a)+\varphi(b)}{2}}^{\varphi(b)}(\varphi(x)-\varphi(a))^{\alpha-1}f(\varphi(x))\,d\varphi(x)\right.\\
&\quad\left.+\int_{\frac{\varphi(a)+\varphi(b)}{2}}^{\varphi(b)}(\varphi(b)-\varphi(x))^{\alpha-1}f(\varphi(x))\,d\varphi(x)\right]
\end{aligned}$$

If use twice the partial integration method for I_2, we have

$$\begin{aligned}
I_2&=\int_{\frac{1}{2}}^{1}\left[(r+1)\left[1-(1-t)^{\alpha+1}-t^{\alpha+1}\right]-r(1-t)(\alpha+1)\right]\\
&\quad\times f''(t\varphi(a)+(1-t)\varphi(b))\,dt\\
&=\left[\tfrac{r(\alpha+1)}{2}-(r+1)\left(1-2^{-\alpha}\right)\right]\tfrac{f'\left(\frac{\varphi(a)+\varphi(b)}{2}\right)}{\varphi(a)-\varphi(b)}\\
&\quad+\tfrac{r(\alpha+1)f\left(\frac{\varphi(a)+\varphi(b)}{2}\right)}{(\varphi(b)-\varphi(a))^{2}}+\tfrac{(\alpha+1)f(\varphi(a))}{(\varphi(b)-\varphi(a))^{2}}-\tfrac{(r+1)(\alpha+1)\alpha\Gamma(\alpha)}{(\varphi(b)-\varphi(a))^{\alpha+2}}\\
&\quad\times\left[\int_{\varphi(a)}^{\frac{\varphi(a)+\varphi(b)}{2}}(\varphi(x)-\varphi(a))^{\alpha-1}f(\varphi(x))\,d\varphi(x)\right.\\
&\quad\left.+\int_{\varphi(a)}^{\frac{\varphi(a)+\varphi(b)}{2}}(\varphi(b)-\varphi(x))^{\alpha-1}f(\varphi(x))\,d\varphi(x)\right].
\end{aligned}$$

By add uping I_1 and I_2, and by multipling with $\frac{(\varphi(b)-\varphi(a))^2}{r(r+1)(\alpha+1)}$,it obtain that:

$$\begin{aligned}&\tfrac{f(\varphi(a))+f(\varphi(b))}{r(r+1)}+\tfrac{2}{r+1}f\left(\tfrac{\varphi(a)+\varphi(b)}{2}\right)\\&-\tfrac{\Gamma(\alpha+1)}{r(\varphi(b)-\varphi(a))^{\alpha}}\left[J^{\alpha}_{\varphi(a)^{+}}f\left(\varphi\left(b\right)\right)+J^{\alpha}_{\varphi(b)^{-}}f\left(\varphi\left(a\right)\right)\right]\\&=\left(\varphi\left(b\right)-\varphi\left(a\right)\right)^{2}\int_0^1 k\left(t\right)f''\left(t\varphi\left(a\right)+\left(1-t\right)\varphi\left(b\right)\right)dt\ .\end{aligned}$$

The proof is done.

Theorem 3.1. *Let I be an interval $a,b\in I$ with $0\le a<b$ and $\varphi:I\to\mathbb{R}$ a continuous increasing function. Let $f:[0,\varphi(b)]\to\mathbb{R}$ be a differentiable mapping and $1<q<\infty$. If $|f''|$ is measurable and $|f''|$ is decreasing and geometric-arithmetically $\varphi-s-$convex on $[0,\varphi(b)]$ for some fixed $\alpha\in(0,\infty)$, $s\in(0,1]$, $0\le a<b$, then the following inequality for fractional integrals holds:*

$$\begin{aligned}&\tfrac{f(\varphi(a))+f(\varphi(b))}{2}-\tfrac{\Gamma(\alpha+1)}{2(\varphi(b)-\varphi(a))^{\alpha}}\left[J^{\alpha}_{\varphi(a)^{+}}f\left(\varphi\left(b\right)\right)+J^{\alpha}_{\varphi(b)^{-}}f\left(\varphi\left(a\right)\right)\right]\\&\le\tfrac{(\varphi(b)-\varphi(a))^{2}\left(\left|f''(\varphi(a))\right|+\left|f''(\varphi(b))\right|\right)}{2(\alpha+1)}\left(\tfrac{1}{s+1}-\tfrac{1}{\alpha+s+1}-B\left(s+1,\alpha+2\right)\right).\end{aligned}\tag{3.4}$$

