

ON HERMITE-HADAMARD INEQUALITIES FOR  
GEOMETRIC-ARITHMETICALLY  $\varphi$ - $s$ -CONVEX FUNCTIONS  
VIA FRACTIONAL INTEGRALS

SÜMEYYE ERMEYDAN AND HÜSEYİN YILDIRIM

ABSTRACT. In this paper, we established integral inequalities of Hermite-Hadamard type involving Riemann-Liouville fractional integrals motivated by the definition of geometric-arithmetically  $s$ -convex and  $v$ -convex functions. We give equality for fractional integral of  $\varphi$ -convex functions on twice differentiable mapping.

Received: 26–August–2016

Accepted: 29–August–2016

1. INTRODUCTION

Fractional calculus was born in 1695. In the past three hundred years, fractional calculus developed in diverse fields from physical sciences and engineering to biological sciences and economics[1 – 8]. Fractional Hermite-Hadamard inequalities involving all kinds of fractional integrals have attracted by many researches. In [12], Shuang et al. introduced a new concept of geometric-arithmetically  $s$ -convex functions and presented interesting Hermite-Hadamard type inequalities for integer integrals of such functions. In [15], Youness introduced a new concept of  $\varphi$ -convex functions. In [16 – 17], YuMei Liao and colleagues gave Riemann-Liouville Hermite-Hadamard integral inequalities for once and twice differentiable geometric-arithmetically  $s$ -convex functions. We establish on Hermite-Hadamard inequalities for twice differentiable geometric-arithmetically  $\varphi - s$ -convex functions via fractional integrals.

2. PRELIMINARIES

In this section, we will give some definitions, lemmas and notations which we use later in this work.

**Definition 2.1.** (see [3]) Let  $f \in L[a, b]$ . The Riemann- Liouville fractional integral  $J_{a+}^{\alpha} f$  and  $J_{b-}^{\alpha} f$  of order  $\alpha > 0$  with  $\alpha \geq 0$  are defined by

$$J_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt, 0 \leq a < x \leq b$$

and

$$(2.1) \quad J_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t) dt, 0 \leq a < x \leq b$$

---

<sup>13<sup>rd</sup></sup> International Intuitionistic Fuzzy Sets and Contemporary Mathematics Conference  
2010 *Mathematics Subject Classification.* 26A33, 26D15, 41A55.

*Key words and phrases.* Fractional Hermite-Hadamard inequalities, geometric-arithmetically  $s$ -convex,  $\varphi$ -convex functions, Riemann-Liouville Fractional Integral.

Here  $\Gamma$  is the gamma function.

**Definition 2.2.** (see [12 – 16]) Let  $f : I \subseteq \mathbb{R}^+ \rightarrow \mathbb{R}^+$  and  $s \in (0, 1]$ . A function  $f(x)$  is said to be geometric-arithmetically  $s$ -convex on  $I$  if for every  $x, y \in I$  and  $t \in [0, 1]$ , we have:

$$(2.2) \quad f(x^t y^{1-t}) \leq t^s f(x) + (1-t)^s f(y)$$

**Definition 2.3.** (see [14]) The incomplete beta function is defined as follows:

$$(2.3) \quad B_x(a, b) = \int_0^x t^{a-1} (1-t)^{b-1} dt.$$

Where  $x \in [0, 1]$ ,  $a, b > 0$ .

**Definition 2.4.** (see [15]) Let  $\varphi : [a, b] \subset \mathbb{R} \rightarrow [a, b]$  A function  $f : [a, b] \rightarrow \mathbb{R}$  is said to be  $\varphi$ -convex on  $[a, b]$  if, for every  $x, y \in [a, b]$  and  $t \in [0, 1]$  the following inequality holds:

$$(2.4) \quad f(t\varphi(x) + (1-t)\varphi(y)) \leq t f(\varphi(x)) + (1-t) f(\varphi(y))$$

**Definition 2.5.** Let  $\varphi : [a, b] \subset \mathbb{R} \rightarrow [a, b]$  and  $s \in (0, 1]$ . A function  $f : [a, b] \rightarrow \mathbb{R}$  is said to be  $\varphi$ - $s$ -convex on  $[a, b]$  if, for every  $x, y \in [a, b]$  and  $t \in [0, 1]$  the following inequality holds:

$$(2.5) \quad f(\varphi(x)^t \varphi(y)^{(1-t)}) \leq t^s f(\varphi(x)) + (1-t)^s f(\varphi(y))$$

**Lemma 2.1.** (see [11]) For  $t \in [0, 1]$ , we have

$$(2.6) \quad \begin{aligned} (1-t)^n &\leq 2^{1-n} - t^n \quad \text{for } n \in [0, 1], \\ (1-t)^n &\geq 2^{1-n} - t^n \quad \text{for } n \in [0, \infty). \end{aligned}$$

The following inequality was used in the proof directly in [12].

**Lemma 2.2.** (see [13]) for  $t \in [0, 1]$  and  $x, y > 0$ , we have

$$(2.7) \quad tx + (1-t)y \geq y^{1-t} x^t.$$

**Lemma 2.3.** (see [10]) Let  $f : [a, b] \rightarrow \mathbb{R}$  be a twice differentiable mapping on  $(a, b)$  with  $a < b$ . If  $f'' \in L[a, b]$ , then the following equality for fractional integrals holds:

$$(2.8) \quad \begin{aligned} &\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} [J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)] \\ &= \frac{(b-a)^2}{2} \int_0^1 \frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{\alpha+1} f''(ta + (1-t)b) dt. \end{aligned}$$

**Lemma 2.4.** (see [9]) Let  $f : [a, b] \rightarrow \mathbb{R}$  be a twice differentiable mapping on  $(a, b)$  with  $a < b$ . If  $f'' \in L[a, b]$ , then

$$(2.9) \quad \begin{aligned} &\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} [J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)] - f\left(\frac{a+b}{2}\right) \\ &= \frac{(b-a)^2}{2} \int_0^1 m(t) f''(ta + (1-t)b) dt. \end{aligned}$$

where

$$m(t) = \begin{cases} t - \frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{\alpha+1}, & t \in [0, \frac{1}{2}), \\ 1-t - \frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{\alpha+1}, & t \in [\frac{1}{2}, 1]. \end{cases}$$

**Lemma 2.5.** (see [9]) Let  $f : [a, b] \rightarrow \mathbb{R}$  be a differentiable mapping on  $(a, b)$  with  $a < b$ . If  $f'' \in L[a, b]$ ,  $r > 0$ , then

$$(2.10) \quad \begin{aligned} &\frac{f(a)+f(b)}{r(r+1)} + \frac{2}{r+1} f\left(\frac{a+b}{2}\right) - \frac{\Gamma(\alpha+1)}{r(b-a)^\alpha} [J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)] \\ &= (b-a)^2 \int_0^1 k(t) f''(ta + (1-t)b) dt. \end{aligned}$$

Where

$$k(t) = \begin{cases} \frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{\alpha+1} - \frac{t}{r+1}, & t \in [0, \frac{1}{2}), \\ \frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{\alpha+1} - \frac{1-t}{r+1}, & t \in [\frac{1}{2}, 1]. \end{cases}$$

YuMei Liao and colleagues based on our study, they have provided the following theorems in [16],[17].

