NEW INTEGRAL INEQUALITIES INVOLVING BETA FUNCTION VIA $P_{\varphi}$-PREINVEX CONVEXITY

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ABSTRACT. In this note, we establish some inequalities, involving the Euler Beta function, of the integral
\[ \int_a^b (x-a)^p (b-x)^q f(x) \, dx \]
for functions when a power of the absolute value is $P_{\varphi}$-preinvex.

Received: 26–August–2016 Accepted: 29–August–2016

1. INTRODUCTION

[27], Wenjun Lio introduced new inequalities for $P_{\varphi}$-convexity. We establish new Hermite-Hadamard inequalities for quasi-preinvex and $P_{\varphi}$-preinvex functions.

Let $I$ be an interval in $\mathbb{R}$. Then $f : I \to \mathbb{R}$ is said to be preinvex convex if
\[ f \left( x + (1-t) e^{i\varphi} \eta (y, x) \right) \leq t f (x) + (1-t) f (y) \]
holds for all $x, y \in I$ and $t \in [0,1]$.

The notion of quasi-preinvex functions generalizes the notion $P_{\varphi}$-preinvex functions. More precisely, a function $f : [a,b] \to \mathbb{R}$ is said to be quasi-preinvex on $[a,b]$ if,
\[ f \left( x + (1-t) e^{i\varphi} \eta (y, x) \right) \leq \max \{ f (x), f (y) \} \]
holds for any $x, y \in [a,b]$ and $t \in [0,1]$. Clearly, any preinvex function is a quasi-preinvex function. Furthermore, there exist quasi-preinvex functions which are not preinvex.

The generalized quadrature formula of Gauss-Jacobi type has the form
\[ \int_a^b (x-a)^p (b-x)^q f(x) \, dx = \sum_{k=0}^m B_{m,k} f (\gamma_k) + R_m [f] \]
for certain $B_{m,k,\gamma_k}$ and rest term $R_m [f]$ (see [22]).

Let $\mathbb{R}^n$ be Euclidian space and $K$ is said to a nonempty closed in $\mathbb{R}^n$. Let $f : K \to \mathbb{R}$, $\varphi : K \to \mathbb{R}$ and $\eta : K \times K \to \mathbb{R}$ be a continuous functions.

Definition 1.1. ([13]) Let $u \in K$. The set $K$ is said to be $\varphi$-invex at $u$ with respect to $\eta$ and $\varphi$ if
\[ u + t e^{i\varphi} \eta (v, u) \in K \]
for all $u, v \in K$ and $t \in [0,1]$. 

13th International Intuitionistic Fuzzy Sets and Contemporary Mathematics Conference
2010 Mathematics Subject Classification. 26A33, 26D15, 41A55.
Key words and phrases. Fractional Hermite-Hadamard inequaities, $\varphi$-preinvex functions, Riemann-Liouville Fractional Integral.
Remark 1.1. Some special cases of Definition 2 are as follows.

1. If \( \varphi = 0 \), then \( K \) is called an invex set.
2. If \( \eta(v, u) = v - u \), then \( K \) is called a \( \varphi \)-convex set.
3. If \( \varphi = 0 \) and \( \eta(v, u) = v - u \), then \( K \) is called a convex set.

Definition 1.2. (see [13]) The set \( K_{\varphi \eta} \) in \( \mathbb{R}^n \) is said to be \( \varphi \)-invex at \( u \) with respect to \( \varphi(\cdot) \), if there exists a bifunction \( \eta(\cdot, \cdot) : K_{\varphi \eta} \times K_{\varphi \eta} \rightarrow \mathbb{R} \), such that

\[
(1.4) \quad u + t e^{i\varphi \eta}(v, u) \in K_{\varphi \eta}, \quad \forall u, v \in K_{\varphi \eta}, \quad t \in [0, 1].
\]

Where \( \mathbb{R}^n \) Euclidian space. The \( \varphi \)-invex set \( K_{\varphi \eta} \) is also called \( \varphi \eta \)-connected set. Note that the convex set with \( \varphi = 0 \) and \( \eta(v, u) = v - u \) is a \( \varphi \)-invex set, but the converse is not true.

Definition 1.3. Let \( f : I \subseteq \mathbb{R} \rightarrow \mathbb{R} \) be a nonnegative function. A function \( f \) on the set \( K_{\varphi \eta} \) is said to be \( P_{\varphi} \)-preinvex function according to \( \varphi \) and bifunction \( \eta \).