□

Proof. By using Definition 2, Lemma 2 and Lemma 6, we have

$$\begin{aligned}&\left|\tfrac{f(\varphi(a))+f(\varphi(b))}{2}-\tfrac{\Gamma(\alpha+1)}{2(\varphi(b)-\varphi(a))^{\alpha}}\left[J^{\alpha}_{\varphi(a)^{+}}f\left(\varphi\left(b\right)\right)+J^{\alpha}_{\varphi(b)^{-}}f\left(\varphi\left(a\right)\right)\right]\right|\\&\le\tfrac{(\varphi(b)-\varphi(a))^2}{2}\int_0^1\left|\tfrac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{\alpha+1}\right|\left|f''\left(t\varphi\left(a\right)+\left(1-t\right)\varphi\left(b\right)\right)\right|dt\\&\le\tfrac{(\varphi(b)-\varphi(a))^2}{2(\alpha+1)}\int_0^1\left(1-\left(1-t\right)^{\alpha+1}-t^{\alpha+1}\right)\left|f''\left(\varphi^{t}\left(a\right)+\varphi^{(1-t)}\left(b\right)\right)\right|dt\\&\le\tfrac{(\varphi(b)-\varphi(a))^2}{2(\alpha+1)}\int_0^1\left(1-\left(1-t\right)^{\alpha+1}-t^{\alpha+1}\right)\\&\quad\times\left[t^{s}\left|f''\varphi\left(a\right)\right|+\left(1-t\right)^{s}\left|f''\varphi\left(b\right)\right|\right]dt\\&\le\tfrac{(\varphi(b)-\varphi(a))^2}{2(\alpha+1)}\left|f''\varphi\left(a\right)\right|\\&\quad\times\left[-\int_0^1 t^{\alpha+s+1}dt+\int_0^1 t^{s}dt-\int_0^1 t^{s}\left(1-t\right)^{\alpha+1}dt\right]\\&\quad+\tfrac{(\varphi(b)-\varphi(a))^2}{2(\alpha+1)}\left|f''\varphi\left(b\right)\right|\\&\quad\times\left[-\int_0^1\left(1-t\right)^{\alpha+s+1}dt+\int_0^1\left(1-t\right)^{s}dt-\int_0^1 t^{\alpha+1}\left(1-t\right)^{s}dt\right]\\&\le\tfrac{(\varphi(b)-\varphi(a))^2}{2(\alpha+1)}\left(\left|f''\varphi\left(a\right)\right|+\left|f''\varphi\left(b\right)\right|\right)\\&\quad\times\left[-\tfrac{1}{\alpha+s+1}+\tfrac{1}{s+1}-B\left(s+1,\alpha+2\right)\right].\end{aligned}$$

The poof is done.

Theorem 3.2. *Let I be an interval $a,b\in I$ with $0\le a<b$ and $\varphi:I\to\mathbb{R}$ a continuous increasing function. Let $f:[0,\varphi(b)]\to\mathbb{R}$ be a differentiable mapping and $1<q<\infty$. If $|f''|^{q}$ is measurable and $|f''|^{q}$ is decreasing and geometric-arithmetically $\varphi-s-$convex on $[0,\varphi(b)]$ for some fixed $\alpha\in(0,\infty)$, $s\in(0,1]$, $0\le a<b$, then the following inequality for fractional integrals holds:*

$$\begin{aligned}&\left|\tfrac{f(\varphi(a))+f(\varphi(b))}{2}-\tfrac{\Gamma(\alpha+1)}{2(\varphi(b)-\varphi(a))^{\alpha}}\left[J^{\alpha}_{\varphi(a)^{+}}f\left(\varphi\left(b\right)\right)+J^{\alpha}_{\varphi(b)^{-}}f\left(\varphi\left(a\right)\right)\right]\right|\\&\le\tfrac{(\varphi(b)-\varphi(a))^{2}\max\left\{1-2^{1-\alpha},2^{1-\alpha}-1\right\}}{2(\alpha+1)}\left(\tfrac{\left|f''(\varphi(a))\right|^{q}+\left|f''(\varphi(b))\right|^{q}}{s+1}\right).\end{aligned}\tag{3.5}$$

□

Proof. To achieve our aim ,we divide our proof into two cases.

Case 1: $\alpha \in (0,1)$, by using Definition 2 , Hölder's inequality and Lemma 6 ,we have

$$\begin{aligned}
&\left|\tfrac{f(\varphi(a))+f(\varphi(b))}{2}-\tfrac{\Gamma(\alpha+1)}{2(\varphi(b)-\varphi(a))^{\alpha}}\left[J^{\alpha}_{\varphi(a)^{+}}f\left(\varphi\left(b\right)\right)+J^{\alpha}_{\varphi(b)^{-}}f\left(\varphi\left(a\right)\right)\right]\right| \\
&\leq \tfrac{(\varphi(b)-\varphi(a))^{2}}{2}\int_{0}^{1}\left|\tfrac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{\alpha+1}\right|\left|f''\left(t\varphi\left(a\right)+\left(1-t\right)\varphi\left(b\right)\right)\right|dt \\
&\leq \tfrac{(\varphi(b)-\varphi(a))^{2}}{2(\alpha+1)}\left(\int_{0}^{1}\left|1-(1-t)^{\alpha+1}-t^{\alpha+1}\right|^{p}\right)^{\frac{1}{p}} \\
&\qquad\qquad \times\left(\int_{0}^{1}\left|f''\left(t\varphi\left(a\right)+\left(1-t\right)\varphi\left(b\right)\right)\right|^{q}dt\right)^{\frac{1}{q}} \\
&\leq \tfrac{(\varphi(b)-\varphi(a))^{2}}{2(\alpha+1)}\left(\int_{0}^{1}\left|1-(1-t)^{\alpha+1}-t^{\alpha+1}\right|^{p}\right)^{\frac{1}{p}} \\
&\qquad\qquad \times\left(\int_{0}^{1}\left|f''\left(\varphi^{t}\left(a\right)\varphi^{(1-t)}\left(b\right)\right)\right|^{q}dt\right)^{\frac{1}{q}} \\
&\leq \tfrac{(\varphi(b)-\varphi(a))^{2}}{2(\alpha+1)}\left(\int_{0}^{1}\left|1-(1-t)^{\alpha+1}-t^{\alpha+1}\right|^{p}\right)^{\frac{1}{p}} \\
&\qquad\qquad \times\left(\int_{0}^{1}\left[t^{s}\left|f''\left(\varphi\left(a\right)\right)\right|+\left(1-t\right)^{s}\left|f''\left(\varphi\left(b\right)\right)\right|\right]dt\right)^{\frac{1}{q}} \\
&\leq \tfrac{(\varphi(b)-\varphi(a))^{2}}{2(\alpha+1)}\left(\tfrac{\left|f''(\varphi(a))\right|^{q}+\left|f''(\varphi(b))\right|^{q}}{s+1}\right)^{\frac{1}{q}}\left(\int_{0}^{1}\left[(1-t)^{\alpha}+t^{\alpha}-1\right]^{p}dt\right)^{\frac{1}{p}} \\
&\leq \tfrac{(\varphi(b)-\varphi(a))^{2}}{2(\alpha+1)}\left(\tfrac{\left|f''(\varphi(a))\right|^{q}+\left|f''(\varphi(b))\right|^{q}}{s+1}\right)^{\frac{1}{q}}\left(\int_{0}^{1}\left[2^{1-\alpha}-1\right]^{p}dt\right)^{\frac{1}{p}} \\
&\leq \tfrac{(\varphi(b)-\varphi(a))^{2}\left(2^{1-\alpha}-1\right)}{2(\alpha+1)}\left(\tfrac{\left|f''(\varphi(a))\right|^{q}+\left|f''(\varphi(b))\right|^{q}}{s+1}\right)^{\frac{1}{q}}.
\end{aligned}$$

