IFSCOM2016[1] Proceeding Book
No. 1 pp. 1-9 (2016)
ISBN: 978-975-6900-54-3

NEW INTEGRAL INEQUALITIES INVOLVING BETA FUNCTION VIA $P\varphi$-PREINVEX CONVEXITY

SEDA KILINÇ, SÜMEYYE ERMEYDAN, AND HÜSEYIN YILDIRIM

Abstract. In this note, we establish some inequalities, involving the Euler Beta function, of the integral $\int_a^b (x-a)^p (b-x)^q f(x)\,dx$ for functions when a power of the absolute value is $P\varphi-$ preinvex.

Received: 26–August–2016 Accepted: 29–August–2016

1. Introduction

[27] , Wenjun Lio introduced new inequalities for P$-$convexity. We establish new Hermite-Hadamard inequalities for quasi-preinvex and $P_\varphi-$preinvex functions.

Let I be an interval in $\mathbb{R}$. Then $f : I \to \mathbb{R}$ is said to be preinvex convex if

$$f\left(x + (1-t)\, e^{i\varphi}\eta(y,x)\right) \le tf(x) + (1-t)\, f(y) \tag{1.1}$$

holds for all $x, y \in I$ and $t \in [0,1]$.

The notion of quasi-preinvex functions generalizes the notion $P_\varphi-$preinvex functions. More precisely, a function $f : [a,b] \to \mathbb{R}$ is said to be quasi-preinvex on $[a,b]$ if,

$$f\left(x + (1-t)\, e^{i\varphi}\eta(y,x)\right) \le \max\{f(x), f(y)\} \tag{1.2}$$

holds for any $x, y \in [a,b]$ and $t \in [0,1]$. Clearly, any preinvex function is a quasi-preinvex function. Furthermore, there exist quasi-preinvex functions which are not preinvex.

The generalized quadrature formula of Gauss-Jacobi type has the form

$$\int_a^b (x-a)^p (b-x)^q f(x)\,dx \quad = \sum_{k=0}^{m} B_{m,k} f(\gamma_k) + \Re_m[f]$$

for certain B_{m,k,γ_k} and rest term $\Re_m[f]$ (*see* [22]).

Let $\mathbb{R}^n$ be Euclidian space and K is said to a nonempty closed in $\mathbb{R}^n$. Let $f : K \to \mathbb{R}$, $\varphi : K \to \mathbb{R}$ and $\eta : K \times K \to \mathbb{R}$ be a continuous functions.

Definition 1.1. ([13]) Let $u \in K$. The set K is said to be $\varphi-$invex at u with respect to η and φ if

$$u + te^{i\varphi}\eta(v,u) \in K \tag{1.3}$$

for all $u, v \in K$ and $t \in [0,1]$.

[1] 3^{rd} International Intuitionistic Fuzzy Sets and Contemporary Mathemathics Conference

2010 *Mathematics Subject Classification.* 26A33, 26D15, 41A55.

Key words and phrases. Fractional Hermite-Hadamard ineauqualities, φ-preinvex functions, Riemann-Liouville Fractional Integral.

Remark 1.1. Some special cases of Definition 2 are as follows.

(1) If $\varphi = 0$, then K is called an invex set.
(2) If $\eta(v,u) = v - u$, then K is called a $\varphi-$convex set.
(3) If $\varphi = 0$ and $\eta(v,u) = v - u$, then K is called a convex set.

Definition 1.2. (*see* [13]) The set $K_{\varphi\eta}$ in $\mathbb{R}^n$ is said to be $\varphi-$invex at u with respect to $\varphi(.)$, if there exists a bifunction $\eta(.,.) : K_{\varphi\eta} \times K_{\varphi\eta} \to \mathbb{R}$, such that

$$u + te^{i\varphi}\eta(v,u) \in K_{\varphi\eta}, \quad \forall u,v \in K_{\varphi\eta}, \ t \in [0,1]. \tag{1.4}$$

Where $\mathbb{R}^n$ Euclidian space. The $\varphi-$invex set $K_{\varphi\eta}$ is also called $\varphi\eta-$connected set. Note that the convex set with $\varphi = 0$ and $\eta(v,u) = v - u$ is a $\varphi-$invex set, but the converse is not true.

Definition 1.3. Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a nonnegative function. A function f on the set $K_{\varphi\eta}$ is said to be $P_\varphi-$preinvex function according to φ and bifunction η. Let $\forall u,v \in I, \ \eta(v,u) > 0$ and $t \in (0,1)$, then

$$f\left(u + te^{i\varphi}\eta(v,u)\right) \leq f(u) + f(v). \tag{1.5}$$

2. Main Results

In this section, we will give lemma which we use later in this work.