**Theorem 2.1.** *Let  $f : [0, b] \rightarrow R$  be a differentiable mapping. If  $|f'|$  is measurable and  $|f'|$  is decreasing and geometric-arithmetically  $s$ -convex on  $[0, b]$  for some fixed  $\alpha \in (0, \infty)$ ,  $s \in (0, 1]$ ,  $0 \leq a < b$ , then the following inequality for fractional integrals holds:*

$$(2.11) \quad \begin{aligned} & \left| \frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} [J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a)] \right| \\ & \leq \frac{(b-a)(2^{\alpha+s}|f'(b)| - |f'(a)| - |f'(b)|)}{2^{\alpha+s+1}(\alpha+s+1)} \\ & \quad + (b-a)|f'(a)| [0.5B(s+1, \alpha+1) - B_{0.5}(\alpha+1, s+1)] \\ & \quad + (b-a)|f'(b)| [B_{0.5}(s+1, \alpha+1) - 0.5B(a+1, \alpha+1)]. \end{aligned}$$

**Theorem 2.2.** *Let  $f : [0, b] \rightarrow R$  be a differentiable mapping. If  $|f''|$  is measurable and  $|f''|$  is decreasing and geometric-arithmetically  $s$ -convex on  $[0, b]$  for some fixed  $\alpha \in (0, \infty)$ ,  $s \in (0, 1]$ ,  $0 \leq a < b$ , then the following inequality for fractional integrals holds:*

$$(2.12) \quad \begin{aligned} & \left| \frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} [J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a)] \right| \\ & \leq \frac{(b-a)^2(|f''(a)|+|f''(b)|)}{2(\alpha+1)} \left( \frac{1}{s+1} - \frac{1}{\alpha+s+1} - B(s+1, \alpha+2) \right). \end{aligned}$$

### 3. MAIN RESULTS

**Lemma 3.1.** *Let  $I$  be an interval  $a, b \in I$  with  $0 \leq a < b$  and  $\varphi : I \rightarrow \mathbb{R}$  a continuous increasing function. Let  $f : [\varphi(a), \varphi(b)] \rightarrow \mathbb{R}$  be a twice differentiable mapping on  $(\varphi(a), \varphi(b))$ . If  $f'' \in L[\varphi(a), \varphi(b)]$ , then the following equality for fractional integral holds:*

$$(3.1) \quad \begin{aligned} & \frac{f(\varphi(a))+f(\varphi(b))}{2} - \frac{\Gamma(\alpha+1)}{2(\varphi(b)-\varphi(a))^\alpha} [J_{\varphi(a)+}^\alpha f(\varphi(b)) + J_{\varphi(b)-}^\alpha f(\varphi(a))] \\ & = \frac{(\varphi(b)-\varphi(a))^2}{2} \int_0^1 \left( \frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{\alpha+1} \right) f''(t\varphi(a) + (1-t)\varphi(b)) dt. \end{aligned}$$

*Proof.* By using Lemma 3 and Definition 5, we have

$$\begin{aligned} I &= \int_0^1 \left( \frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{\alpha+1} \right) (f''(t\varphi(a) + (1-t)\varphi(b))) dt \\ &= \frac{1}{\varphi(b)-\varphi(a)} \left[ \int_0^1 ((1-t)^\alpha - t^\alpha) f'(t\varphi(a) + (1-t)\varphi(b)) dt \right] \\ &= \frac{f(\varphi(a))+f(\varphi(b))}{(\varphi(b)-\varphi(a))^2} - \frac{\alpha}{(\varphi(b)-\varphi(a))^2} \\ & \quad \times \left( \int_0^1 (1-t)^{\alpha-1} f(t\varphi(a) + (1-t)\varphi(b)) dt \right. \\ & \quad \left. + \int_0^1 t^{\alpha-1} f(t\varphi(a) + (1-t)\varphi(b)) dt \right) \\ &= \frac{f(\varphi(a))+f(\varphi(b))}{(\varphi(b)-\varphi(a))^2} - \frac{\alpha\Gamma(\alpha)}{(\varphi(b)-\varphi(a))^{\alpha+2}} \\ & \quad \times \left( \int_{\varphi(a)}^{\varphi(b)} (\varphi(x) - \varphi(a))^{\alpha-1} f(\varphi(x)) d\varphi(x) \right. \\ & \quad \left. + \int_{\varphi(a)}^{\varphi(b)} (\varphi(b) - \varphi(x))^{\alpha-1} f(\varphi(x)) d\varphi(x) \right) \\ &= \frac{f(\varphi(a))+f(\varphi(b))}{(\varphi(b)-\varphi(a))^2} - \frac{\Gamma(\alpha+1)}{(\varphi(b)-\varphi(a))^{\alpha+2}} [J_{\varphi(a)+}^\alpha f(\varphi(b)) + J_{\varphi(b)-}^\alpha f(\varphi(a))]. \end{aligned}$$

If  $I$  by multiplying  $\frac{(\varphi(b)-\varphi(a))^2}{2}$ , it obtain that

$$\begin{aligned} & \frac{(\varphi(b)-\varphi(a))^2}{2} \int_0^1 \left( \frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{\alpha+1} \right) f'' t\varphi(a) + (1-t)\varphi(b) dt \\ & = \frac{f(\varphi(a))+f(\varphi(b))}{2} - \frac{\Gamma(\alpha+1)}{2(\varphi(b)-\varphi(a))^\alpha} \left[ J_{\varphi(a)^+}^\alpha f(\varphi(b)) + J_{\varphi(b)^-}^\alpha f(\varphi(a)) \right]. \end{aligned}$$

The proof is done. □

**Lemma 3.2.** *Let  $I$  be an interval  $a, b \in I$  with  $0 \leq a < b$  and  $\varphi : I \rightarrow \mathbb{R}$  a continuous increasing function. Let  $f : [\varphi(a), \varphi(b)] \rightarrow \mathbb{R}$  be a twice differentiable mapping on  $(\varphi(a), \varphi(b))$ . If  $f'' \in L[\varphi(a), \varphi(b)]$ , then the following equality for fractional integral holds:*

$$\begin{aligned} (3.2) \quad & \frac{\Gamma(\alpha+1)}{2(\varphi(b)-\varphi(a))^\alpha} \left[ J_{\varphi(a)^+}^\alpha f(\varphi(b)) + J_{\varphi(b)^-}^\alpha f(\varphi(a)) \right] - f\left(\frac{\varphi(a)+\varphi(b)}{2}\right) \\ & = \frac{(\varphi(b)-\varphi(a))^2}{2} \int_0^1 m(t) f''(t\varphi(a) + (1-t)\varphi(b)) dt, \end{aligned}$$

where

$$m(t) = \begin{cases} t - \frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{\alpha+1}, & t \in [0, \frac{1}{2}), \\ 1-t - \frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{\alpha+1}, & t \in [\frac{1}{2}, 1]. \end{cases}$$

*Proof.* By using Lemma 4 and Definition 5, we have:

$$\begin{aligned} I & = \int_0^1 m(t) f''(t\varphi(a) + (1-t)\varphi(b)) dt \\ & = \int_0^{\frac{1}{2}} \left( t - \frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{\alpha+1} \right) (f''(t\varphi(a) + (1-t)\varphi(b))) dt \\ & \quad + \int_{\frac{1}{2}}^1 \left( 1-t - \frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{\alpha+1} \right) (f''(t\varphi(a) + (1-t)\varphi(b))) dt \\ & = I_1 + I_2. \end{aligned}$$

If use twice the partial integration method for  $I_1$ , we have

$$\begin{aligned} I_1 & = \int_0^{\frac{1}{2}} \left( t - \frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{\alpha+1} \right) f''(t\varphi(a) + (1-t)\varphi(b)) dt \\ & = \frac{(\alpha-1+2^{1-\alpha})f'(\frac{\varphi(a)+\varphi(b)}{2})}{2(\alpha+1)(\varphi(a)-\varphi(b))} - \frac{f(\frac{\varphi(a)+\varphi(b)}{2})}{(\varphi(a)-\varphi(b))^2} + \frac{\Gamma(\alpha+1)}{(\varphi(a)-\varphi(b))^{\alpha+2}} \\ & \quad \times \left[ \int_{\frac{\varphi(a)+\varphi(b)}{2}}^{\varphi(b)} (\varphi(x) - \varphi(a))^{\alpha-1} f(\varphi(x)) d\varphi(x) \right. \\ & \quad \left. + \int_{\frac{\varphi(a)+\varphi(b)}{2}}^{\varphi(a)} (\varphi(b) - \varphi(x))^{\alpha-1} f(\varphi(x)) d\varphi(x) \right]. \end{aligned}$$