Where $\frac{1}{p}+\frac{1}{q}=1$.

Case 2: $\alpha \in [1,\infty)$, by using Definition 2, Hölder's inequality and Lemma 6, we have

$$\begin{aligned}
&\left|\tfrac{f(\varphi(a))+f(\varphi(b))}{2}-\tfrac{\Gamma(\alpha+1)}{2(\varphi(b)-\varphi(a))^{\alpha}}\left[J^{\alpha}_{\varphi(a)^{+}}f\left(\varphi\left(b\right)\right)+J^{\alpha}_{\varphi(b)^{-}}f\left(\varphi\left(a\right)\right)\right]\right| \\
&\leq \tfrac{(\varphi(b)-\varphi(a))^{2}}{2(\alpha+1)}\left(\tfrac{\left|f''(\varphi(a))\right|^{q}+\left|f''(\varphi(b))\right|^{q}}{s+1}\right)^{\frac{1}{q}} \\
&\qquad\qquad \times\left(\int_{0}^{1}\left[1-(1-t)^{\alpha}-t^{\alpha}\right]^{p}dt\right)^{\frac{1}{p}} \\
&\leq \tfrac{(\varphi(b)-\varphi(a))^{2}\left(1-2^{1-\alpha}\right)}{2(\alpha+1)}\left(\tfrac{\left|f''(\varphi(a))\right|^{q}+\left|f''(\varphi(b))\right|^{q}}{s+1}\right)^{\frac{1}{q}}.
\end{aligned}$$

The proof is done.

Theorem 3.3. *Let I be an interval $a,b \in I$ with $0 \leq a < b$ and $\varphi : I \to \mathbb{R}$ a continuous increasing function. Let $f : [0,\varphi(b)] \to \mathbb{R}$ be a differentiable mapping and $1 < q < \infty$. If $|f''|$ is measurable and $|f''|$ is decreasing and geometric-arithmetically $\varphi - s-$convex on $[0,\varphi(b)]$ for some fixed $\alpha \in (0,\infty)$, $s \in (0,1]$, $0 \leq a < b$, then the* *following inequality for fractional integrals holds:*

$$
\begin{aligned}
&\left|\tfrac{\Gamma(\alpha+1)}{2(\varphi(b)-\varphi(a))^{\alpha}}\left[J^{\alpha}_{\varphi(a)^{+}}f(\varphi(b))+J^{\alpha}_{\varphi(b)^{-}}f(\varphi(a))\right]-f\left(\tfrac{\varphi(a)+\varphi(b)}{2}\right)\right|\\
&\le \tfrac{(\varphi(b)-\varphi(a))^{2}\left|f''(\varphi(a))\right|}{2(\alpha+1)}\\
&\qquad\times\left[\tfrac{\alpha-\alpha 2^{-s-1}-2^{-s-1}}{1+s}-\tfrac{\alpha+1}{2+s}+2B(s+1,\alpha+2)+\tfrac{1}{\alpha+s+2}\right]\\
&\quad+\tfrac{(\varphi(b)-\varphi(a))^{2}\left|f''(\varphi(a))\right|}{2(\alpha+1)}\\
&\qquad\times\left[\tfrac{\alpha 2^{-s-1}+2^{-s-1}-1}{1+s}+\tfrac{1}{\alpha+s+2}+2B(s+1,\alpha+2)\right].
\end{aligned}
\tag{3.6}
$$