Lemma 2.1. *Let $f : [a,b] \subset [0,\infty) \to \mathbb{R}$ be continuous on $[a,b]$. Where is $f \in L\left(\left[a, a + e^{i\varphi}\eta(b,a)\right]\right)$, $p,q > 0$ and $a < b$. Then the following equality holds,*

$$\int_a^{a+e^{i\varphi}\eta(b,a)} (x-a)^p \left(a + e^{i\varphi}\eta(b,a) - x\right)^q f(x)\,dx$$
$$= \left[e^{i\varphi}\eta(b,a)\right]^{p+q+1} \int_0^1 (1-t)^p t^q f\left(a + (1-t)e^{i\varphi}\eta(b,a)\right) dt.$$

Proof. By using Definition 3, if left-hand side of equality use $x = a + (1-t)e^{i\varphi}\eta(b,a)$, we have

$$\int_a^{a+e^{i\varphi}\eta(b,a)} (x-a)^p \left(a + e^{i\varphi}\eta(b,a) - x\right)^q f(x)\,dx$$
$$= \int_0^1 \left((1-t)e^{i\varphi}\eta(b,a)\right)^p \left(te^{i\varphi}\eta(b,a)\right)^q f\left(a + (1-t)e^{i\varphi}\eta(b,a)\right) e^{i\varphi}\eta(b,a)\,dt$$
$$= \left[e^{i\varphi}\eta(b,a)\right]^{p+q+1} \int_0^1 (1-t)^p t^q f\left(a + (1-t)e^{i\varphi}\eta(b,a)\right) dt,$$

the proof is done. □

Remark 2.1. If we consider $\eta(b,a) = b - a$ and $\varphi = 0$ in Lemma 1, we obtain Lemma 1 in [27],

$$\int_a^b (x-a)^p (b-x)^q f(x)\,dx$$
$$= [b-a]^{p+q+1} \int_0^1 (1-t)^p t^q f(at + (1-t)b)\,dt.$$

Theorem 2.1. *Let $f:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$. Where is $f\in L\left(\left[a,a+e^{i\varphi}\eta(b,a)\right]\right)$ and $0\leq a<b<\infty$. If f is quasi-preinvex on $[a,b]$, then for some fixed $p,q>0$, we have*

$$\int_{a}^{a+e^{i\varphi}\eta(b,a)}(x-a)^{p}\left(a+e^{i\varphi}\eta(b,a)-x\right)^{q}f(x)\,dx \leq \left(e^{i\varphi}\eta(b,a)\right)^{p+q+1}\beta(p+1,q+1)\max\{f(a),f(b)\},$$

here $\beta(x,y)$ is the Euler Beta function.

Proof. By using inequality in (1.2), if left-hand side of equality use $x=a+(1-t)e^{i\varphi}\eta(b,a)$, we have

$$\begin{aligned}
&\int_{a}^{a+e^{i\varphi}\eta(b,a)}(x-a)^{p}\left(a+e^{i\varphi}\eta(b,a)-x\right)^{q}f(x)\,dx\\
&=\int_{0}^{1}\left((1-t)e^{i\varphi}\eta(b,a)\right)^{p}\left(te^{i\varphi}\eta(b,a)\right)^{q}f\left(a+(1-t)e^{i\varphi}\eta(b,a)\right)e^{i\varphi}\eta(b,a)\,dt\\
&=\left[e^{i\varphi}\eta(b,a)\right]^{p+q+1}\int_{0}^{1}(1-t)^{p}t^{q}f\left(a+(1-t)e^{i\varphi}\eta(b,a)\right)dt\\
&\leq\left(e^{i\varphi}\eta(b,a)\right)^{p+q+1}\beta(p+1,q+1)\max\{f(a),f(b)\},
\end{aligned}$$

the proof is done. □

Remark 2.2. If we consider $\eta(b,a)=b-a$ and $\varphi=0$ in Theorem 1, we obtain Theorem 1 in [27]

$$\int_{a}^{b}(x-a)^{p}(b-x)^{q}f(x)\,dx \leq (b-a)^{p+q+1}\beta(p+1,q+1)\max\{f(a),f(b)\}.$$