Let \( \forall u, v \in I, \quad \eta(v, u) > 0 \) and \( t \in (0, 1) \), then

\[
(1.5) \quad f(u + t e^{i\varphi \eta}(v, u)) \leq f(u) + f(v).
\]

2. Main Results

In this section, we will give lemma which we use later in this work.

Lemma 2.1. Let \( f : [a, b] \subset [0, \infty) \rightarrow \mathbb{R} \) be continuous on \([a, b]\). Where is \( f \in L\left([a, a + e^{i\varphi \eta}(b, a)]\right) \), \( p, q > 0 \) and \( a < b \). Then the following equality holds,

\[
\int_{a}^{a + e^{i\varphi \eta}(b, a)} (x - a)^p (a + e^{i\varphi \eta}(b, a) - x)^q f(x) \, dx
= [e^{i\varphi \eta}(b, a)]^{p+q+1} \int_{0}^{1} (1 - t)^p t^q f(a + (1 - t) e^{i\varphi \eta}(b, a)) \, dt.
\]

Proof. By using Definition 3, if left-hand side of equality use \( x = a + (1 - t) e^{i\varphi \eta}(b, a) \), we have

\[
\int_{a}^{a + e^{i\varphi \eta}(b, a)} (x - a)^p (a + e^{i\varphi \eta}(b, a) - x)^q f(x) \, dx
= \int_{0}^{1} ((1 - t) e^{i\varphi \eta}(b, a))^p (te^{i\varphi \eta}(b, a))^q f(a + (1 - t) e^{i\varphi \eta}(b, a)) e^{i\varphi \eta}(b, a) \, dt
= [e^{i\varphi \eta}(b, a)]^{p+q+1} \int_{0}^{1} (1 - t)^p t^q f(a + (1 - t) e^{i\varphi \eta}(b, a)) \, dt,
\]
the proof is done.

Remark 2.1. If we consider \( \eta(b, a) = b - a \) and \( \varphi = 0 \) in Lemma 1, we obtain Lemma 1 in [27],

\[
\int_{a}^{b} (x - a)^p (b - x)^q f(x) \, dx
= [b - a]^{p+q+1} \int_{0}^{1} (1 - t)^p t^q f(at + (1 - t) b) \, dt.
\]
Theorem 2.1. Let \( f : [a, b] \to \mathbb{R} \) be continuous on \([a, b]\).
Where is \( f \in L \left( \left[ a, a + e^{i\varphi} \eta(b, a) \right] \right) \) and \( 0 \leq a < b < \infty \). If \( f \) is quasi-preinvex on \([a, b] \), then for some fixed \( p, q > 0 \), we have

\[
\int_a^{a + e^{i\varphi} \eta(b,a)} (x - a)^p \left( a + e^{i\varphi} \eta(b,a) - x \right)^q f(x) \, dx \\
\leq (e^{i\varphi} \eta(b,a))^{p+q+1} \beta(p + 1, q + 1) \max \{ f(a), f(b) \},
\]

here \( \beta(x, y) \) is the Euler Beta function.

Proof. By using inequality in (1.2), if left-hand side of equality use \( x = a + (1 - t) e^{i\varphi} \eta(b,a) \), we have

\[
\int_a^{a + e^{i\varphi} \eta(b,a)} (x - a)^p \left( a + e^{i\varphi} \eta(b,a) - x \right)^q f(x) \, dx \\
= \int_0^1 ((1 - t) e^{i\varphi} \eta(b,a))^p (te^{i\varphi} \eta(b,a))^q f(a + (1 - t) e^{i\varphi} \eta(b,a)) e^{i\varphi} \eta(b,a) \, dt \\
= \left[ e^{i\varphi} \eta(b,a) \right]^{p+q+1} \int_0^1 (1 - t)^p t^q f(a + (1 - t) e^{i\varphi} \eta(b,a)) \, dt \\
\leq \left( e^{i\varphi} \eta(b,a) \right)^{p+q+1} \beta(p + 1, q + 1) \max \{ f(a), f(b) \},
\]

the proof is done. \( \square \)