□

Proof. By using Definition 5 and Lemma 7, we have

$$
\begin{aligned}
&\left|\tfrac{\Gamma(\alpha+1)}{2(\varphi(b)-\varphi(a))^{\alpha}}\left[J^{\alpha}_{\varphi(a)^{+}}f(\varphi(b))+J^{\alpha}_{\varphi(b)^{-}}f(\varphi(a))\right]-f\left(\tfrac{\varphi(a)+\varphi(b)}{2}\right)\right|\\
&\le \tfrac{(\varphi(b)-\varphi(a))^{2}}{2}\int_0^1|m(t)|\left|f''(t\varphi(a)+(1-t)\varphi(b))\right|dt\\
&\le \tfrac{(\varphi(b)-\varphi(a))^{2}}{2}\int_0^1|m(t)|\left[f''\left(\varphi^{t}(a)\varphi^{(1-t)}(b)\right)\right]dt\\
&\le \tfrac{(\varphi(b)-\varphi(a))^{2}}{2}\int_0^1|m(t)|\left[t^{s}\left|f''(\varphi(a))\right|+(1-t)^{s}\left|f''(\varphi(b))\right|\right]dt\\
&\le \tfrac{(\varphi(b)-\varphi(a))^{2}}{2}\int_0^{\frac12}\left|t-\tfrac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{\alpha+1}\right|\left[t^{s}\left|f''(\varphi(a))\right|+(1-t)^{s}\left|f''(\varphi(b))\right|\right]dt\\
&\quad+\tfrac{(\varphi(b)-\varphi(a))^{2}}{2}\int_{\frac12}^1\left|1-t-\tfrac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{\alpha+1}\right|\left[t^{s}\left|f''(\varphi(a))\right|+(1-t)^{s}\left|f''(\varphi(b))\right|\right]dt\\
&\le \tfrac{(\varphi(b)-\varphi(a))^{2}}{2(\alpha+1)}\left|f''(\varphi(a))\right|\left[\int_0^{\frac12}\left[-t^{s}+(\alpha+1)t^{s+1}+t^{s}(1-t)^{\alpha+1}+t^{\alpha+s+1}\right]dt\right.\\
&\quad\left.+\int_{\frac12}^1\left[\alpha t^{s}-(\alpha+1)t^{s+1}+t^{s}(1-t)^{\alpha+1}+t^{\alpha+s+1}\right]dt\right]+\tfrac{(\varphi(b)-\varphi(a))^{2}}{2(\alpha+1)}\left|f''(\varphi(b))\right|\\
&\quad\times\left[\int_0^{\frac12}\left[-(1-t)^{s}+(\alpha+1)t(1-t)^{s+1}+(1-t)^{\alpha+s+1}+t^{\alpha+s}(1-t)^{s}\right]dt\right.\\
&\quad\left.+\int_{\frac12}^1\left[\alpha(1-t)^{s}-(\alpha+1)t(1-t)^{s}+(1-t)^{\alpha+s+1}+t^{\alpha+1}(1-t)^{s}\right]dt\right]\\
&\le \tfrac{(\varphi(b)-\varphi(a))^{2}}{2(\alpha+1)}\left|f''(\varphi(a))\right|\left[\tfrac{\alpha-\alpha 2^{-s-1}-2^{-s-1}}{1+s}-\tfrac{\alpha+1}{s+2}+2B(s+1,\alpha+2)+\tfrac{1}{\alpha+s+2}\right]\\
&\quad+\tfrac{(\varphi(b)-\varphi(a))^{2}}{2(\alpha+1)}\left|f''(\varphi(b))\right|\left[\tfrac{\alpha 2^{-s-1}+2^{-s-1}-1}{1+s}+2B(\alpha+2,s+1)+\tfrac{1}{\alpha+s+2}\right].
\end{aligned}
$$

Theorem 3.4. *Let I be an interval $a,b\in I$ with $0\le a<b$ and $\varphi: I\to\mathbb{R}$ a continuous increasing function. Let $f:[0,\varphi(b)]\to\mathbb{R}$ be a differentiable mapping and $1<q<\infty$. If $|f''|$ is measurable and $|f''|^{q}$ is decreasing and geometric-arithmetically $\varphi-s-$convex on $[0,\varphi(b)]$ for some fixed $\alpha\in(0,\infty)$, $s\in(0,1]$, $0\le a<b$, then the following inequality for fractional integrals holds:*

$$
\begin{aligned}
&\tfrac{\Gamma(\alpha+1)}{2(\varphi(b)-\varphi(a))^{\alpha}}\left[J^{\alpha}_{\varphi(a)^{+}}f(\varphi(b))+J^{\alpha}_{\varphi(b)^{-}}f(\varphi(a))\right]-f\left(\tfrac{\varphi(a)+\varphi(b)}{2}\right)\\
&\le \tfrac{(\varphi(b)-\varphi(a))^{2}}{2(\alpha+1)}\left(\tfrac{\left|f''(\varphi(a))\right|^{q}+\left|f''(\varphi(b))\right|^{q}}{s+1}\right)^{\frac1q}\left(\tfrac{(\alpha+1)2^{-p-1}+(\alpha+0.5)^{p+1}-\alpha^{p+1}}{p+1}\right)^{\frac1p}.
\end{aligned}
\tag{3.7}
$$

Where $\frac1p+\frac1q=1$.