Theorem 2.2. *Let $f:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$. Where is $f\in L\left(\left[a,a+e^{i\varphi}\eta(b,a)\right]\right)$, $p,q>0$ and $0\leq a<b<\infty$. If $|f|$ is P_{φ}-preinvex on $[a,b]$, then following inequality, we have*

$$\int_{a}^{a+e^{i\varphi}\eta(b,a)}(x-a)^{p}\left(a+e^{i\varphi}\eta(b,a)-x\right)^{q}f(x)\,dx \leq \left(e^{i\varphi}\eta(b,a)\right)^{p+q+1}\beta(p+1,q+1)\left(|f(a)|+|f(b)|\right),$$

Proof. By using Definition 3, if left-hand side of equality use $x=a+(1-t)e^{i\varphi}\eta(b,a)$, we have

$$\begin{aligned}
&\int_{a}^{a+e^{i\varphi}\eta(b,a)}(x-a)^{p}\left(a+e^{i\varphi}\eta(b,a)-x\right)^{q}f(x)\,dx\\
&\leq\left[e^{i\varphi}\eta(b,a)\right]^{p+q+1}\int_{0}^{1}\left|(1-t)^{p}t^{q}\right|\left|f\left(a+(1-t)e^{i\varphi}\eta(b,a)\right)\right|dt\\
&\leq\left[e^{i\varphi}\eta(b,a)\right]^{p+q+1}\int_{0}^{1}\left[(1-t)^{p}t^{q}\right]\left(|f(a)|+|f(b)|\right)dt\\
&\leq\left(e^{i\varphi}\eta(b,a)\right)^{p+q+1}\beta(p+1,q+1)\left(|f(a)|+|f(b)|\right),
\end{aligned}$$

the proof is done. □

Remark 2.3. If we consider $\eta(b,a) = b - a$ and $\varphi = 0$ in Theorem 2, we obtain Theorem 4 in [27],

$$\int_a^b (x-a)^p (b-x)^q f(x)\,dx \leq (b-a)^{p+q+1} \beta(p+1,q+1)\left(|f(a)| + |f(b)|\right).$$

Theorem 2.3. *Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$. Where is $f \in L\left(\left[a, a + e^{i\varphi}\eta(b,a)\right]\right)$, $p, q > 0$ and $0 \leq a < b < \infty$. If $|f|^{\frac{k}{k-1}}$ is quasi-preinvex on $[a,b]$ and $k > 1$, then following inequality, we have*

$$\int_a^{a+e^{i\varphi}\eta(b,a)} (x-a)^p \left(a + e^{i\varphi}\eta(b,a) - x\right)^q f(x)\,dx \leq \left(e^{i\varphi}\eta(b,a)\right)^{p+q+1} \left[\beta(kp+1,kq+1)\right]^{\frac{1}{k}} \left(\max\left\{|f(a)|^{\frac{k}{k-1}}, |f(b)|^{\frac{k}{k-1}}\right\}\right)^{\frac{k-1}{k}}.$$

Proof. By using lemma 1, quasi-preinvex of $|f|^{\frac{k}{k-1}}$ and Hölder's inequality, we obtain

$$\begin{aligned}
&\int_a^{a+e^{i\varphi}\eta(b,a)} (x-a)^p \left(a + e^{i\varphi}\eta(b,a) - x\right)^q f(x)\,dx \\
&\leq \left[e^{i\varphi}\eta(b,a)\right]^{p+q+1} \int_0^1 \left|(1-t)^p t^q\right| \left|f\left(a + (1-t)e^{i\varphi}\eta(b,a)\right)\right| dt \\
&\leq \left[e^{i\varphi}\eta(b,a)\right]^{p+q+1} \left(\int_0^1 \left|(1-t)^p t^q\right|^k dt\right)^{\frac{1}{k}} \left(\int_0^1 \left|f\left(a + (1-t)e^{i\varphi}\eta(b,a)\right)\right|^{\frac{k}{k-1}} dt\right)^{\frac{k-1}{k}} \\
&\leq \left[e^{i\varphi}\eta(b,a)\right]^{p+q+1} \left(\int_0^1 (1-t)^{kp} t^{kq} dt\right)^{\frac{1}{k}} \left(\int_0^1 \max\left(|f(a)|^{\frac{k}{k-1}}, |f(b)|^{\frac{k}{k-1}}\right) dt\right)^{\frac{k-1}{k}} \\
&\leq \left(e^{i\varphi}\eta(b,a)\right)^{p+q+1} \beta^{\frac{1}{k}}(kp+1,kq+1) \left[\max\left(|f(a)|^{\frac{k}{k-1}}, |f(b)|^{\frac{k}{k-1}}\right)\right]^{\frac{k-1}{k}},
\end{aligned}$$

