

ON A SOCIAL ECONOMIC MODEL

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ABSTRACT. Distribution of saving for a family sets on a region satisfies Kolmogorov equation given by

$$\frac{\partial u}{\partial t} = -\frac{\partial}{\partial x} ((c + F)u) + \frac{1}{2} \frac{\partial^2}{\partial x^2} (bu) + f$$

where $u = u(x, t)$ is density distribution of family saving. Boundary condition defined by distribution of minimum saving and total family saving are considered for the model. By the separation of variables method, eigenvalues and eigenfunctions of problem are obtained and solution is written. In addition, numerical methods are applied to problem and errors of numerical methods are presented.

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1. INTRODUCTION

Nonlocal boundary conditions are dealt with some wave, diffusion and any other physical equations [Cannon, Van der Hoek, Ionkin, Kamynin, etc...]. Generally these type of problems are solved by numeric methods or reducing point boundary conditions. In this study we consider a family saving model used in economy. This problem is expressed with diffusion equation with integral boundary conditions.

Suppose that $x(t)$ shows saving of a family at time t and satisfy the differential equation

$$(1.1) \quad dx = F(x, t) dt + G(x, t) dX, \quad G \geq 0$$

where X is Markov process, $F(x, t)$ denotes rate of change of the family saving and $G(x, t) dX$ denotes random change of family income.

For a family set let us assume that equation (1.1) describes the saving of all families by ignoring the dynamic of individual family saving. The density distribution of the saving of families $u(x, t)$ satisfies

$$(1.2) \quad \frac{\partial u}{\partial t} = -\frac{\partial}{\partial x} ((c(x, t) + F(x, t)) u) + \frac{1}{2} \frac{\partial^2}{\partial x^2} (b(x, t)u) + f(x, t)$$

with initial condition

$$(1.3) \quad u(x, 0) = \varphi(x), \quad 0 \leq x \leq l$$

and boundary conditions

$$(1.4) \quad u(0, t) = 0$$

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$$(1.5) \quad \int_0^l x u(x, t) dx = K(t), \quad t \geq 0$$

where $c(x, t)$, $b(x, t)$, $K(t)$ are continuously differentiable functions. $K(t)$ in (1.5) describes total amount of family saving in $[0, l]$ [5].

We will consider special case of problem (1.1)-(1.5) on region $D = (0 < t < \infty) \times (0 < x < l)$

$$(1.6) \quad \frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} + f(x, t)$$

$$(1.7) \quad u(x, 0) = \varphi(x), \quad 0 \leq x \leq l$$

$$(1.8) \quad u(0, t) = 0$$

$$(1.9) \quad \int_0^l x u(x, t) dx = K(t), \quad 0 \leq x \leq l$$

where $f(x, t)$, $K(t)$, $\varphi(x)$ are continuously differentiable function on region D and a is given constant. Compatibility conditions of this problem is $\int_0^l \varphi(x) dx = K(0)$.

In order to obtain classical solution of problem (1.6)-(1.9), we transform boundary conditions into homegenous ones by the transformation

$$(1.10) \quad u(x, t) = v(x, t) + 3K(t)x$$

Carrying out substitution (1.10) in (1.1)-(1.5) problem gives

$$(1.11) \quad \frac{\partial v}{\partial t} = a^2 \frac{\partial^2 v}{\partial x^2} + F(x, t)$$

$$(1.12) \quad v(x, 0) = \psi(x)$$

$$(1.13) \quad v(0, t) = 0$$

$$(1.14) \quad \int_0^l x v(x, t) dx = 0$$

where $F(x, t) = f(x, t) - 3K'(t)x$ and $\psi(x) = \varphi(x) - 3K(0)x$

This problem has homegenous boundary conditions. Due to linearity, problem can split into two auxilary problem:

i)

$$(1.15) \quad \frac{\partial v}{\partial t} = a^2 \frac{\partial^2 v}{\partial x^2}$$

$$(1.16) \quad v(x, 0) = \psi(x)$$

$$(1.17) \quad v(0, t) = 0$$

$$(1.18) \quad \int_0^1 xv(x,t)dx = 0$$

ii)

$$(1.19) \quad \frac{\partial v}{\partial t} = a^2 \frac{\partial^2 v}{\partial x^2} + F(x,t)$$

$$(1.20) \quad v(x,0) = 0$$

$$(1.21) \quad v(x,t) = 0$$

$$(1.22) \quad \int_0^1 xv(x,t)dx = 0$$

Integrating both sides of (1.15) with respect to x from 0 to 1 and using integration by parts, integral boundary condition turns into Neumann boundary condition

$$v_x(1,t) - v(1,t) = 0$$

Thus problem (1.15)-(1.18) becomes

$$(1.23) \quad \frac{\partial v}{\partial t} = a^2 \frac{\partial^2 v}{\partial x^2}$$

$$(1.24) \quad v(x,0) = \psi(x)$$

$$(1.25) \quad v(0,t) = 0$$

$$(1.26) \quad v_x(1,t) - v(1,t) = 0$$

By the Fourier method, Sturm Liouville problem and ODE are ,respectively, obtained as

$$(1.27) \quad X''(x) + \lambda X(x) = 0$$

$$(1.28) \quad X(0) = 0$$

$$(1.29) \quad X'(1) - X(1) = 0$$

and

$$(1.30) \quad T'(t) + \lambda a^2 T(t) = 0$$

Sturm Liouville problem (1.27)-(1.29)is self adjoint and boundary conditions are regular, moreover strongly regular. Then eigenfunctions of Sturm Liouville problem are Riesz basis on $L^2[0, 1]$.