□

Proof. By using Lemma 2, Hölder's inequality and Lemma 7, we have

$$
\begin{aligned}
&\left|\frac{\Gamma(\alpha+1)}{2(\varphi(b)-\varphi(a))^{\alpha}}\left[J^{\alpha}_{\varphi(a)^+}f(\varphi(b))+J^{\alpha}_{\varphi(b)^-}f(\varphi(a))\right]-f\left(\frac{\varphi(a)+\varphi(b)}{2}\right)\right| \\
&\leq \frac{(\varphi(b)-\varphi(a))^2}{2}\int_0^1 |m(t)|\,|f''(t\varphi(a)+(1-t)\varphi(b))|\,dt \\
&\leq \frac{(\varphi(b)-\varphi(a))^2}{2}\left(\int_0^1 |m(t)|^p\,dt\right)^{\frac{1}{p}} \\
&\quad\times\left(\int_0^1 |f''(t\varphi(a)+(1-t)\varphi(b))|^q\,dt\right)^{\frac{1}{q}} \\
&\leq \frac{(\varphi(b)-\varphi(a))^2}{2}\left(\int_0^1 |m(t)|^p\,dt\right)^{\frac{1}{p}} \\
&\quad\times\left(\int_0^1 \left|f''\left(\varphi^t(a)\,\varphi^{(1-t)}(b)\right)\right|^q dt\right)^{\frac{1}{q}} \\
&\leq \frac{(\varphi(b)-\varphi(a))^2}{2}\left(\int_0^1 |m(t)|^p\,dt\right)^{\frac{1}{p}} \\
&\quad\times\left(\int_0^1 \left[t^s\,|f''(\varphi(a))|^q+(1-t)^s\,|f''(\varphi(b))|^q\right]dt\right)^{\frac{1}{q}} \\
&\leq \frac{(\varphi(b)-\varphi(a))^2}{2(\alpha+1)}\left(\frac{\left|f''(\varphi(a))\right|^q+\left|f''(\varphi(b))\right|^q}{s+1}\right)^{\frac{1}{q}} \\
&\quad\times\left(\int_0^{\frac{1}{2}}\left|t-\frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{\alpha+1}\right|^p dt+\int_{\frac{1}{2}}^1\left|1-t-\frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{\alpha+1}\right|^p dt\right)^{\frac{1}{p}} \\
&\leq \frac{(\varphi(b)-\varphi(a))^2}{2(\alpha+1)}\left(\frac{\left|f''(\varphi(a))\right|^q+\left|f''(\varphi(b))\right|^q}{s+1}\right)^{\frac{1}{q}} \\
&\quad\dot{u}\times\left((\alpha+1)\int_0^{\frac{1}{2}} t^p dt+\int_{\frac{1}{2}}^1(\alpha-t+1)^p\,dt\right)^{\frac{1}{p}} \\
&\leq \frac{(\varphi(b)-\varphi(a))^2}{2(\alpha+1)}\left(\frac{\left|f''(\varphi(a))\right|^q+\left|f''(\varphi(b))\right|^q}{s+1}\right)^{\frac{1}{q}}\left(\frac{(\alpha+1)2^{-p-1}+(\alpha+0.5)^{p+1}-\alpha^{p+1}}{p+1}\right)^{\frac{1}{p}}.
\end{aligned}
$$

The proof is done.

Theorem 3.5. *Let I be an interval $a, b \in I$ with $0 \leq a < b$ and $\varphi : I \to \mathbb{R}$ a continuous increasing function. Let $f : [0, \varphi(b)] \to \mathbb{R}$ be a differentiable mapping and $1 < q < \infty$. If $|f''|$ is measurable and $|f''|$ is decreasing and geometric-arithmetically $\varphi - s-$convex on $[0, \varphi(b)]$ for some fixed $\alpha \in (0, \infty)$, $s \in (0, 1]$, $0 \leq a < b$, then the* *following inequality for fractional integrals holds:*

(3.8)

$$
\begin{aligned}
&\left|\tfrac{f(\varphi(a))+f(\varphi(b))}{r(r+1)}+\tfrac{2}{r+1}f\left(\tfrac{\varphi(a)+\varphi(b)}{2}\right)-\tfrac{\Gamma(\alpha+1)}{r(\varphi(b)-\varphi(a))^{\alpha}}\left[J^{\alpha}_{\varphi(a)^{+}}f\left(\varphi\left(b\right)\right)+J^{\alpha}_{\varphi(b)^{-}}f\left(\varphi\left(a\right)\right)\right]\right|\\
&\le \tfrac{(\varphi(b)-\varphi(a))^{2}}{r(r+1)(\alpha+1)}\max\left\{\left[r+1-\left(r+1\right)2^{-\alpha}\right]\left(\tfrac{2^{-s-1}\left|f''(\varphi(a))\right|+\left(1-2^{-s-1}\right)\left|f''(\varphi(b))\right|}{s+1}\right)\right.\\
&-r\left(\alpha+1\right)\left[\tfrac{2^{-s-2}(s+1)\left|f''(\varphi(a))\right|+(1-t)^{s+1}(-ts-3t+1)f''(\varphi(a))}{(s+1)(s+2)}\right],\\
&\left.r\left(\alpha+1\right)\left[\tfrac{2^{-s-2}(s+1)\left|f''(\varphi(a))\right|+(1-t)^{s+1}(-ts-3t+1)f''(\varphi(a))}{(s+1)(s+2)}\right]\right\}\\
&+\tfrac{(\varphi(b)-\varphi(a))^{2}}{r(r+1)(\alpha+1)}\max\left\{\left(r+1-\left(r+1\right)2^{-\alpha}-r\left(\alpha+1\right)\right)\right.\\
&\times\left[\tfrac{\left(1-2^{-s-1}\right)\left|f''(\varphi(a))\right|-2^{-s-1}\left|f''(\varphi(b))\right|}{s+1}\right]\\
&+r\left(\alpha+1\right)\left[\tfrac{\left(1-2^{-s-1}\right)(s+1)\left|f''(\varphi(a))\right|+2^{-s-2}(s+3)\left|f''(\varphi(b))\right|}{(s+1)(s+2)}\right],\\
&\quad r\left(\alpha+1\right)\left[\tfrac{\left(1-2^{-s-1}\right)\left|f''(\varphi(a))\right|-2^{-s-1}\left|f''(\varphi(b))\right|}{s+1}\right.\\
&\quad\left.\left.-\tfrac{\left(1-2^{-s-2}\right)(s+1)\left|f''(\varphi(a))\right|+2^{-s-2}(s+3)\left|f''(\varphi(b))\right|}{(s+1)(s+2)}\right]\right\}.
\end{aligned}
$$