the proof is done. ∎

Remark 2.4. If we consider $\eta(b,a) = b - a$ and $\varphi = 0$ in Theorem 3, we obtain Theorem 2 in [27],

$$\int_a^b (x-a)^p (b-x)^q f(x)\,dx \leq (b-a)^{p+q+1} \left[\beta(kp+1,kq+1)\right]^{\frac{1}{k}} \left(\max\left\{|f(a)|^{\frac{k}{k-1}}, |f(b)|^{\frac{k}{k-1}}\right\}\right)^{\frac{k-1}{k}}.$$

Theorem 2.4. *Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$. Where is $f \in L\left(\left[a, a + e^{i\varphi}\eta(b,a)\right]\right)$, $p, q > 0$ and $0 \leq a < b < \infty$. If $|f|^{\frac{k}{k-1}}$ is P_φ-preinvex on $[a,b]$ and $k > 1$, then following inequality, we have*

$$\int_a^{a+e^{i\varphi}\eta(b,a)} (x-a)^p \left(a + e^{i\varphi}\eta(b,a) - x\right)^q f(x)\,dx \leq \left(e^{i\varphi}\eta(b,a)\right)^{p+q+1} \left[\beta(kp+1,kq+1)\right]^{\frac{1}{k}} \left(|f(a)|^{\frac{k}{k-1}} + |f(b)|^{\frac{k}{k-1}}\right)^{\frac{k-1}{k}}.$$

Proof. By using lemma 1, P_{φ}-preinvex of $|f|^{\frac{k}{k-1}}$ and Hölder's inequality, we obtain

$$\begin{aligned}
&\int_{a}^{a+e^{i\varphi}\eta(b,a)} (x-a)^{p}\left(a+e^{i\varphi}\eta(b,a)-x\right)^{q} f(x)\,dx \\
&\leq \left[e^{i\varphi}\eta(b,a)\right]^{p+q+1}\int_{0}^{1}\left|(1-t)^{p}t^{q}\right|\left|f\left(a+(1-t)e^{i\varphi}\eta(b,a)\right)\right|dt \\
&\leq \left[e^{i\varphi}\eta(b,a)\right]^{p+q+1}\left(\int_{0}^{1}\left|(1-t)^{p}t^{q}\right|^{k}dt\right)^{\frac{1}{k}}\left(\int_{0}^{1}\left|f\left(a+(1-t)e^{i\varphi}\eta(b,a)\right)\right|^{\frac{k}{k-1}}dt\right)^{\frac{k-1}{k}} \\
&\leq \left[e^{i\varphi}\eta(b,a)\right]^{p+q+1}\left(\int_{0}^{1}(1-t)^{kp}t^{kq}dt\right)^{\frac{1}{k}}\left(\int_{0}^{1}\left(|f(a)|^{\frac{k}{k-1}}+|f(b)|^{\frac{k}{k-1}}\right)dt\right)^{\frac{k-1}{k}} \\
&\leq \left(e^{i\varphi}\eta(b,a)\right)^{p+q+1}\beta^{\frac{1}{k}}(kp+1,kq+1)\left[|f(a)|^{\frac{k}{k-1}}+|f(b)|^{\frac{k}{k-1}}\right]^{\frac{k-1}{k}},
\end{aligned}$$

The proof is done. □

Remark 2.5. If we consider $\eta(b,a)=b-a$ and $\varphi=0$ in Theorem 4, we obtain Theorem 5 in [27],

$$\begin{aligned}
&\int_{a}^{b}(x-a)^{p}(b-x)^{q}f(x)\,dx \\
&\qquad\leq (b-a)^{p+q+1}\left[\beta(kp+1,kq+1)\right]^{\frac{1}{k}}\left(|f(a)|^{\frac{k}{k-1}}+|f(b)|^{\frac{k}{k-1}}\right)^{\frac{k-1}{k}}.
\end{aligned}$$