Remark 2.2. If we consider \( \eta(b,a) = b - a \) and \( \varphi = 0 \) in Theorem 1, we obtain Theorem 1 in [27]

\[
\int_a^b (x - a)^p (b - x)^q f(x) \, dx \\
\leq (b - a)^{p+q+1} \beta(p + 1, q + 1) \max \{ f(a), f(b) \}.
\]

Theorem 2.2. Let \( f : [a, b] \to \mathbb{R} \) be continuous on \([a, b]\).
Where is \( f \in L \left( \left[ a, a + e^{i\varphi} \eta(b,a) \right] \right), p, q > 0 \) and \( 0 \leq a < b < \infty \). If \( |f| \) is \( \psi \) quasi-preinvex on \([a, b] \), then following inequality, we have

\[
\int_a^{a + e^{i\varphi} \eta(b,a)} (x - a)^p \left( a + e^{i\varphi} \eta(b,a) - x \right)^q f(x) \, dx \\
\leq (e^{i\varphi} \eta(b,a))^{p+q+1} \beta(p + 1, q + 1) \max \{ |f(a)|, |f(b)| \},
\]

Proof. By using Definition 3, if left-hand side of equality use \( x = a + (1 - t) e^{i\varphi} \eta(b,a) \), we have

\[
\int_a^{a + e^{i\varphi} \eta(b,a)} (x - a)^p \left( a + e^{i\varphi} \eta(b,a) - x \right)^q f(x) \, dx \\
\leq \left[ e^{i\varphi} \eta(b,a) \right]^{p+q+1} \int_0^1 |(1 - t)^p t^q| f(a + (1 - t) e^{i\varphi} \eta(b,a)) \, dt \\
\leq \left[ e^{i\varphi} \eta(b,a) \right]^{p+q+1} \int_0^1 [(1 - t)^p t^q] \max \{ |f(a)|, |f(b)| \} \, dt \\
\leq \left( e^{i\varphi} \eta(b,a) \right)^{p+q+1} \beta(p + 1, q + 1) \max \{ |f(a)|, |f(b)| \},
\]

the proof is done. \( \square \)
Remark 2.3. If we consider $\eta(b,a) = b - a$ and $\varphi = 0$ in Theorem 2, we obtain
Theorem 4 in [27],
\[
\int_a^b (x-a)^p (b-x)^q \ f \ (x) \ dx \\
\leq (b-a)^{p+q+1} \ (p+1, q+1) \ (|f(a)| + |f(b)|).
\]

Theorem 2.3. Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$.
Where is $f \in L \left([a, a + e^{i\varphi} \eta(b,a)]\right)$, $p, q > 0$ and $0 < a < b < \infty$. If $|f|^{\frac{k}{k+1}}$ is
quasi-preinvex on $[a, b]$ and $k > 1$, then following inequality, we have
\[
\int_a^b (x-a)^p (a + e^{i\varphi} \eta(b,a) - x)^q f(x) \ dx \\
\leq \left( e^{i\varphi} \eta(b,a) \right)^{p+q+1} \left[ \beta (kp + 1, kq + 1) \right] \left( \max \left\{ |f(a)|^{\frac{k}{k+1}}, |f(b)|^{\frac{k}{k+1}} \right\} \right)^{\frac{k-1}{k}}.
\]

Proof. By using lemma 1, quasi-preinvex of $|f|^{\frac{k}{k+1}}$ and H"older’s inequality, we obtain
\[
\int_a^b (x-a)^p (a + e^{i\varphi} \eta(b,a) - x)^q f(x) \ dx \\
\leq \left[ e^{i\varphi} \eta(b,a) \right]^{p+q+1} \left[ \int_0^1 (1-t)^p \ |f(a + (1-t) e^{i\varphi} \eta(b,a))| \ dt \right] \\
\leq \left[ e^{i\varphi} \eta(b,a) \right]^{p+q+1} \left[ \int_0^1 (1-t)^p \ t^q \ dt \ \max \left\{ |f(a)|^{\frac{k}{k+1}}, |f(b)|^{\frac{k}{k+1}} \right\} \ right]^{\frac{k-1}{k+1}}.
\]

Remark 2.4. If we consider $\eta(b,a) = b - a$ and $\varphi = 0$ in Theorem 3, we obtain
Theorem 2 in [27],
\[
\int_a^b (x-a)^p (b-x)^q \ f \ (x) \ dx \\
\leq (b-a)^{p+q+1} \ (p+1, q+1) \ (|f(a)| + |f(b)|).
\]