□

Proof. By using Definition 3, Lemma 3 and Lemma 8, we have

$$
\begin{aligned}
&\left|\tfrac{f(\varphi(a))+f(\varphi(b))}{r(r+1)}+\tfrac{2}{r+1}f\left(\tfrac{\varphi(a)+\varphi(b)}{2}\right)-\tfrac{\Gamma(\alpha+1)}{r(\varphi(b)-\varphi(a))^{\alpha}}\left[J^{\alpha}_{\varphi(a)^{+}}f\left(\varphi\left(b\right)\right)+J^{\alpha}_{\varphi(b)^{-}}f\left(\varphi\left(a\right)\right)\right]\right|\\
&\le\left(\varphi\left(b\right)-\varphi\left(a\right)\right)^{2}\int_{0}^{1}\left|k\left(t\right)\right|\left[t^{s}\left|f''\left(\varphi\left(a\right)\right)\right|+\left(1-t\right)^{s}\left|f''\left(\varphi\left(b\right)\right)\right|\right]dt\\
&\le\tfrac{(\varphi(b)-\varphi(a))^{2}}{r(r+1)(\alpha+1)}\int_{0}^{\frac{1}{2}}\left|\left(r+1\right)\left[1-\left(1-t\right)^{\alpha+1}-t^{\alpha+1}\right]-tr\left(\alpha+1\right)\right|\\
&\quad\times\left[t^{s}\left|f''\left(\varphi\left(a\right)\right)\right|+\left(1-t\right)^{s}\left|f''\left(\varphi\left(b\right)\right)\right|\right]dt\\
&\quad+\tfrac{(\varphi(b)-\varphi(a))^{2}}{r(r+1)(\alpha+1)}\int_{\frac{1}{2}}^{1}\left|\left(r+1\right)\left[1-\left(1-t\right)^{\alpha+1}-t^{\alpha+1}\right]-r\left(\alpha+1\right)\left(1-t\right)\right|\\
&\quad\times\left[t^{s}\left|f''\left(\varphi\left(a\right)\right)\right|+\left(1-t\right)^{s}\left|f''\left(\varphi\left(b\right)\right)\right|\right]dt
\end{aligned}
$$

$$
\begin{aligned}
&\le\tfrac{(\varphi(b)-\varphi(a))^{2}}{r(r+1)(\alpha+1)}\max\left\{\left[r+1-\left(r+1\right)2^{-\alpha}\right]\left(\tfrac{2^{-s-1}\left|f''(\varphi(a))\right|+\left(1-2^{-s-1}\right)\left|f''(\varphi(b))\right|}{s+1}\right)\right.\\
&-r\left(\alpha+1\right)\left[\tfrac{2^{-s-2}(s+1)\left|f''(\varphi(a))\right|+(1-t)^{s+1}(-ts-3t+1)f''(\varphi(a))}{(s+1)(s+2)}\right],\\
&\left.r\left(\alpha+1\right)\left[\tfrac{2^{-s-2}(s+1)\left|f''(\varphi(a))\right|+(1-t)^{s+1}(-ts-3t+1)f''(\varphi(a))}{(s+1)(s+2)}\right]\right\}\\
&+\tfrac{(\varphi(b)-\varphi(a))^{2}}{r(r+1)(\alpha+1)}\max\left(r+1-\left(r+1\right)2^{-\alpha}-r\left(\alpha+1\right)\right)\left[\tfrac{\left(1-2^{-s-1}\right)\left|f''(\varphi(a))\right|-2^{-s-1}\left|f''(\varphi(b))\right|}{s+1}\right]\\
&+r\left(\alpha+1\right)\left[\tfrac{\left(1-2^{-s-1}\right)(s+1)\left|f''(\varphi(a))\right|+2^{-s-2}(s+3)\left|f''(\varphi(b))\right|}{(s+1)(s+2)}\right],\\
&r\left(\alpha+1\right)\left[\tfrac{\left(1-2^{-s-1}\right)\left|f''(\varphi(a))\right|-2^{-s-1}\left|f''(\varphi(b))\right|}{s+1}\right.\\
&\left.-\tfrac{\left(1-2^{-s-2}\right)(s+1)\left|f''(\varphi(a))\right|+2^{-s-2}(s+3)\left|f''(\varphi(b))\right|}{(s+1)(s+2)}\right]
\end{aligned}
$$