Theorem 2.5. *Let $f:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$. Where is $f\in L\left(\left[a,a+e^{i\varphi}\eta(b,a)\right]\right)$, $p,q>0$ and $0\leq a<b<\infty$. If $|f|^{l}$ is quasi-preinvex on $[a,b]$ and $l\geq 1$, then following inequality, we have*

$$\begin{aligned}
&\int_{a}^{a+e^{i\varphi}\eta(b,a)} (x-a)^{p}\left(a+e^{i\varphi}\eta(b,a)-x\right)^{q} f(x)\,dx \\
&\qquad\leq \left(e^{i\varphi}\eta(b,a)\right)^{p+q+1}\left[\beta(p+1,q+1)\right]\left(\max\left\{|f(a)|^{l},|f(b)|^{l}\right\}\right)^{\frac{1}{l}}.
\end{aligned}$$

Proof. By using lemma 1, quasi-preinvex of $|f|^{l}$ and Power Mean inequality, we obtain

$$\begin{aligned}
&\int_{a}^{a+e^{i\varphi}\eta(b,a)} (x-a)^{p}\left(a+e^{i\varphi}\eta(b,a)-x\right)^{q} f(x)\,dx \\
&\leq \left[e^{i\varphi}\eta(b,a)\right]^{p+q+1}\int_{0}^{1}\left|(1-t)^{p}t^{q}\right|\left|f\left(a+(1-t)e^{i\varphi}\eta(b,a)\right)\right|dt \\
&\leq \left[e^{i\varphi}\eta(b,a)\right]^{p+q+1}\left(\int_{0}^{1}\left|(1-t)^{p}t^{q}\right|dt\right)^{1-\frac{1}{l}}\left(\int_{0}^{1}\left|(1-t)^{p}t^{q}\right|\left|f\left(a+(1-t)e^{i\varphi}\eta(b,a)\right)\right|^{l}dt\right)^{\frac{1}{l}} \\
&\leq \left[e^{i\varphi}\eta(b,a)\right]^{p+q+1}\left(\beta(p+1,q+1)\right)^{1-\frac{1}{l}}\left(\int_{0}^{1}\left|(1-t)^{p}t^{q}\right|\max\left(|f(a)|^{l},|f(b)|^{l}\right)dt\right)^{\frac{1}{l}} \\
&\leq \left(e^{i\varphi}\eta(b,a)\right)^{p+q+1}\beta(p+1,q+1)\left[\max\left(|f(a)|^{l},|f(b)|^{l}\right)\right]^{\frac{1}{l}},
\end{aligned}$$

the proof is done. □

Remark 2.6. If we consider $\eta(b,a) = b-a$ and $\varphi = 0$ in Theorem 5, we obtain Theorem 3 in [27],

$$\int_a^b (x-a)^p (b-x)^q f(x)\,dx \le (b-a)^{p+q+1} [\beta(p+1,q+1)] \left(\max\left\{|f(a)|^l, |f(b)|^l\right\}\right)^{\frac{1}{l}}.$$

Theorem 2.6. *Let $f:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$. Where is $f \in L\left(\left[a, a+e^{i\varphi}\eta(b,a)\right]\right)$, $p,q>0$ and $0\le a<b<\infty$. If $|f|^l$ is quasi-preinvex on $[a,b]$ and $l\ge 1$, then following inequality, we have*

$$\int_a^{a+e^{i\varphi}\eta(b,a)} (x-a)^p \left(a+e^{i\varphi}\eta(b,a)-x\right)^q f(x)\,dx \le \left(e^{i\varphi}\eta(b,a)\right)^{p+q+1} [\beta(p+1,q+1)] \left(|f(a)|^l + |f(b)|^l\right)^{\frac{1}{l}}.$$

Proof. By using lemma 1, P_φ-preinvex of $|f|^l$ and Power Mean inequality, we obtain

$$\begin{aligned}
&\int_a^{a+e^{i\varphi}\eta(b,a)} (x-a)^p \left(a+e^{i\varphi}\eta(b,a)-x\right)^q f(x)\,dx \\
&\le \left[e^{i\varphi}\eta(b,a)\right]^{p+q+1} \int_0^1 |(1-t)^p t^q| \left|f\left(a+(1-t)e^{i\varphi}\eta(b,a)\right)\right| dt \\
&\le \left[e^{i\varphi}\eta(b,a)\right]^{p+q+1} \left(\int_0^1 |(1-t)^p t^q|\,dt\right)^{1-\frac{1}{l}} \left(\int_0^1 |(1-t)^p t^q| \left|f\left(a+(1-t)e^{i\varphi}\eta(b,a)\right)\right|^l dt\right)^{\frac{1}{l}} \\
&\le \left[e^{i\varphi}\eta(b,a)\right]^{p+q+1} (\beta(p+1,q+1))^{1-\frac{1}{l}} \left(\int_0^1 |(1-t)^p t^q| \left(|f(a)|^l + |f(b)|^l\right) dt\right)^{\frac{1}{l}} \\
&\le \left(e^{i\varphi}\eta(b,a)\right)^{p+q+1} \beta(p+1,q+1) \left[|f(a)|^l + |f(b)|^l\right]^{\frac{1}{l}},
\end{aligned}$$

the proof is done.