Theorem 2.4. Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$.
Where is $f \in L \left([a, a + e^{i\varphi} \eta(b,a)]\right)$, $p, q > 0$ and $0 < a < b < \infty$. If $|f|^{\frac{k}{k+1}}$ is
$P_2$-preinvex on $[a, b]$ and $k > 1$, then following inequality, we have
\[
\int_a^b (x-a)^p (a + e^{i\varphi} \eta(b,a) - x)^q f(x) \ dx \\
\leq \left( e^{i\varphi} \eta(b,a) \right)^{p+q+1} \left[ \beta (kp + 1, kq + 1) \right] \left( \max \left\{ |f(a)|^{\frac{k}{k+1}}, |f(b)|^{\frac{k}{k+1}} \right\} \right)^{\frac{k-1}{k}}.
\]
Proof. By using lemma 1, $P_{\varphi}$-preinvex of $|f|^{\frac{1}{r}}$ and Hölder’s inequality, we obtain

\[ \int_a^b (x-a)^p \left( a + e^{i\varphi} \eta(b, a) - x \right)^q f(x) \, dx \]

\leq \left[ e^{i\varphi} \eta(b, a) \right]^{p+q+1} \int_0^1 \frac{1}{(1-t)^p \, t^q} \left| \int f \left( a + (1-t) e^{i\varphi} \eta(b, a) \right) \, dt \right| \, dt

\leq \left[ e^{i\varphi} \eta(b, a) \right]^{p+q+1} \frac{1}{(1-t)^p \, t^q} \left( \frac{1}{f(a+1-t \, e^{i\varphi} \eta(b, a))} \right)^{\frac{k}{k-1}} dt

\leq \left[ e^{i\varphi} \eta(b, a) \right]^{p+q+1} \frac{1}{(1-t)^p \, t^q} \left( \frac{1}{f(a)} + \frac{1}{f(b)} \right)^{\frac{k}{k-1}}

\leq \left( e^{i\varphi} \eta(b, a) \right)^{p+q+1} \beta^{\frac{1}{p+q+1}} \left( e^{i\varphi} \eta(b, a) \right) \left( \frac{1}{(1-t)^p \, t^q} \right)^{\frac{k}{k-1}} dt.

The proof is done.

Remark 2.5. If we consider $\eta(b, a) = b - a$ and $\varphi = 0$ in Theorem 4, we obtain Theorem 5 in [27],

\[ \int_a^b (x-a)^p \left( b - a \right)^q f(x) \, dx \]

\leq \left( b - a \right)^{p+q+1} \left( \frac{1}{(1-t)^p \, t^q} \right)^{\frac{k}{k-1}} \left( \frac{1}{(1-t)^p \, t^q} \right) dt.

Theorem 2.5. Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$. Where $f \in L \left( [a, a + e^{i\varphi} \eta(b, a)] \right)$, $p, q > 0$ and $0 \leq a < b < \infty$. If $|f|^l$ is quasi-preinvex on $[a, b]$ and $l \geq 1$, then following inequality, we have

\[ \int_a^b (x-a)^p \left( a + e^{i\varphi} \eta(b, a) - x \right)^q f(x) \, dx \]

\leq \left( e^{i\varphi} \eta(b, a) \right)^{p+q+1} \left( \frac{1}{(1-t)^p \, t^q} \right)^{\frac{k}{k-1}} \left( \frac{1}{(1-t)^p \, t^q} \right) dt.

Proof. By using lemma 1, quasi-preinvex of $|f|^l$ and Power Mean inequality, we obtain

\[ \int_a^b (x-a)^p \left( a + e^{i\varphi} \eta(b, a) - x \right)^q f(x) \, dx \]

\leq \left[ e^{i\varphi} \eta(b, a) \right]^{p+q+1} \left( \frac{1}{(1-t)^p \, t^q} \right)^{\frac{k}{k-1}} \left( \frac{1}{(1-t)^p \, t^q} \right) dt

\leq \left( e^{i\varphi} \eta(b, a) \right)^{p+q+1} \left( \frac{1}{(1-t)^p \, t^q} \right)^{\frac{k}{k-1}} \left( \frac{1}{(1-t)^p \, t^q} \right) dt