Charecteristic equation of Sturm Liouville problem is

$$\tan k = k$$

Thus the problem has the eigenvalues λ_n , $n = 0, 1, 2..$ such that $\lambda_0 = 0$ and by using Langrange-Burmman formula

$$k_n = \frac{(2n+1)\pi}{2} - \left(\frac{(2n+1)\pi}{2}\right)^{-1} - \frac{2}{3} \left(\frac{(2n+1)\pi}{2}\right)^{-3} - \frac{13}{15} \left(\frac{(2n+1)\pi}{2}\right)^{-5} - \frac{146}{105} \left(\frac{(2n+1)\pi}{2}\right)^{-7} + O\left(\frac{1}{n^9}\right)$$

where $\sqrt{\lambda_n} = k_n$.

Corresponding eigenfunctions are obtained by

$$\begin{aligned} X_0(x) &= cx \\ X_n(x) &= \sin(k_n x), n = 1, 2, \dots \end{aligned}$$

Hence solution of problem (1.23)-(1.26) is

$$v_1(x, t) = A_0 x + \sum_{n=1}^{\infty} A_n e^{-a^2 k_n^2 t} \sin(k_n x)$$

where

$$\begin{aligned} A_0 &= 3 \int_0^1 x \psi(x) dx \\ A_n &= \frac{1}{\frac{1}{2} - \frac{\sin(2k_n)}{4k_n}} \int_0^1 \psi(x) \sin(k_n x) dx, n = 1, 2, \dots \end{aligned}$$

Solution of problem (1.19)-(1.22) can easily obtained by

$$v_2(x, t) = \left[\int_0^t F_0(\tau) d\tau \right] x + \sum_{n=1}^{\infty} \left[\int_0^t F_n(\tau) e^{-a^2 k_n^2 (t-\tau)} d\tau \right] \sin(k_n(x))$$

where

$$\begin{aligned} F_n(\tau) &= \int_0^1 F(x, \tau) \sin(k_n x) dx \\ F_0(\tau) &= \int_0^1 F(x, \tau) x dx \end{aligned}$$

2. NUMERICAL SOLUTION

For the numerical solution of problem, we will use the Method of Lines method and Crank-Nicolson method presented in [8],[9] respectively. In both methods, Simpson's rule is used to approximate the integral in (1.18) numerically. We display here a few of numerical results.

Example 1..

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2} + (x^2 - 2)e^t \\ u(x, 0) &= x^2 \\ u(0, t) &= 0 \\ \int_0^1 x u(x, t) dx &= \frac{e^t}{4} \end{aligned}$$

Exact solution is $u(x, t) = x^2 e^t$. The computed results at various spatial lengths are shown in Table 2. This table exhibits the absolute relative error results for $u(0.5, 0.5)$.

Example 2:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{-2(x^2 + t + 1)}{(t + 1)^3}$$

TABLE 1. Relative Error at $u(0.5,0.5)$ in Example 1

Spatial Length	MOL Method	Crank-Nicolson Method		
h=0.1	1.6291E-5	4.8730E-5		
h=0.05	2.553E-6	1.2398E-5		
h=0.025	3.6733E-7	3.1067E-6		
h=0.0125	4.8533E-8	7.7695E-7		

TABLE 2. Relative Error at $u(0.5,0.5)$ in Example 2

Spatial Length	MOL Method	Crank-Nicolson Method		
h=0.1	1.4572E-4	3.7936E-4		
h=0.05	2.2868E-5	1.0927E-4		
h=0.025	3.3245E-6	2.7647E-5		
h=0.0125	4.468E-7	6.9193E-6		

$$u(x, 0) = x^2$$

$$u(0, t) = 0$$

$$\int_0^1 xu(x, t)dx = \frac{1}{4(t+1)^2}$$

Exact solution is $u(x, t) = \frac{x^2}{(t+1)^2}$. The computed results at various spatial lengths are shown in Table 2. This table exhibits the absolute relative error results for $u(0.5, 0.5)$.

We studied a special case of family saving model which is diffusion equation with nonlocal boundary condition. Analytic solution of this problem is obtained. Moreover by applying the Method of Lines method [8] and Crank Nicolson [9], numerical solution of problem is found.

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