If use twice the partial integration method for  $I_2$ , we have

$$\begin{aligned} I_2 & = \int_{\frac{1}{2}}^1 \left( 1-t - \frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{\alpha+1} \right) f''(t\varphi(a) + (1-t)\varphi(b)) dt \\ & = - \left( \frac{(\alpha-1+2^{1-\alpha})f'(\frac{\varphi(a)+\varphi(b)}{2})}{2(\alpha+1)(\varphi(a)-\varphi(b))} \right) - \frac{f(\frac{\varphi(a)+\varphi(b)}{2})}{(\varphi(a)-\varphi(b))^2} \\ & \quad + \frac{\Gamma(\alpha+1)}{(\varphi(a)-\varphi(b))^{\alpha+2}} \left[ \int_{\varphi(a)}^{\frac{\varphi(a)+\varphi(b)}{2}} (\varphi(x) - \varphi(a))^{\alpha-1} f(\varphi(x)) d\varphi(x) \right. \\ & \quad \left. + \int_{\varphi(a)}^{\frac{\varphi(a)+\varphi(b)}{2}} (\varphi(b) - \varphi(x))^{\alpha-1} f(\varphi(x)) d\varphi(x) \right], \end{aligned}$$

by add uping  $I_1$  and  $I_2$ , and by multiplying  $\frac{(\varphi(b)-\varphi(a))^2}{2}$  with  $I$ , it obtain that:

$$\begin{aligned} & \frac{\Gamma(\alpha+1)}{2(\varphi(b)-\varphi(a))^\alpha} \left[ J_{\varphi(a)^+}^\alpha f(\varphi(b)) + J_{\varphi(b)^-}^\alpha f(\varphi(a)) \right] - f\left(\frac{\varphi(a)+\varphi(b)}{2}\right) \\ & = \frac{(\varphi(b)-\varphi(a))^2}{2} \int_0^1 m(t) f''(t\varphi(a) + (1-t)\varphi(b)) dt. \end{aligned}$$

The proof is done.

**Lemma 3.3.** *Let  $I$  be an interval  $a, b \in I$  with  $0 \leq a < b$  and  $\varphi : I \rightarrow \mathbb{R}$  a continuous increasing function. Let  $f : [\varphi(a), \varphi(b)] \rightarrow \mathbb{R}$  be a twice differentiable mapping on  $(\varphi(a), \varphi(b))$ . If  $f'' \in L[\varphi(a), \varphi(b)]$ , then the following equality for fractional integral holds:*

$$(3.3) \quad \begin{aligned} & \frac{f(\varphi(a))+f(\varphi(b))}{r(r+1)} + \frac{2}{r+1} f\left(\frac{\varphi(a)+\varphi(b)}{2}\right) - \frac{\Gamma(\alpha+1)}{r(\varphi(b)-\varphi(a))^\alpha} \\ & \quad \times \left[ J_{\varphi(a)+}^\alpha f(\varphi(b)) + J_{\varphi(b)-}^\alpha f(\varphi(a)) \right] \\ & = (\varphi(b) - \varphi(a))^2 \int_0^1 k(t) f''(t\varphi(a) + (1-t)\varphi(b)) dt. \end{aligned}$$

Where

$$k(t) = \begin{cases} \frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{\alpha+1} - \frac{t}{r+1}, & t \in [0, \frac{1}{2}), \\ \frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{\alpha+1} - \frac{1-t}{r+1}, & t \in [\frac{1}{2}, 1]. \end{cases}$$

□

*Proof.* By using Definition 5 and Lemma 5, we have

$$\begin{aligned} I &= \int_0^1 k(t) f''(t\varphi(a) + (1-t)\varphi(b)) dt \\ &= \frac{1}{r(r+1)(\alpha+1)} \\ & \quad \times \int_0^{\frac{1}{2}} \left[ (r+1) \left[ 1 - (1-t)^{\alpha+1} - t^{\alpha+1} \right] - rt(\alpha+1) \right] \\ & \quad \times f''(t\varphi(a) + (1-t)\varphi(b)) dt \\ & \quad + \frac{1}{r(r+1)(\alpha+1)} \\ & \quad \times \int_{\frac{1}{2}}^1 \left[ (r+1) \left[ 1 - (1-t)^{\alpha+1} - t^{\alpha+1} \right] - r(1-t)(\alpha+1) \right] \\ & \quad \times f''(t\varphi(a) + (1-t)\varphi(b)) dt \\ & = I_1 + I_2. \end{aligned}$$

If use twice the partial integration method for  $I_1$ , we have

$$\begin{aligned} I_1 &= \int_0^{\frac{1}{2}} \left[ (r+1) \left[ 1 - (1-t)^{\alpha+1} - t^{\alpha+1} \right] - rt(\alpha+1) \right] \\ & \quad \times f''(t\varphi(a) + (1-t)\varphi(b)) dt \\ &= \left[ (r+1)(1-2^{-\alpha}) - \frac{r(\alpha+1)}{2} \right] \frac{f'(\frac{\varphi(a)+\varphi(b)}{2})}{\varphi(a)-\varphi(b)} \\ & \quad + \frac{r(\alpha+1)f(\frac{\varphi(a)+\varphi(b)}{2})}{(\varphi(b)-\varphi(a))^2} + \frac{(\alpha+1)f(\varphi(b))}{(\varphi(b)-\varphi(a))^2} - \frac{(r+1)(\alpha+1)\alpha\Gamma(\alpha)}{(\varphi(b)-\varphi(a))^{\alpha+2}} \\ & \quad \times \left[ \int_{\frac{\varphi(a)+\varphi(b)}{2}}^{\varphi(b)} (\varphi(x) - \varphi(a))^{\alpha-1} f(\varphi(x)) d\varphi(x) \right. \\ & \quad \left. + \int_{\frac{\varphi(a)+\varphi(b)}{2}}^{\varphi(b)} (\varphi(b) - \varphi(x))^{\alpha-1} f(\varphi(x)) d\varphi(x) \right] \end{aligned}$$

If use twice the partial integration method for  $I_2$ , we have

$$\begin{aligned} I_2 &= \int_{\frac{1}{2}}^1 \left[ (r+1) \left[ 1 - (1-t)^{\alpha+1} - t^{\alpha+1} \right] - r(1-t)(\alpha+1) \right] \\ & \quad \times f''(t\varphi(a) + (1-t)\varphi(b)) dt \\ &= \left[ \frac{r(\alpha+1)}{2} - (r+1)(1-2^{-\alpha}) \right] \frac{f'(\frac{\varphi(a)+\varphi(b)}{2})}{\varphi(a)-\varphi(b)} \\ & \quad + \frac{r(\alpha+1)f(\frac{\varphi(a)+\varphi(b)}{2})}{(\varphi(b)-\varphi(a))^2} + \frac{(\alpha+1)f(\varphi(a))}{(\varphi(b)-\varphi(a))^2} - \frac{(r+1)(\alpha+1)\alpha\Gamma(\alpha)}{(\varphi(b)-\varphi(a))^{\alpha+2}} \\ & \quad \times \left[ \int_{\varphi(a)}^{\frac{\varphi(a)+\varphi(b)}{2}} (\varphi(x) - \varphi(a))^{\alpha-1} f(\varphi(x)) d\varphi(x) \right. \\ & \quad \left. + \int_{\varphi(a)}^{\frac{\varphi(a)+\varphi(b)}{2}} (\varphi(b) - \varphi(x))^{\alpha-1} f(\varphi(x)) d\varphi(x) \right]. \end{aligned}$$

By add uping  $I_1$  and  $I_2$ , and by multiplying with  $\frac{(\varphi(b)-\varphi(a))^2}{r(r+1)(\alpha+1)}$ , it obtain that:

$$\begin{aligned} & \frac{f(\varphi(a))+f(\varphi(b))}{r(r+1)} + \frac{2}{r+1} f\left(\frac{\varphi(a)+\varphi(b)}{2}\right) \\ & - \frac{\Gamma(\alpha+1)}{r(\varphi(b)-\varphi(a))^\alpha} \left[ J_{\varphi(a)^+}^\alpha f(\varphi(b)) + J_{\varphi(b)^-}^\alpha f(\varphi(a)) \right] \\ & = (\varphi(b) - \varphi(a))^2 \int_0^1 k(t) f''(t\varphi(a) + (1-t)\varphi(b)) dt . \end{aligned}$$

The proof is done.