\leq \left[ e^{i\varphi} \eta(b, a) \right]^{p+q+1} \left( \frac{1}{(1-t)^p \, t^q} \right)^{\frac{k}{k-1}} \left( \frac{1}{(1-t)^p \, t^q} \right) dt

\leq \left( e^{i\varphi} \eta(b, a) \right)^{p+q+1} \beta^{\frac{1}{p+q+1}} \left( \frac{1}{(1-t)^p \, t^q} \right)^{\frac{k}{k-1}} dt

\leq \left( e^{i\varphi} \eta(b, a) \right)^{p+q+1} \beta^{\frac{1}{p+q+1}} \left( \frac{1}{(1-t)^p \, t^q} \right)^{\frac{k}{k-1}} dt,

the proof is done.
Remark 2.6. If we consider \( \eta(b,a) = b - a \) and \( \varphi = 0 \) in Theorem 5, we obtain Theorem 3 in [27],
\[
\int_a^b (x-a)^p (b-x)^q f(x) \, dx 
\leq (b-a)^{p+q+1} [\beta (p+1, q+1)] \left( \max \left\{|f(a)|^l, |f(b)|^l\right\} \right)^{\frac{1}{l}}.
\]

Theorem 2.6. Let \( f : [a, b] \to \mathbb{R} \) be continuous on \([a, b]\).
Where is \( f \in L \left( [a, a + e^{i\varphi} \eta(b, a)] \right) \), \( p, q > 0 \) and \( 0 \leq a < b < \infty \). If \( |f|^l \) is quasi-preinvex on \([a, b]\) and \( l \geq 1 \), then following inequality, we have
\[
a + e^{i\varphi} \eta(b,a) \int_a^b (x-a)^p (a + e^{i\varphi} \eta(b, a) - x)^q f(x) \, dx 
\leq \left( e^{i\varphi} \eta(b,a) \right)^{p+q+1} [\beta (p+1, q+1)] \left( |f(a)|^l + |f(b)|^l \right)^{\frac{1}{l}}.
\]
Proof. By using lemma 1, \( P_\varphi \)-preinvex of \( |f|^l \) and Power Mean inequality, we obtain
\[
a + e^{i\varphi} \eta(b,a) \int_a^b (x-a)^p (a + e^{i\varphi} \eta(b, a) - x)^q f(x) \, dx 
\leq \left( e^{i\varphi} \eta(b,a) \right)^{p+q+1} \left( \int_0^1 (1-t)^p t^q |f(a + (1-t) e^{i\varphi} \eta(b, a))| \, dt \right)^{\frac{1}{l}} 
\leq \left( e^{i\varphi} \eta(b,a) \right)^{p+q+1} \left( \int_0^1 (1-t)^p t^q \left( |f(a) + (1-t) e^{i\varphi} \eta(b, a))|^l \right) \, dt \right)^{\frac{1}{l}} 
\leq \left( e^{i\varphi} \eta(b,a) \right)^{p+q+1} \beta (p+1, q+1) \left[ |f(a)|^l + |f(b)|^l \right]^{\frac{1}{l}},
\]
the proof is done.

In this section some new integral inequalities for functions of several variables on preinvex subsets of \( \mathbb{R}^n \) will be given. First we recall the notion of \( P_\varphi \)-preinvex convexity for functions on a preinvex subset \( U \) of \( \mathbb{R}^n \).

Remark 2.7. If we consider \( \eta(b,a) = b - a \) and \( \varphi = 0 \) in Theorem 6, we obtain Theorem 6 in [27],
\[
\int_a^b (x-a)^p (b-x)^q f(x) \, dx 
\leq (b-a)^{p+q+1} [\beta (p+1, q+1)] \left( |f(a)|^l + |f(b)|^l \right)^{\frac{1}{l}}.
\]

Definition 2.1. The functions \( f : U \to \mathbb{R} \) is said to be \( P_\varphi \)-preinvex convexity on \( U \) if it is nonnegative and, for all \( x, y \in U \) and \( \lambda \in [0, 1] \), satisfies the inequality
\[
f(x + (1-\lambda) e^{i\varphi} \eta(y,x)) \leq f(x) + f(y).
\]

The following proposition will be used throughout this section.