Where we have used the following inequality:

$$\begin{aligned}
&\int_0^{\frac{1}{2}} \left|(r+1)\left[1-(1-t)^{\alpha+1}-t^{\alpha+1}\right]-tr(\alpha+1)\right| \\
&\quad \times \left[t^s\left|f''(\varphi(a))\right|+(1-t)^s\left|f''(\varphi(b))\right|\right]dt \\
&\quad \leq \left[(r+1)\left(1-2^{-\alpha}\right)\right]\left[\frac{2^{-s-1}\left|f''(\varphi(a))\right|+\left(1-2^{-s-1}\right)\left|f''(\varphi(b))\right|}{s+1}\right] \\
&\quad -r(\alpha+1)\left[\frac{2^{-s-2}\left|f''(\varphi(a))\right|}{s+2}+\frac{1-(s+3)2^{-s-2}\left|f''(\varphi(b))\right|}{(s+2)(s+1)}\right]
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^{\frac{1}{2}} \left|-r-1+(r+1)\left[(1-t)^{\alpha+1}+t^{\alpha+1}\right]+tr(\alpha+1)\right| \\
&\quad \times \left[t^s\left|f''(\varphi(a))\right|+(1-t)^s\left|f''(\varphi(b))\right|\right]dt \\
&\quad \leq r(\alpha+1)\left[\frac{2^{-s-2}\left|f''(\varphi(a))\right|}{s+2}+\frac{1-(s+3)2^{-s-2}\left|f''(\varphi(b))\right|}{(s+2)(s+1)}\right],
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\frac{1}{2}}^1 \left|r+1+tr(\alpha+1)-(r+1)\left[(1-t)^{\alpha+1}+t^{\alpha+1}\right]-r(\alpha+1)\right| \\
&\quad \times \left[t^s\left|f''(\varphi(a))\right|+(1-t)^s\left|f''(\varphi(b))\right|\right]dt \\
&\leq \left[r+1-(r+1)2^{-\alpha}-r(\alpha+1)\right]\left[\frac{\left(1-2^{-s-1}\right)\left|f''(\varphi(a))\right|-2^{-s-1}\left|f''(\varphi(b))\right|}{s+1}\right] \\
&\quad +r(\alpha+1)\left[\frac{2^{-s-2}\left|f''(\varphi(a))\right|}{s+2}+\frac{(s+3)2^{-s-2}\left|f''(\varphi(b))\right|}{(s+2)(s+1)}\right],
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\frac{1}{2}}^1 \left|-r-1-tr(\alpha+1)+(r+1)\left[(1-t)^{\alpha+1}+t^{\alpha+1}\right]+r(\alpha+1)\right| \\
&\quad \times \left[t^s\left|f''(\varphi(a))\right|+(1-t)^s\left|f''(\varphi(b))\right|\right]dt \\
&\leq r(\alpha+1) \\
&\quad \times \left[\frac{\left(1-2^{-s-1}\right)\left|f''(\varphi(a))\right|-2^{-s-1}\left|f''(\varphi(b))\right|}{s+1}-\frac{\left(1-2^{-s-2}\right)\left|f''(\varphi(a))\right|}{s+2}-\frac{(s+3)2^{-s-2}\left|f''(\varphi(b))\right|}{(s+2)(s+1)}\right].
\end{aligned}$$

The proof is done.

Theorem 3.6. *Let I be an interval $a,b\in I$ with $0\leq a<b$ and $\varphi: I\to\mathbb{R}$ a continuous increasing function. Let $f:[0,\varphi(b)]\to\mathbb{R}$ be a differentiable mapping and $1<q<\infty$. If $|f''|$ is measurable and $|f''|^q$ is decreasing and geometric-arithmetically $\varphi-s-$convex on $[0,\varphi(b)]$ for some fixed $\alpha\in(0,\infty)$, $s\in(0,1]$, $0\leq a<b$, then the following inequality for fractional integrals holds:*

(3.9)

$$\begin{aligned}
&\left|\frac{f(\varphi(a))+f(\varphi(b))}{r(r+1)}+\frac{2}{r+1}f\left(\frac{\varphi(a)+\varphi(b)}{2}\right)-\frac{\Gamma(\alpha+1)}{r(\varphi(b)-\varphi(a))^\alpha}\left[J^\alpha_{\varphi(a)^+}f(\varphi(b))+J^\alpha_{\varphi(b)^-}f(\varphi(a))\right]\right| \\
&\leq \frac{(\varphi(b)-\varphi(a))^2}{\left[r(r+1)(\alpha+1)\right]^{1+p^{-1}}}\left(\frac{\left|f''(\varphi(a))\right|^q+\left|f''(\varphi(b))\right|^q}{s+1}\right)^{\frac{1}{q}} \\
&\quad \times\left(\max\left\{\left[(r+1)\left(1-2^{-\alpha}\right)\right]^{p+1}-\left[\frac{2+r(1-\alpha)-(1+r)2^{-\alpha}}{2}\right]^{p+1},\left[r(\alpha+1)\right]^{p+1}2^{-p-1}\right\}\right. \\
&\quad \left.+\max\left\{\left[r+1-(r+1)2^{-\alpha}\right]^{p+1}-\left[\frac{2+r(1-\alpha)-(1+r)2^{-\alpha}}{2}\right]^{p+1},\left[\frac{r(\alpha+1)}{2}\right]^{p+1}\right\}\right)^{\frac{1}{p}}.
\end{aligned}$$

Where $\frac{1}{p}+\frac{1}{q}=1$.