**Theorem 3.1.** *Let  $I$  be an interval  $a, b \in I$  with  $0 \leq a < b$  and  $\varphi : I \rightarrow \mathbb{R}$  a continuous increasing function. Let  $f : [0, \varphi(b)] \rightarrow \mathbb{R}$  be a differentiable mapping and  $1 < q < \infty$ . If  $|f''|$  is measurable and  $|f''|$  is decreasing and geometric-arithmetically  $\varphi - s$ -convex on  $[0, \varphi(b)]$  for some fixed  $\alpha \in (0, \infty)$ ,  $s \in (0, 1]$ ,  $0 \leq a < b$ , then the following inequality for fractional integrals holds:*

$$\begin{aligned} (3.4) \quad & \frac{f(\varphi(a))+f(\varphi(b))}{2} - \frac{\Gamma(\alpha+1)}{2(\varphi(b)-\varphi(a))^\alpha} \left[ J_{\varphi(a)^+}^\alpha f(\varphi(b)) + J_{\varphi(b)^-}^\alpha f(\varphi(a)) \right] \\ & \leq \frac{(\varphi(b)-\varphi(a))^2 (|f''(\varphi(a))| + |f''(\varphi(b))|)}{2(\alpha+1)} \left( \frac{1}{s+1} - \frac{1}{\alpha+s+1} - B(s+1, \alpha+2) \right) . \end{aligned}$$

□

*Proof.* By using Definition 2, Lemma 2 and Lemma 6, we have

$$\begin{aligned} & \left| \frac{f(\varphi(a))+f(\varphi(b))}{2} - \frac{\Gamma(\alpha+1)}{2(\varphi(b)-\varphi(a))^\alpha} \left[ J_{\varphi(a)^+}^\alpha f(\varphi(b)) + J_{\varphi(b)^-}^\alpha f(\varphi(a)) \right] \right| \\ & \leq \frac{(\varphi(b)-\varphi(a))^2}{2} \int_0^1 \left| \frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{\alpha+1} \right| |f''(t\varphi(a) + (1-t)\varphi(b))| dt \\ & \leq \frac{(\varphi(b)-\varphi(a))^2}{2(\alpha+1)} \int_0^1 \left( 1 - (1-t)^{\alpha+1} - t^{\alpha+1} \right) |f''(\varphi^t(a) + \varphi^{(1-t)}(b))| dt \\ & \leq \frac{(\varphi(b)-\varphi(a))^2}{2(\alpha+1)} \int_0^1 \left( 1 - (1-t)^{\alpha+1} - t^{\alpha+1} \right) \\ & \quad \times [t^s |f''\varphi(a)| + (1-t)^s |f''\varphi(b)|] dt \\ & \leq \frac{(\varphi(b)-\varphi(a))^2}{2(\alpha+1)} |f''\varphi(a)| \\ & \quad \times \left[ -\int_0^1 t^{\alpha+s+1} dt + \int_0^1 t^s dt - \int_0^1 t^s (1-t)^{\alpha+1} dt \right] \\ & \quad + \frac{(\varphi(b)-\varphi(a))^2}{2(\alpha+1)} |f''\varphi(b)| \\ & \quad \times \left[ -\int_0^1 (1-t)^{\alpha+s+1} dt + \int_0^1 (1-t)^s dt - \int_0^1 t^{\alpha+1} (1-t)^s dt \right] \\ & \leq \frac{(\varphi(b)-\varphi(a))^2}{2(\alpha+1)} (|f''\varphi(a)| + |f''\varphi(b)|) \\ & \quad \times \left[ -\frac{1}{\alpha+s+1} + \frac{1}{s+1} - B(s+1, \alpha+2) \right] . \end{aligned}$$

The poof is done.

**Theorem 3.2.** *Let  $I$  be an interval  $a, b \in I$  with  $0 \leq a < b$  and  $\varphi : I \rightarrow \mathbb{R}$  a continuous increasing function. Let  $f : [0, \varphi(b)] \rightarrow \mathbb{R}$  be a differentiable mapping and  $1 < q < \infty$ . If  $|f''|^q$  is measurable and  $|f''|^q$  is decreasing and geometric-arithmetically  $\varphi - s$ -convex on  $[0, \varphi(b)]$  for some fixed  $\alpha \in (0, \infty)$ ,  $s \in (0, 1]$ ,  $0 \leq a < b$ , then the following inequality for fractional integrals holds:*

$$\begin{aligned} (3.5) \quad & \left| \frac{f(\varphi(a))+f(\varphi(b))}{2} - \frac{\Gamma(\alpha+1)}{2(\varphi(b)-\varphi(a))^\alpha} \left[ J_{\varphi(a)^+}^\alpha f(\varphi(b)) + J_{\varphi(b)^-}^\alpha f(\varphi(a)) \right] \right| \\ & \leq \frac{(\varphi(b)-\varphi(a))^2 \max\{1-2^{1-\alpha}, 2^{1-\alpha}-1\}}{2(\alpha+1)} \left( \frac{|f''(\varphi(a))|^q + |f''(\varphi(b))|^q}{s+1} \right) . \end{aligned}$$

□

*Proof.* To achieve our aim ,we divide our proof into two cases.

Case 1:  $\alpha \in (0, 1)$ , by using Definition 2 , Hölder's inequality and Lemma 6 ,we have

$$\begin{aligned}
& \left| \frac{f(\varphi(a))+f(\varphi(b))}{2} - \frac{\Gamma(\alpha+1)}{2(\varphi(b)-\varphi(a))^\alpha} \left[ J_{\varphi(a)^+}^\alpha f(\varphi(b)) + J_{\varphi(b)^-}^\alpha f(\varphi(a)) \right] \right| \\
& \leq \frac{(\varphi(b)-\varphi(a))^2}{2} \int_0^1 \left| \frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{\alpha+1} \right| |f''(t\varphi(a) + (1-t)\varphi(b))| dt \\
& \leq \frac{(\varphi(b)-\varphi(a))^2}{2(\alpha+1)} \left( \int_0^1 |1-(1-t)^{\alpha+1}-t^{\alpha+1}|^p dt \right)^{\frac{1}{p}} \\
& \quad \times \left( \int_0^1 |f''(t\varphi(a) + (1-t)\varphi(b))|^q dt \right)^{\frac{1}{q}} \\
& \leq \frac{(\varphi(b)-\varphi(a))^2}{2(\alpha+1)} \left( \int_0^1 |1-(1-t)^{\alpha+1}-t^{\alpha+1}|^p dt \right)^{\frac{1}{p}} \\
& \quad \times \left( \int_0^1 |f''(\varphi^t(a)\varphi^{1-t}(b))|^q dt \right)^{\frac{1}{q}} \\
& \leq \frac{(\varphi(b)-\varphi(a))^2}{2(\alpha+1)} \left( \int_0^1 |1-(1-t)^{\alpha+1}-t^{\alpha+1}|^p dt \right)^{\frac{1}{p}} \\
& \quad \times \left( \int_0^1 [t^s |f''(\varphi(a))| + (1-t)^s |f''(\varphi(b))|] dt \right)^{\frac{1}{q}} \\
& \leq \frac{(\varphi(b)-\varphi(a))^2}{2(\alpha+1)} \left( \frac{|f''(\varphi(a))|^q + |f''(\varphi(b))|^q}{s+1} \right)^{\frac{1}{q}} \left( \int_0^1 [(1-t)^\alpha + t^\alpha - 1]^p dt \right)^{\frac{1}{p}} \\
& \leq \frac{(\varphi(b)-\varphi(a))^2}{2(\alpha+1)} \left( \frac{|f''(\varphi(a))|^q + |f''(\varphi(b))|^q}{s+1} \right)^{\frac{1}{q}} \left( \int_0^1 [2^{1-\alpha} - 1]^p dt \right)^{\frac{1}{p}} \\
& \leq \frac{(\varphi(b)-\varphi(a))^2 (2^{1-\alpha} - 1)}{2(\alpha+1)} \left( \frac{|f''(\varphi(a))|^q + |f''(\varphi(b))|^q}{s+1} \right)^{\frac{1}{q}} .
\end{aligned}$$

Where  $\frac{1}{p} + \frac{1}{q} = 1$ .