In this section some new integral inequalities for functions of several variables on preinvex subsets of $\mathbb{R}^n$ will be given. First we recall the notion of P_φ-preinvex convexity for functions on a preinvex subset U of $\mathbb{R}^n$. □

Remark 2.7. If we consider $\eta(b,a) = b-a$ and $\varphi = 0$ in Theorem 6, we obtain Theorem 6 in [27],

$$\int_a^b (x-a)^p (b-x)^q f(x)\,dx \le (b-a)^{p+q+1} [\beta(p+1,q+1)] \left(|f(a)|^l + |f(b)|^l\right)^{\frac{1}{l}}.$$

Definition 2.1. The functions $f: U\to\mathbb{R}$ is said to be P_φ-preinvex convexity on U if it is nonnegative and, for all $x,y\in U$ and $\lambda\in[0,1]$, satisfies the inequality

$$f\left(x+(1-\lambda)e^{i\varphi}\eta(y,x)\right) \le f(x)+f(y).$$

The following proposition will be used throughout this section.

Proposition 2.1. *Let $U\subseteq\mathbb{R}$ be a preinvex subset of $\mathbb{R}$ and $f:U\to\mathbb{R}$ be a function. Then f is P_φ-preinvex on U if and only if, for every $x,y\in U$, the function $\varphi:[0,1]\to\mathbb{R}$, defined by*

$$\varphi(t) := f\left(x+te^{i\varphi}\eta(y,x)\right),$$

is P_{φ}-convex on I with $I=[0,1]$.

Theorem 2.7. *Let $U\subseteq\mathbb{R}$ be a preinvex subset of $\mathbb{R}$. Assume that $f:U\to\mathbb{R}^{+}$ is a P_{φ}-preinvex function on U. Then, for every $x,y\in U$ and every $[a,b]\in[0,1]$ with $a<b$, the following inequality holds:*

$$\int_{a}^{a+e^{i\varphi}\eta(b,a)}(t-a)^{p}\left(a+e^{i\varphi}\eta(b,a)-t\right)^{q}f\left(x+te^{i\varphi}\eta(y,x)\right)dt$$
$$\leq\left(e^{i\varphi}\eta(b,a)\right)^{p+q+1}\beta(p+1,q+1)\left[f\left(x+ae^{i\varphi}\eta(y,x)\right)+f\left(x+be^{i\varphi}\eta(y,x)\right)\right].$$

Proof. Let $x,y\in U$ and every $[a,b]\in[0,1]$ with $a<b$. Since $f:U\to\mathbb{R}^{+}$ is a P_{φ}-preinvex function, by Proposition 1 the function $\varphi:[0,1]\to\mathbb{R}^{+}$ defined by

$$\varphi(t):=f\left(x+te^{i\varphi}\eta(y,x)\right),$$

is P_{φ}-preinvex on I with $I=[0,1]$. Appling Theorem 4 to the function φ implies that

$$\int_{a}^{a+e^{i\varphi}\eta(b,a)}(t-a)^{p}\left(a+e^{i\varphi}\eta(b,a)-t\right)^{q}\varphi(t)\,dt$$
$$\leq\left(e^{i\varphi}\eta(b,a)\right)^{p+q+1}\beta(p+1,q+1)\left[|\varphi(a)|+|\varphi(b)|\right]$$
$$\leq\left(e^{i\varphi}\eta(b,a)\right)^{p+q+1}\beta(p+1,q+1)\left[f\left(x+ae^{i\varphi}\eta(y,x)\right)+f\left(x+be^{i\varphi}\eta(y,x)\right)\right],$$

the proof is done. □

Remark 2.8. If we consider $\eta(b,a)=b-a$ and $\varphi=0$ in Theorem 7, we obtain Theorem 7 in [27],

$$\int_{a}^{b}(t-a)^{p}(b-t)^{q}f((1-t)x+ty)\,dt$$
$$(b-a)^{p+q+1}\beta(p+1,q+1)\left[f((1-a)x+ay)+f((1-b)x+by)\right].$$