Proposition 2.1. Let \( U \subseteq \mathbb{R} \) be a preinvex subset of \( \mathbb{R} \) and \( f : U \to \mathbb{R} \) be a function. Then \( f \) is \( P_\varphi \)-preinvex on \( U \) if and only if, for every \( x, y \in U \), the function \( \varphi : [0, 1] \to \mathbb{R} \), defined by
\[
\varphi(t) := f(x + te^{i\varphi} \eta(y,x)),
\]
is $P_\varphi$-convex on $I$ with $I = [0, 1]$.

**Theorem 2.7.** Let $U \subseteq \mathbb{R}$ be a preinvex subset of $\mathbb{R}$. Assume that $f : U \to \mathbb{R}^+$ is a $P_\varphi$-preinvex function on $U$. Then, for every $x, y \in U$ and every $[a, b] \in [0, 1]$ with $a < b$, the following inequality holds:

$$\int_a^b (t - a)^p (x + te^{i\varphi}(y, x)) \, dt \leq (e^{i\varphi}(a))^{p+q+1} \beta (p + 1, q + 1) \left[ f(x + ae^{i\varphi}(y, x)) + f(x + be^{i\varphi}(y, x)) \right].$$

**Proof.** Let $x, y \in U$ and every $[a, b] \in [0, 1]$ with $a < b$. Since $f : U \to \mathbb{R}^+$ is a $P_\varphi$-preinvex function, by Proposition 1 the function $\varphi : [0, 1] \to \mathbb{R}^+$ defined by

$$\varphi(t) := f(x + te^{i\varphi}(y, x)),$$

is $P_\varphi$-preinvex on $I$ with $I = [0, 1]$. Applying Theorem 4 to the function $\varphi$ implies that

$$\int_a^b (t - a)^p (x + te^{i\varphi}(y, x)) \varphi(t) \, dt \leq (e^{i\varphi}(a))^{p+q+1} \beta (p + 1, q + 1) \left[ \varphi(a) + \varphi(b) \right]$$

$$\leq (e^{i\varphi}(a))^{p+q+1} \beta (p + 1, q + 1) \left[ f(x + ae^{i\varphi}(y, x)) + f(x + be^{i\varphi}(y, x)) \right],$$

the proof is done. \qed

**Remark 2.8.** If we consider $\eta(b, a) = b - a$ and $\varphi = 0$ in Theorem 7, we obtain Theorem 7 in [27],

$$\int_a^b (t - a)^p (b - t)^q f((1 - t)x + ty) \, dt$$

$$\leq (b - a)^{p+q+1} \beta (p + 1, q + 1) \left[ f((1 - a)x + ay) + f((1 - b)x + by) \right].$$

**Theorem 2.8.** Let $U \subseteq \mathbb{R}$ be a preinvex subset of $\mathbb{R}$ and let $k > 1$. Assume that $f^{\frac{1}{k-1}} : U \to \mathbb{R}^+$ is a $P_\varphi$-preinvex function on $U$. Then, for every $x, y \in U$ and every $[a, b] \in [0, 1]$ with $a < b$, the following inequality holds:

$$\int_a^b (t - a)^p (x + te^{i\varphi}(y, x)) \, dt \leq (e^{i\varphi}(a))^{p+q+1} \beta (kp + 1, kq + 1) \left[ f^{\frac{k}{k-1}}(x + ae^{i\varphi}(y, x)) + f^{\frac{k}{k-1}}(x + be^{i\varphi}(y, x)) \right]^{\frac{1}{k-1}}.$$
with \( a < b \), the following inequality holds:

\[
\int_{a}^{b} (x-a)^p \left( a + e^{i\varphi} \eta(b,a) - x \right)^q f \left( x + te^{i\varphi} \eta(y,x) \right) dx \\
\leq (e^{i\varphi} \eta(b,a))^{p+q+1} \left( p + 1, q + 1 \right) \left( f' \left( x + ae^{i\varphi} \eta(y,x) \right) + f' \left( x + be^{i\varphi} \eta(y,x) \right) \right)^{\frac{1}{\eta}}.
\]

**Remark 2.10.** If we consider \( \eta(b,a) = b - a \) and \( \varphi = 0 \) in Theorem 9, we obtain Theorem 9 in [27],

\[
\int_{a}^{b} (t-a)^p (b-t)^q f \left( (1-t)x + ty \right) dt \\
\leq (b-a)^{p+q+1} \beta(p+1,q+1) \left( f' \left( (1-a)x + ay \right) + f' \left( (1-b)x + by \right) \right)^{\frac{1}{\eta}}.
\]

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