Case 2:  $\alpha \in [1, \infty)$ , by using Definition 2, Hölder's inequality and Lemma 6, we have

$$\begin{aligned}
& \left| \frac{f(\varphi(a))+f(\varphi(b))}{2} - \frac{\Gamma(\alpha+1)}{2(\varphi(b)-\varphi(a))^\alpha} \left[ J_{\varphi(a)^+}^\alpha f(\varphi(b)) + J_{\varphi(b)^-}^\alpha f(\varphi(a)) \right] \right| \\
& \leq \frac{(\varphi(b)-\varphi(a))^2}{2(\alpha+1)} \left( \frac{|f''(\varphi(a))|^q + |f''(\varphi(b))|^q}{s+1} \right)^{\frac{1}{q}} \\
& \quad \times \left( \int_0^1 [1-(1-t)^\alpha - t^\alpha]^p dt \right)^{\frac{1}{p}} \\
& \leq \frac{(\varphi(b)-\varphi(a))^2 (1-2^{1-\alpha})}{2(\alpha+1)} \left( \frac{|f''(\varphi(a))|^q + |f''(\varphi(b))|^q}{s+1} \right)^{\frac{1}{q}} .
\end{aligned}$$

The proof is done.

**Theorem 3.3.** Let  $I$  be an interval  $a, b \in I$  with  $0 \leq a < b$  and  $\varphi : I \rightarrow \mathbb{R}$  a continuous increasing function. Let  $f : [0, \varphi(b)] \rightarrow \mathbb{R}$  be a differentiable mapping and  $1 < q < \infty$ . If  $|f''|$  is measurable and  $|f''|$  is decreasing and geometric-arithmetically  $\varphi - s$ -convex on  $[0, \varphi(b)]$  for some fixed  $\alpha \in (0, \infty)$ ,  $s \in (0, 1]$ ,  $0 \leq a < b$ , then the

following inequality for fractional integrals holds:

$$\begin{aligned}
 & \left| \frac{\Gamma(\alpha+1)}{2(\varphi(b)-\varphi(a))^\alpha} \left[ J_{\varphi(a)^+}^\alpha f(\varphi(b)) + J_{\varphi(b)^-}^\alpha f(\varphi(a)) \right] - f\left(\frac{\varphi(a)+\varphi(b)}{2}\right) \right| \\
 & \leq \frac{(\varphi(b)-\varphi(a))^2 |f''(\varphi(a))|}{2(\alpha+1)} \\
 (3.6) \quad & \times \left[ \frac{\alpha - \alpha 2^{-s-1} - 2^{-s-1}}{1+s} - \frac{\alpha+1}{2+s} + 2B(s+1, \alpha+2) + \frac{1}{\alpha+s+2} \right] \\
 & + \frac{(\varphi(b)-\varphi(a))^2 |f''(\varphi(b))|}{2(\alpha+1)} \\
 & \times \left[ \frac{\alpha 2^{-s-1} + 2^{-s-1} - 1}{1+s} + \frac{1}{\alpha+s+2} + 2B(s+1, \alpha+2) \right].
 \end{aligned}$$

□

*Proof.* By using Definition 5 and Lemma 7, we have

$$\begin{aligned}
 & \left| \frac{\Gamma(\alpha+1)}{2(\varphi(b)-\varphi(a))^\alpha} \left[ J_{\varphi(a)^+}^\alpha f(\varphi(b)) + J_{\varphi(b)^-}^\alpha f(\varphi(a)) \right] - f\left(\frac{\varphi(a)+\varphi(b)}{2}\right) \right| \\
 & \leq \frac{(\varphi(b)-\varphi(a))^2}{2} \int_0^1 |m(t)| |f''(t\varphi(a) + (1-t)\varphi(b))| dt \\
 & \leq \frac{(\varphi(b)-\varphi(a))^2}{2} \int_0^1 |m(t)| [f''(\varphi^t(a)\varphi^{1-t}(b))] dt \\
 & \leq \frac{(\varphi(b)-\varphi(a))^2}{2} \int_0^1 |m(t)| [t^s |f''(\varphi(a))| + (1-t)^s |f''(\varphi(b))|] dt \\
 & \leq \frac{(\varphi(b)-\varphi(a))^2}{2} \int_0^{\frac{1}{2}} \left| t - \frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{\alpha+1} \right| [t^s |f''(\varphi(a))| + (1-t)^s |f''(\varphi(b))|] dt \\
 & \quad + \frac{(\varphi(b)-\varphi(a))^2}{2} \int_{\frac{1}{2}}^1 \left| 1-t - \frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{\alpha+1} \right| [t^s |f''(\varphi(a))| + (1-t)^s |f''(\varphi(b))|] dt \\
 & \leq \frac{(\varphi(b)-\varphi(a))^2}{2(\alpha+1)} |f''(\varphi(a))| \left[ \int_0^{\frac{1}{2}} [-t^s + (\alpha+1)t^{s+1} + t^s(1-t)^{\alpha+1} + t^{\alpha+s+1}] dt \right. \\
 & \quad \left. + \int_{\frac{1}{2}}^1 [\alpha t^s - (\alpha+1)t^{s+1} + t^s(1-t)^{\alpha+1} + t^{\alpha+s+1}] dt \right] + \frac{(\varphi(b)-\varphi(a))^2}{2(\alpha+1)} |f''(\varphi(b))| \\
 & \quad \times \left[ \int_0^{\frac{1}{2}} [-(1-t)^s + (\alpha+1)t(1-t)^{s+1} + (1-t)^{\alpha+s+1} + t^{\alpha+s}(1-t)^s] dt \right. \\
 & \quad \left. + \int_{\frac{1}{2}}^1 [\alpha(1-t)^s - (\alpha+1)t(1-t)^s + (1-t)^{\alpha+s+1} + t^{\alpha+1}(1-t)^s] dt \right] \\
 & \leq \frac{(\varphi(b)-\varphi(a))^2}{2(\alpha+1)} |f''(\varphi(a))| \left[ \frac{\alpha - \alpha 2^{-s-1} - 2^{-s-1}}{1+s} - \frac{\alpha+1}{s+2} + 2B(s+1, \alpha+2) + \frac{1}{\alpha+s+2} \right] \\
 & \quad + \frac{(\varphi(b)-\varphi(a))^2}{2(\alpha+1)} |f''(\varphi(b))| \left[ \frac{\alpha 2^{-s-1} + 2^{-s-1} - 1}{1+s} + 2B(\alpha+2, s+1) + \frac{1}{\alpha+s+2} \right].
 \end{aligned}$$

**Theorem 3.4.** Let  $I$  be an interval  $a, b \in I$  with  $0 \leq a < b$  and  $\varphi : I \rightarrow \mathbb{R}$  a continuous increasing function. Let  $f : [0, \varphi(b)] \rightarrow \mathbb{R}$  be a differentiable mapping and  $1 < q < \infty$ . If  $|f''|$  is measurable and  $|f''|^q$  is decreasing and geometric-arithmetically  $\varphi - s$ -convex on  $[0, \varphi(b)]$  for some fixed  $\alpha \in (0, \infty)$ ,  $s \in (0, 1)$ ,  $0 \leq a < b$ , then the following inequality for fractional integrals holds:

$$\begin{aligned}
 (3.7) \quad & \frac{\Gamma(\alpha+1)}{2(\varphi(b)-\varphi(a))^\alpha} \left[ J_{\varphi(a)^+}^\alpha f(\varphi(b)) + J_{\varphi(b)^-}^\alpha f(\varphi(a)) \right] - f\left(\frac{\varphi(a)+\varphi(b)}{2}\right) \\
 & \leq \frac{(\varphi(b)-\varphi(a))^2}{2(\alpha+1)} \left( \frac{|f''(\varphi(a))|^q + |f''(\varphi(b))|^q}{s+1} \right)^{\frac{1}{q}} \left( \frac{(\alpha+1)2^{-p-1} + (\alpha+0.5)^{p+1} - \alpha^{p+1}}{p+1} \right)^{\frac{1}{p}}.
 \end{aligned}$$

Where  $\frac{1}{p} + \frac{1}{q} = 1$ .