Theorem 2.8. *Let $U\subseteq\mathbb{R}$ be a preinvex subset of $\mathbb{R}$ and let $k>1$. Assume that $f^{\frac{k}{k-1}}:U\to\mathbb{R}^{+}$ is a P_{φ}-preinvex function on U. Then, for every $x,y\in U$ and every $[a,b]\in[0,1]$ with $a<b$, the following inequality holds:*

$$\int_{a}^{a+e^{i\varphi}\eta(b,a)}(x-a)^{p}\left(a+e^{i\varphi}\eta(b,a)-x\right)^{q}f\left(x+te^{i\varphi}\eta(y,x)\right)dx$$
$$\leq\left(e^{i\varphi}\eta(b,a)\right)^{p+q+1}\left[\beta(kp+1,kq+1)\right]^{\frac{1}{k}}\left(f^{\frac{k}{k-1}}\left(x+ae^{i\varphi}\eta(y,x)\right)+f^{\frac{k}{k-1}}\left(x+be^{i\varphi}\eta(y,x)\right)\right)^{\frac{k-1}{k}}.$$

Remark 2.9. If we consider $\eta(b,a)=b-a$ and $\varphi=0$ in Theorem 8, we obtain Theorem 8 in [27],

$$\int_{a}^{b}(t-a)^{p}(b-t)^{q}f((1-t)x+ty)\,dt$$
$$\leq(b-a)^{p+q+1}\left[\beta(kp+1,kq+1)\right]^{\frac{1}{k}}\left(f^{\frac{k}{k-1}}((1-a)x+ay)+f^{\frac{k}{k-1}}((1-b)x+by)\right)^{\frac{k-1}{k}}.$$

Theorem 2.9. *Let $U\subseteq\mathbb{R}$ be a preinvex subset of $\mathbb{R}$ and let $k>1$. Assume that $f^{l}:U\to\mathbb{R}^{+}$ is a P_{φ}-preinvex function on U. Then, for every $x,y\in U$ and every*

$[a,b]\in[0,1]$ *with* $a<b$*, the following inequality holds:*

$$\int\limits_{a}^{a+e^{i\varphi}\eta(b,a)} (x-a)^{p}\left(a+e^{i\varphi}\eta(b,a)-x\right)^{q} f\left(x+te^{i\varphi}\eta(y,x)\right)dx$$
$$\leq \left(e^{i\varphi}\eta(b,a)\right)^{p+q+1}\beta(p+1,q+1)\left(f^{l}\left(x+ae^{i\varphi}\eta(y,x)\right)+f^{l}\left(x+be^{i\varphi}\eta(y,x)\right)\right)^{\frac{1}{l}}.$$

Remark 2.10. If we consider $\eta(b,a)=b-a$ and $\varphi=0$ in Theorem 9, we obtain Theorem 9 in [27],

$$\int\limits_{a}^{b}(t-a)^{p}(b-t)^{q}f((1-t)x+ty)\,dt$$
$$\leq (b-a)^{p+q+1}\beta(p+1,q+1)\left(f^{l}((1-a)x+ay)+f^{l}((1-b)x+by)\right)^{\frac{1}{l}}.$$

References

[1] A. O. Akdemir and M. E. Özdemir, "Some Hadamard- type inequalities for coordinated P-convex functions and Godunova-Levin functions,' AIP Conf. Proc., vol. 1309, pp. 7-15,2010.

[2] M. Alomari, M. Darus, and S.S. Dragomir, 'Inequalities of Hermite-Hadamard's type for functions whose derivates absolute values are quasi-convex. " RGMIA Res. Rep. Coll., vol. 12, p.11, Article 14,2009.

[3] A. Barani and S. Barani, "Hermite-Hadamard type inequalities for functions when a power of the absolute value of the first derivative is P-convex, " Bull. Aust. Math. Soc., vol. 86, no. 1, pp. 126-134, 2012.

[4] S. S. Dragomirr, "On some new inequalities of Hermite-Hadamard type for m-convex functions. "Tamkang J. Math., vol. 33, no. 1, pp. 55-65, 2012

[5] S. S. Dagomir and C. E. M. Pearce, Selected toics on Hermite-Hadamard inequalities and applications, ser. RGMIA Monographs. Victoria University, 2000.

[6] S. S. Dragomir, J. E. Pećarić, and L. Persson, "Some inequalities of Hadamard type" Soochow J. Math., vol. 21, no. 3, pp. 335-341, 1995.

[7] S. S. Dragomir and C. Pearce, "Quasi-convex functions and Hadamar's inequality" Bull. Aust. Math. Soc., vol.57 no. 3, pp. 377-385, 1998.