□

Proof. By using Hölder's inequality and Lemma 7,we have

$$
\begin{aligned}
&\left|\frac{f(\varphi(a))+f(\varphi(b))}{r(r+1)}+\frac{2}{r+1}f\left(\frac{\varphi(a)+\varphi(b)}{2}\right)-\frac{\Gamma(\alpha+1)}{r(\varphi(b)-\varphi(a))^{\alpha}}\left[J^{\alpha}_{\varphi(a)^{+}}f(\varphi(b))+J^{\alpha}_{\varphi(b)^{-}}f(\varphi(a))\right]\right|\\
&\leq(\varphi(b)-\varphi(a))^{2}\int_{0}^{1}|k(t)|\,|f''(t\varphi(a)+(1-t)\varphi(b))|\,dt\\
&\leq(\varphi(b)-\varphi(a))^{2}\left(\int_{0}^{1}|k(t)|^{p}dt\right)^{\frac{1}{p}}\left(\int_{0}^{1}|f''(t\varphi(a)+(1-t)\varphi(b))|^{q}dt\right)^{\frac{1}{q}}\\
&\leq(\varphi(b)-\varphi(a))^{2}\left(\int_{0}^{1}|k(t)|^{p}dt\right)^{\frac{1}{p}}\left(\int_{0}^{1}\left|f''\left(\varphi^{t}(a)\varphi^{(1-t)}(b)\right)\right|^{q}dt\right)^{\frac{1}{q}}\\
&\leq(\varphi(b)-\varphi(a))^{2}\left(\int_{0}^{1}|k(t)|^{p}dt\right)^{\frac{1}{p}}\left(\int_{0}^{1}\left[t^{s}|f''(\varphi(a))|^{q}+(1-t)^{s}|f''(\varphi(b))|^{q}\right]dt\right)^{\frac{1}{q}}\\
&\leq(\varphi(b)-\varphi(a))^{2}\left(\frac{|f''(\varphi(a))|^{q}+|f''(\varphi(b))|^{q}}{s+1}\right)^{\frac{1}{q}}\left(\int_{0}^{1}|k(t)|^{p}dt\right)^{\frac{1}{p}}\\
&\leq(\varphi(b)-\varphi(a))^{2}\left(\frac{|f''(\varphi(a))|^{q}+|f''(\varphi(b))|^{q}}{s+1}\right)^{\frac{1}{q}}\\
&\quad\times\left(\int_{0}^{\frac{1}{2}}\left|\frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{r(\alpha+1)}-\frac{t}{r+1}\right|^{p}dt+\int_{\frac{1}{2}}^{1}\left|\frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{r(\alpha+1)}-\frac{1-t}{r+1}\right|^{p}dt\right)^{\frac{1}{p}}\\
&\leq\frac{(\varphi(b)-\varphi(a))^{2}}{[r(r+1)(\alpha+1)]^{1+p^{-1}}}\left(\frac{|f''(\varphi(a))|^{q}+|f''(\varphi(b))|^{q}}{s+1}\right)^{\frac{1}{q}}\\
&\quad\left(\times\max\left\{[(r+1)[1-2^{-\alpha}]]^{p+1}-\left[\frac{2+r(1-\alpha)-(1+r)2^{-\alpha-1}}{2}\right]^{p+1},[r(\alpha+1)]^{p+1}2^{-p-1}\right\}\right.\\
&\quad\left.+\max\left\{[(r+1)[1-2^{-\alpha}]]^{p+1}-\left[\frac{2+r(1-\alpha)+(1+r)2^{-\alpha-1}}{2}\right]^{p+1},\left[\frac{r(\alpha+1)}{2}\right]^{p+1}\right\}\right)^{\frac{1}{p}}.
\end{aligned}
$$

Where we have used the following inequalities:

$$
\int_{0}^{\frac{1}{2}}\left|(r+1)\left[1-(1-t)^{\alpha+1}-t^{\alpha+1}\right]-tr(\alpha+1)\right|^{p}dt \leq\frac{\left[(r+1)\left[1-2^{-\alpha}\right]\right]^{p+1}-\left[\frac{2+r(1-\alpha)-(1+r)2^{-\alpha-1}}{2}\right]^{p+1}}{r(\alpha+1)(p+1)},
$$

and

$$
\int_{0}^{\frac{1}{2}}\left|-r-1+(r+1)\left[(1-t)^{\alpha+1}+t^{\alpha+1}\right]+tr(\alpha+1)\right|^{p}dt \leq\frac{[r(\alpha+1)]^{p+1}2^{-p-1}}{r(\alpha+1)(p+1)},
$$

and

$$
\int_{\frac{1}{2}}^{1}\left|(r+1)\left[1-(1-t)^{\alpha+1}-t^{\alpha+1}\right]-r(1-t)(\alpha+1)\right|^{p}d \leq\frac{\left[(r+1)\left[1-2^{-\alpha}\right]\right]^{p+1}-\left[\frac{2+r(1-\alpha)+(1+r)2^{-\alpha-1}}{2}\right]^{p+1}}{r(\alpha+1)(p+1)},
$$

and

$$
\int_{\frac{1}{2}}^{1}\left|-r-1-tr(\alpha+1)+(r+1)\left[(1-t)^{\alpha+1}+t^{\alpha+1}\right]+r(\alpha+1)\right|^{p}dt \leq\frac{\left[\frac{r(\alpha+1)}{2}\right]^{p+1}}{r(\alpha+1)(p+1)}.
$$

The proof is done.

□

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