□



*Proof.* By using Lemma 2, Hölder's inequality and Lemma 7, we have

$$\begin{aligned}
& \left| \frac{\Gamma(\alpha+1)}{2(\varphi(b)-\varphi(a))^\alpha} \left[ J_{\varphi(a)+}^\alpha f(\varphi(b)) + J_{\varphi(b)-}^\alpha f(\varphi(a)) \right] - f\left(\frac{\varphi(a)+\varphi(b)}{2}\right) \right| \\
& \leq \frac{(\varphi(b)-\varphi(a))^2}{2} \int_0^1 |m(t)| |f''(t\varphi(a) + (1-t)\varphi(b))| dt \\
& \leq \frac{(\varphi(b)-\varphi(a))^2}{2} \left( \int_0^1 |m(t)|^p dt \right)^{\frac{1}{p}} \\
& \quad \times \left( \int_0^1 |f''(t\varphi(a) + (1-t)\varphi(b))|^q dt \right)^{\frac{1}{q}} \\
& \leq \frac{(\varphi(b)-\varphi(a))^2}{2} \left( \int_0^1 |m(t)|^p dt \right)^{\frac{1}{p}} \\
& \quad \times \left( \int_0^1 |f''(\varphi^t(a)\varphi^{(1-t)}(b))|^q dt \right)^{\frac{1}{q}} \\
& \leq \frac{(\varphi(b)-\varphi(a))^2}{2} \left( \int_0^1 |m(t)|^p dt \right)^{\frac{1}{p}} \\
& \quad \times \left( \int_0^1 [t^s |f''(\varphi(a))|^q + (1-t)^s |f''(\varphi(b))|^q] dt \right)^{\frac{1}{q}} \\
& \leq \frac{(\varphi(b)-\varphi(a))^2}{2(\alpha+1)} \left( \frac{|f''(\varphi(a))|^q + |f''(\varphi(b))|^q}{s+1} \right)^{\frac{1}{q}} \\
& \quad \times \left( \int_0^{\frac{1}{2}} \left| t - \frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{\alpha+1} \right|^p dt + \int_{\frac{1}{2}}^1 \left| 1-t - \frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{\alpha+1} \right|^p dt \right)^{\frac{1}{p}} \\
& \leq \frac{(\varphi(b)-\varphi(a))^2}{2(\alpha+1)} \left( \frac{|f''(\varphi(a))|^q + |f''(\varphi(b))|^q}{s+1} \right)^{\frac{1}{q}} \\
& \quad \times \left( (\alpha+1) \int_0^{\frac{1}{2}} t^p dt + \int_{\frac{1}{2}}^1 (\alpha-t+1)^p dt \right)^{\frac{1}{p}} \\
& \leq \frac{(\varphi(b)-\varphi(a))^2}{2(\alpha+1)} \left( \frac{|f''(\varphi(a))|^q + |f''(\varphi(b))|^q}{s+1} \right)^{\frac{1}{q}} \left( \frac{(\alpha+1)2^{-p-1} + (\alpha+0.5)^{p+1} - \alpha^{p+1}}{p+1} \right)^{\frac{1}{p}}.
\end{aligned}$$

The proof is done.

**Theorem 3.5.** *Let  $I$  be an interval  $a, b \in I$  with  $0 \leq a < b$  and  $\varphi : I \rightarrow \mathbb{R}$  a continuous increasing function. Let  $f : [0, \varphi(b)] \rightarrow \mathbb{R}$  be a differentiable mapping and  $1 < q < \infty$ . If  $|f''|$  is measurable and  $|f''|$  is decreasing and geometric-arithmetically  $\varphi - s$ -convex on  $[0, \varphi(b)]$  for some fixed  $\alpha \in (0, \infty)$ ,  $s \in (0, 1]$ ,  $0 \leq a < b$ , then the*

following inequality for fractional integrals holds:

$$\begin{aligned}
 (3.8) \quad & \left| \frac{f(\varphi(a))+f(\varphi(b))}{r(r+1)} + \frac{2}{r+1} f\left(\frac{\varphi(a)+\varphi(b)}{2}\right) - \frac{\Gamma(\alpha+1)}{r(\varphi(b)-\varphi(a))^\alpha} \left[ J_{\varphi(a)+}^\alpha f(\varphi(b)) + J_{\varphi(b)-}^\alpha f(\varphi(a)) \right] \right| \\
 & \leq \frac{(\varphi(b)-\varphi(a))^2}{r(r+1)(\alpha+1)} \max \left\{ [r+1 - (r+1)2^{-\alpha}] \left( \frac{2^{-s-1}|f''(\varphi(a))| + (1-2^{-s-1})|f''(\varphi(b))|}{s+1} \right) \right. \\
 & \quad - r(\alpha+1) \left[ \frac{2^{-s-2}(s+1)|f''(\varphi(a))| + (1-t)^{s+1}(-ts-3t+1)f''(\varphi(a))}{(s+1)(s+2)} \right], \\
 & \quad \left. r(\alpha+1) \left[ \frac{2^{-s-2}(s+1)|f''(\varphi(a))| + (1-t)^{s+1}(-ts-3t+1)f''(\varphi(a))}{(s+1)(s+2)} \right] \right\} \\
 & \quad + \frac{(\varphi(b)-\varphi(a))^2}{r(r+1)(\alpha+1)} \max \{ (r+1 - (r+1)2^{-\alpha} - r(\alpha+1)) \\
 & \quad \times \left[ \frac{(1-2^{-s-1})|f''(\varphi(a))| - 2^{-s-1}|f''(\varphi(b))|}{s+1} \right] \\
 & \quad + r(\alpha+1) \left[ \frac{(1-2^{-s-1})(s+1)|f''(\varphi(a))| + 2^{-s-2}(s+3)|f''(\varphi(b))|}{(s+1)(s+2)} \right], \\
 & \quad r(\alpha+1) \left[ \frac{(1-2^{-s-1})|f''(\varphi(a))| - 2^{-s-1}|f''(\varphi(b))|}{s+1} \right. \\
 & \quad \left. - \frac{(1-2^{-s-2})(s+1)|f''(\varphi(a))| + 2^{-s-2}(s+3)|f''(\varphi(b))|}{(s+1)(s+2)} \right] \left. \right\}.
 \end{aligned}$$