[8] A. Hāzy, "Bernstein- Doetsch type results for h-convex functions,"Math. Inequal., vol. 14, no. 3, pp. 499-508,2011.

[9] A. Hāzy, "Bernstein- Doetsch type results for (k, h)-convex functions,"Miskolc Math. Notes, vol. 13, no. 2, pp. 325-336, 2012.

[10] V.N.Huy and Nguyen Thanh Chung, "Some generalizations of the Fejēr and Hermide-Hadamard inequalities in Hölder spaces ," J. Appl. Math. Inform., vol. 29, no. 3-4, pp. 859-868. 2011.

[11] D.A. Ion, "Some estimates on the Hermite- Hadamard inequality through quasi-convex functions,"An. Univ. Craiova, Ser. Mat. Inf., vol. 34, pp. 83-88, 2007.

[12] W. Liu, "New integral inequalities via $(\alpha,m)-$convexity and quasi-convexity, "Hacet. J. Math. Stat., vol. 42, no. 3, pp. 289-297, 2013.

[13] W. Liu, "Some Ostrowski type inequalities via Riemann-Liouville fractional intagrals for $h-$convex functions," J. Comput. Anal. Appl., vol. 16, no. 5, pp. 998-1004, 2014.

[14] W. Liu, "Some Simpson type inequalities for $h-$convex and $(\alpha,m)-$convex functions," J. Comput. Anal. Appl., vol. 16, no. 5, pp. 1005-1012, 2014

[15] B. Micherda and T. Rajba, " On some Hermite-Hadamard-Fejèr inequalities for $(k,h)-$convex functions," Math. Inequal. Appl. vol. 15, no. 4, pp. 931-940, 2012.

[16] M. E. Özdemir and C. Yildiz, " New inequalities for Hermite- Hadamard and simpson type and applications, " arXiv, no. 1103.1965v1 [math. CA].

[17] M. E. Özdemir, E. Set, and M.Alomari, "Integral inequalities via several kinds of convexity,"Creat. Math. Inform., vol. 20, no. 1, pp. 62-73, 2011.

[18] M. Z. Sarikaya, A. Saglam, and H. Yildirim, " On some Hadamard-type inequalities for $h-$convex functions," J. Math. Inequal., vol.2, no.3, pp. 335-341,2008.

[19] M. Z. Sarikaya, E. Set, and M. E. Özdemir, " On some new inequalities of Hadamard type involving $h-$convex functions," Acta Math. Univ. Comen., New Ser., vol. 79, no. 2, pp. 265-272, 2010.

[20] E. Set, M. Sardari, M.E.Özdemir, and J. Rooin, "On generalizations of the Hadamard inequality for $(\alpha, m)-$convex functions," RGMIA Res. Rep. Coll., vol. 12,p. 10, Article 4, 2009
[21] E. Set, M. E. Özdemir, and S. S. Dragomir," On Hadamard type inequalities involvineg several kinds of convexity, " J. Inequal. Appl., vol. 2010, p. 12,2010.
[22] D. D. Stancu , G. Coman, and P. Blaga, Analiză numerică şi teoria aproximării Vol. II. Cluj-Napoca: Presa Universitară Clujeannă, 2012.
[23] K. L. Tseng, G. S. Yang, and S. S. Dragamir, "On quasiconvex functions and Hadamard's inequality," RGMIA Res. Rep. Coll., vol. 6, NO. 3,p. 12, Article 1, 2003.
[24] M. Tunç, "On some nequalities via $h-$convexity," Miskolc Math. Notes, vol. 14, no. 3, pp. 1041-1057, 2013.
[25] M. Tunç," Ostrowski type inequalities via $h-$convex functions with applications to special means, " J.Inequal. Appl., vol. 2013, p. 10, 2013.
[26] S. Varošanec, " On $h-$convexity, " J. Math. Appl., vol. 326, no. 1, pp. 303-311,2007.
[27] W. Liu, "New Integral Inequalities Involving Beta Function via P-Convexity," Miskolc Matematical Notes, Vol. 15 No. 2, pp. 585-591, 2014.

Department of Mathematics, Faculty of Science and Arts, University of Kahramanmaraş Sütçü İmam, 46000, Kahramanmaraş, Turkey

E-mail address: sedaaa_kilinc@hotmail.com

E-mail address: sumeyye_ermeydan@hotmail.com

E-mail address: hyildir@ksu.edu.tr