□

*Proof.* By using Definition 3, Lemma 3 and Lemma 8, we have

$$\begin{aligned}
 & \left| \frac{f(\varphi(a))+f(\varphi(b))}{r(r+1)} + \frac{2}{r+1} f\left(\frac{\varphi(a)+\varphi(b)}{2}\right) - \frac{\Gamma(\alpha+1)}{r(\varphi(b)-\varphi(a))^\alpha} \left[ J_{\varphi(a)+}^\alpha f(\varphi(b)) + J_{\varphi(b)-}^\alpha f(\varphi(a)) \right] \right| \\
 & \leq (\varphi(b) - \varphi(a))^2 \int_0^1 |k(t)| [t^s |f''(\varphi(a))| + (1-t)^s |f''(\varphi(b))|] dt \\
 & \leq \frac{(\varphi(b)-\varphi(a))^2}{r(r+1)(\alpha+1)} \int_0^{\frac{1}{2}} \left| (r+1) \left[ 1 - (1-t)^{\alpha+1} - t^{\alpha+1} \right] - tr(\alpha+1) \right| \\
 & \quad \times [t^s |f''(\varphi(a))| + (1-t)^s |f''(\varphi(b))|] dt \\
 & \quad + \frac{(\varphi(b)-\varphi(a))^2}{r(r+1)(\alpha+1)} \int_{\frac{1}{2}}^1 \left| (r+1) \left[ 1 - (1-t)^{\alpha+1} - t^{\alpha+1} \right] - r(\alpha+1)(1-t) \right| \\
 & \quad \times [t^s |f''(\varphi(a))| + (1-t)^s |f''(\varphi(b))|] dt \\
 & \leq \frac{(\varphi(b)-\varphi(a))^2}{r(r+1)(\alpha+1)} \max \left\{ [r+1 - (r+1)2^{-\alpha}] \left( \frac{2^{-s-1}|f''(\varphi(a))| + (1-2^{-s-1})|f''(\varphi(b))|}{s+1} \right) \right. \\
 & \quad - r(\alpha+1) \left[ \frac{2^{-s-2}(s+1)|f''(\varphi(a))| + (1-t)^{s+1}(-ts-3t+1)f''(\varphi(a))}{(s+1)(s+2)} \right], \\
 & \quad \left. r(\alpha+1) \left[ \frac{2^{-s-2}(s+1)|f''(\varphi(a))| + (1-t)^{s+1}(-ts-3t+1)f''(\varphi(a))}{(s+1)(s+2)} \right] \right\} \\
 & \quad + \frac{(\varphi(b)-\varphi(a))^2}{r(r+1)(\alpha+1)} \max (r+1 - (r+1)2^{-\alpha} - r(\alpha+1)) \left[ \frac{(1-2^{-s-1})|f''(\varphi(a))| - 2^{-s-1}|f''(\varphi(b))|}{s+1} \right] \\
 & \quad + r(\alpha+1) \left[ \frac{(1-2^{-s-1})(s+1)|f''(\varphi(a))| + 2^{-s-2}(s+3)|f''(\varphi(b))|}{(s+1)(s+2)} \right], \\
 & \quad r(\alpha+1) \left[ \frac{(1-2^{-s-1})|f''(\varphi(a))| - 2^{-s-1}|f''(\varphi(b))|}{s+1} \right. \\
 & \quad \left. - \frac{(1-2^{-s-2})(s+1)|f''(\varphi(a))| + 2^{-s-2}(s+3)|f''(\varphi(b))|}{(s+1)(s+2)} \right]
 \end{aligned}$$

Where we have used the following inequality:

$$\begin{aligned} & \int_0^{\frac{1}{2}} \left| (r+1) \left[ 1 - (1-t)^{\alpha+1} - t^{\alpha+1} \right] - tr(\alpha+1) \right| \\ & \quad \times [t^s |f''(\varphi(a))| + (1-t)^s |f''(\varphi(b))|] dt \\ & \leq [(r+1)(1-2^{-\alpha})] \left[ \frac{2^{-s-1} |f''(\varphi(a))| + (1-2^{-s-1}) |f''(\varphi(b))|}{s+1} \right] \\ & \quad - r(\alpha+1) \left[ \frac{2^{-s-2} |f''(\varphi(a))|}{s+2} + \frac{1-(s+3)2^{-s-2} |f''(\varphi(b))|}{(s+2)(s+1)} \right] \end{aligned}$$

and

$$\begin{aligned} & \int_0^{\frac{1}{2}} \left| -r-1 + (r+1) \left[ (1-t)^{\alpha+1} + t^{\alpha+1} \right] + tr(\alpha+1) \right| \\ & \quad \times [t^s |f''(\varphi(a))| + (1-t)^s |f''(\varphi(b))|] dt \\ & \leq r(\alpha+1) \left[ \frac{2^{-s-2} |f''(\varphi(a))|}{s+2} + \frac{1-(s+3)2^{-s-2} |f''(\varphi(b))|}{(s+2)(s+1)} \right], \end{aligned}$$

and

$$\begin{aligned} & \int_{\frac{1}{2}}^1 \left| r+1 + tr(\alpha+1) - (r+1) \left[ (1-t)^{\alpha+1} + t^{\alpha+1} \right] - r(\alpha+1) \right| \\ & \quad \times [t^s |f''(\varphi(a))| + (1-t)^s |f''(\varphi(b))|] dt \\ & \leq [r+1 - (r+1)2^{-\alpha} - r(\alpha+1)] \left[ \frac{(1-2^{-s-1}) |f''(\varphi(a))| - 2^{-s-1} |f''(\varphi(b))|}{s+1} \right] \\ & \quad + r(\alpha+1) \left[ \frac{2^{-s-2} |f''(\varphi(a))|}{s+2} + \frac{(s+3)2^{-s-2} |f''(\varphi(b))|}{(s+2)(s+1)} \right], \end{aligned}$$

and

$$\begin{aligned} & \int_{\frac{1}{2}}^1 \left| -r-1 - tr(\alpha+1) + (r+1) \left[ (1-t)^{\alpha+1} + t^{\alpha+1} \right] + r(\alpha+1) \right| \\ & \quad \times [t^s |f''(\varphi(a))| + (1-t)^s |f''(\varphi(b))|] dt \\ & \leq r(\alpha+1) \\ & \quad \times \left[ \frac{(1-2^{-s-1}) |f''(\varphi(a))| - 2^{-s-1} |f''(\varphi(b))|}{s+1} - \frac{(1-2^{-s-2}) |f''(\varphi(a))|}{s+2} - \frac{(s+3)2^{-s-2} |f''(\varphi(b))|}{(s+2)(s+1)} \right]. \end{aligned}$$

The proof is done.

**Theorem 3.6.** *Let  $I$  be an interval  $a, b \in I$  with  $0 \leq a < b$  and  $\varphi : I \rightarrow \mathbb{R}$  a continuous increasing function. Let  $f : [0, \varphi(b)] \rightarrow \mathbb{R}$  be a differentiable mapping and  $1 < q < \infty$ . If  $|f''|$  is measurable and  $|f''|^q$  is decreasing and geometric-arithmetically  $\varphi - s$ -convex on  $[0, \varphi(b)]$  for some fixed  $\alpha \in (0, \infty)$ ,  $s \in (0, 1)$ ,  $0 \leq a < b$ , then the following inequality for fractional integrals holds:*

$$\begin{aligned} (3.9) \quad & \left| \frac{f(\varphi(a)) + f(\varphi(b))}{r(r+1)} + \frac{2}{r+1} f\left(\frac{\varphi(a) + \varphi(b)}{2}\right) - \frac{\Gamma(\alpha+1)}{r(\varphi(b) - \varphi(a))^\alpha} \left[ J_{\varphi(a)^+}^\alpha f(\varphi(b)) + J_{\varphi(b)^-}^\alpha f(\varphi(a)) \right] \right| \\ & \leq \frac{(\varphi(b) - \varphi(a))^2}{[r(r+1)(\alpha+1)]^{1+p-1}} \left( \frac{|f''(\varphi(a))|^q + |f''(\varphi(b))|^q}{s+1} \right)^{\frac{1}{q}} \\ & \quad \times \left( \max \left\{ [(r+1)(1-2^{-\alpha})]^{p+1} - \left[ \frac{2+r(1-\alpha)-(1+r)2^{-\alpha}}{2} \right]^{p+1}, [r(\alpha+1)]^{p+1} 2^{-p-1} \right\} \right. \\ & \quad \left. + \max \left\{ [r+1 - (r+1)2^{-\alpha}]^{p+1} - \left[ \frac{2+r(1-\alpha)-(1+r)2^{-\alpha}}{2} \right]^{p+1}, \left[ \frac{r(\alpha+1)}{2} \right]^{p+1} \right\} \right)^{\frac{1}{p}}. \end{aligned}$$

Where  $\frac{1}{p} + \frac{1}{q} = 1$ .

□

*Proof.* By using Hölder's inequality and Lemma 7, we have

$$\begin{aligned}
 & \left| \frac{f(\varphi(a))+f(\varphi(b))}{r(r+1)} + \frac{2}{r+1} f\left(\frac{\varphi(a)+\varphi(b)}{2}\right) - \frac{\Gamma(\alpha+1)}{r(\varphi(b)-\varphi(a))^\alpha} \left[ J_{\varphi(a)^+}^\alpha f(\varphi(b)) + J_{\varphi(b)^-}^\alpha f(\varphi(a)) \right] \right| \\
 & \leq (\varphi(b) - \varphi(a))^2 \int_0^1 |k(t)| |f''(t\varphi(a) + (1-t)\varphi(b))| dt \\
 & \leq (\varphi(b) - \varphi(a))^2 \left( \int_0^1 |k(t)|^p dt \right)^{\frac{1}{p}} \left( \int_0^1 |f''(t\varphi(a) + (1-t)\varphi(b))|^q dt \right)^{\frac{1}{q}} \\
 & \leq (\varphi(b) - \varphi(a))^2 \left( \int_0^1 |k(t)|^p dt \right)^{\frac{1}{p}} \left( \int_0^1 |f''(\varphi^t(a)\varphi^{(1-t)}(b))|^q dt \right)^{\frac{1}{q}} \\
 & \leq (\varphi(b) - \varphi(a))^2 \left( \int_0^1 |k(t)|^p dt \right)^{\frac{1}{p}} \left( \int_0^1 [t^s |f''(\varphi(a))|^q + (1-t)^s |f''(\varphi(b))|^q] dt \right)^{\frac{1}{q}} \\
 & \leq (\varphi(b) - \varphi(a))^2 \left( \frac{|f''(\varphi(a))|^q + |f''(\varphi(b))|^q}{s+1} \right)^{\frac{1}{q}} \left( \int_0^1 |k(t)|^p dt \right)^{\frac{1}{p}} \\
 & \leq (\varphi(b) - \varphi(a))^2 \left( \frac{|f''(\varphi(a))|^q + |f''(\varphi(b))|^q}{s+1} \right)^{\frac{1}{q}} \\
 & \quad \times \left( \int_0^{\frac{1}{2}} \left| \frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{r(\alpha+1)} - \frac{t}{r+1} \right|^p dt + \int_{\frac{1}{2}}^1 \left| \frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{r(\alpha+1)} - \frac{1-t}{r+1} \right|^p dt \right)^{\frac{1}{p}} \\
 & \leq \frac{(\varphi(b)-\varphi(a))^2}{[r(r+1)(\alpha+1)]^{1+p-1}} \left( \frac{|f''(\varphi(a))|^q + |f''(\varphi(b))|^q}{s+1} \right)^{\frac{1}{q}} \\
 & \quad \left( \times \max \left\{ [(r+1)[1-2^{-\alpha}]]^{p+1} - \left[ \frac{2+r(1-\alpha)-(1+r)2^{-\alpha-1}}{2} \right]^{p+1}, [r(\alpha+1)]^{p+1} 2^{-p-1} \right\} \right. \\
 & \quad \left. + \max \left\{ [(r+1)[1-2^{-\alpha}]]^{p+1} - \left[ \frac{2+r(1-\alpha)+(1+r)2^{-\alpha-1}}{2} \right]^{p+1}, \left[ \frac{r(\alpha+1)}{2} \right]^{p+1} \right\} \right)^{\frac{1}{p}}.
 \end{aligned}$$

Where we have used the following inequalities:

$$\begin{aligned}
 & \int_0^{\frac{1}{2}} \left| (r+1) \left[ 1 - (1-t)^{\alpha+1} - t^{\alpha+1} \right] - tr(\alpha+1) \right|^p dt \\
 & \leq \frac{[(r+1)[1-2^{-\alpha}]]^{p+1} - \left[ \frac{2+r(1-\alpha)-(1+r)2^{-\alpha-1}}{2} \right]^{p+1}}{r(\alpha+1)(p+1)},
 \end{aligned}$$

and

$$\begin{aligned}
 & \int_{\frac{1}{2}}^1 \left| -r-1 + (r+1) \left[ (1-t)^{\alpha+1} + t^{\alpha+1} \right] + tr(\alpha+1) \right|^p dt \\
 & \leq \frac{[r(\alpha+1)]^{p+1} 2^{-p-1}}{r(\alpha+1)(p+1)},
 \end{aligned}$$

and

$$\begin{aligned}
 & \int_{\frac{1}{2}}^1 \left| (r+1) \left[ 1 - (1-t)^{\alpha+1} - t^{\alpha+1} \right] - r(1-t)(\alpha+1) \right|^p dt \\
 & \leq \frac{[(r+1)[1-2^{-\alpha}]]^{p+1} - \left[ \frac{2+r(1-\alpha)+(1+r)2^{-\alpha-1}}{2} \right]^{p+1}}{r(\alpha+1)(p+1)},
 \end{aligned}$$

and

$$\begin{aligned}
 & \int_{\frac{1}{2}}^1 \left| -r-1 - tr(\alpha+1) + (r+1) \left[ (1-t)^{\alpha+1} + t^{\alpha+1} \right] + r(\alpha+1) \right|^p dt \\
 & \leq \frac{\left[ \frac{r(\alpha+1)}{2} \right]^{p+1}}{r(\alpha+1)(p+1)}.
 \end{aligned}$$

The proof is done.

□

## REFERENCES

- [1] Baleanu, D, Machado, JAT, Luo, ACJ: Fractional Dynamics and Control. Springer, New York (2012)
- [2] Diethelm, K: The Analysis of Fractional Differential Equations. Lecture Notes in Mathematics.(2010)
- [3] Kilbas, AA, Srivastava, HM , Trujillo, JJ: Theory and Applications of Fractional Differential Equations.Elsevier, Amsterdam (2006)
- [4] Lakshmikantham, V, Leela, S, Devi, JV: Theory of Fractional Dynamic Systems. Cambridge Scientific Publishers, Cambridge (2009)
- [5] Miller, KS, Ross, B: An Introduction to the Fractional Calculus and Differential Equations. Wiley, New York (1993)
- [6] Michalski, MW: Derivates of noninteger Order and Their Applications. Diss. Math. CC-CXXVIII. Inst. Math., Polish Acad.Sci.(1993)
- [7] Podlubny, I: Fractional Differential Equations.Academic Press, San Diego (1999)
- [8] Tarasov, VE. Fractional Dynamics: Application of Fractional Calculus to Dynamics of Particles, Fields and Media.Springer, Berlin (2011)
- [9] Zhang, Y, Wang, J: On some new Hermite-Hadamard inequalities involving Riemann-Liouville fractional integrals. J.Inequal. Appl. 2013, 220 (2013)
- [10] Wang, J, Li,X, Feckan, M, Zhou, Y: Hermite-Hadamard-type inequalities for Riemann-Liouville fractional integrals via two kinds of convexity. Appl. Anal. (2012).doi: 10.1080/00036811.2012.727986
- [11] Deng, J, Wang, J: Fractional Hermite-Hadamard inequalities for  $(\alpha, m)$ -logarithmically convex functions.J. Inqual.Appl. 2013, 364 (2013). doi: 10.1186/1029-242X-2013-364
- [12] Shuang, Y, Yin, HP, QI, F: Hermite-Hadamard type integral inequalities for geometric-arithmetically  $s$ -convex functions. Analysis 33, 197-208 (2013)
- [13] Liao, Y, Deng, J, Wang, j: Riemann-Liouville fractional Hermite-Hadamard inequalities. Part I: for once differentiable Geometric-Arithmetically  $s$ -convex functions. J.Inequal. Appl. 2013, 443 (2013)
- [14] DiDonato, AR, Jarnagin, MP: The efficient calculation of the incomplete beta-function ratio for half-integer values of the parameters. Math. Comput. 21, 652-662 (1967)
- [15] E. A. Youness, E-convex sets, E-convex functions and E-convex programming, J. of Optim.Theory and Appl., 102, No.2(1999), 439-450
- [16] Liao, Y, Deng, J, Wang, j: Riemann-Liouville fractional Hermite-Hadamard inequalities. Part I: for once differentiable Geometric-Arithmetically  $s$ -convex functions. J.Inequal. Appl. 2013, 443 (2013)
- [17] Liao, Y, Deng, J, Wang, j: Riemann-Liouville fractional Hermite-Hadamard inequalities. Part II: for twice differentiable Geometric-Arithmetically  $s$ -convex functions. J.Inequal. Appl. 2013, 517 (2013)

DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE AND ARTS, UNIVERSITY OF KAHRAMANMARAŞ  
 SÜTÇÜ İMAM, 46100, KAHRAMANMARAŞ, TURKEY  
*E-mail address:* [sumeyye\\_ermeydan@hotmail.com](mailto:sumeyye_ermeydan@hotmail.com)  
*E-mail address:* [hyildir@ksu.edu.tr](mailto:hyildir@ksu.edu.tr)